Financial Mathematics
Conditional probability and independence
Definition: The **conditional probability** of $P$ given $Q$ is

$$
\Pr[P | Q] := \frac{\Pr[P \& Q]}{\Pr[Q]}
$$

Warning: Only defined when $\Pr[Q] \neq 0$.

Is $P$ likely or unlikely? Given that you’re told $Q$ happened, is $P$ likely or unlikely?
Definition: The **conditional probability** of $P$ given $Q$ is

$$\Pr[P \mid Q] := \frac{\Pr[P \land Q]}{\Pr[Q]}$$

**Warning:** Only defined when $\Pr[Q] \neq 0$.

---

$C_1 := \text{same distr. coin-flipping standard}$

$C_2 := \text{standard}$

Key point: Finding out $C_1 = 1$ has no influence on the prob. that $C_2 = 1$.

$$\Pr[(C_2 = 1) \mid (C_1 = 1)] = \frac{0.25}{0.5} = 0.5 = \Pr[C_2 = 1]$$
Definition: The **conditional probability** of \( P \) given \( Q \) is

\[
\Pr[P \mid Q] := \frac{\Pr[P \amp Q]}{\Pr[Q]}
\]

**Warning:** Only defined when \( \Pr[Q] \neq 0 \).

Assume \( \Pr[Q] \neq 0 \):

\( P \amp Q \) are **independent** (events)

if \( \Pr[P \mid Q] = \Pr[P] \),

i.e.: if \( \frac{\Pr[P \amp Q]}{\Pr[Q]} = \Pr[P] \),

i.e.: if \( \Pr[P \amp Q] = (\Pr[P])(\Pr[Q]) \).

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Definition: Assume $\Pr[Q] \neq 0$. $P$ & $Q$ are **independent** (events) if $\Pr[P \& Q] = (\Pr[P])(\Pr[Q])$.

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"The probability of both is the product of the probabilities"

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$6$
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Definition: $S \& T$ are **independent** (PCRVs) if, $\forall A, B \subset \mathbb{R}$, $S \in A$ is independent of $T \in B$.

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These are independent

$C_1 \in \{1\}$ is independent of $C_2 \in \{1\}$.

$C_1 \in \{-1\}$ is independent of $C_2 \in \{1\}$.

$C_1$ and $C_2$ independent
Def’ns: $P, Q, R$ are independent (events) if
$P, Q, R$ are pairwise-independent
and $\Pr[P \& Q \& R] = (\Pr[P])(\Pr[Q])(\Pr[R])$.

$S, T, U$ are independent (PCRVs) if,
$\forall A, B, C \subseteq \mathbb{R}, S \in A, T \in B$ and $U \in C$ are indep.
$\text{etc.}, \text{etc.}, \text{etc.}$

**Definition:**
$P \& Q$ are independent (events)
if $\Pr[P \& Q] = (\Pr[P])(\Pr[Q])$.

**Definition:**
$S \& T$ are independent (PCRVs)
if, $\forall A, B \subseteq \mathbb{R}$,
$S \in A$ is independent of $T \in B$.

$\Pr[(C_2 = 1) \mid (C_1 = 1)] = \frac{0.25}{0.5} = 0.5$

$\text{these are independent}$

$C_1 \in \{1\}$ is independent of $C_2 \in \{1\}$.

$C_1 \in \{-1\}$ is independent of $C_2 \in \{1\}$.

$C_1$ and $C_2$ independent
Exercise: Graph $C_4$.

Fact: $C_1, C_2, C_3, \ldots$ are pairwise independent.

Stronger: Any finite set of $C_1, C_2, \ldots$ is an independent set.
Definition: \( \forall n \in \mathbb{Z}, n > 0, \)

\[
C_1 := \frac{-1}{1}^{1/2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{1} \cdot \cdots \cdot D_n := C_1^{1/2} \cdot \cdots \cdot C_n^{1/2}
\]
Definition: \( \forall \) integers \( n > 0 \),
\[ D_n := C_1 + \cdots + C_n \]
models (\#heads) – (\#tails) after \( n \) flips of a fair coin

\[ C_1 := \]
\[ C_2 := \]
\[ D_2 := \]

\[ 50\% \]
\[ 50\% \]
\[ 50\% \]
\[ 50\% \]
\[ 50\% \]
\[ 25\% \]
\[ 50\% \]
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Definition: \( \forall \text{ integers } n > 0, \)
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models (\#heads) \(-\) (\#tails) after \( n \) flips of a fair coin.

Fact: independent \( \Rightarrow \) uncorrelated. \( ^{pf \text{ omitted}} \)
I.e., \( S, T \) independent \( \Rightarrow \)
\[ \text{Var}[S + T] = \text{Var}[S] + \text{Var}[T]. \]

\( C_1, \ldots, C_n \) are all standard (i.e., mean 0, variance 1)
\[ \mathbb{E}[D_n] = (\mathbb{E}[C_1]) + \cdots + (\mathbb{E}[C_n]) \]
\[ = 0 + \cdots + 0 = 0 \]

\[ \text{Var}[D_n] = (\text{Var}[C_1]) + \cdots + (\text{Var}[C_n]) \]
\[ = 1 + \cdots + 1 = n \]

\[ \mathbb{E} \left[ \frac{D_n}{\sqrt{n}} \right] = 0 \quad \text{and} \quad \text{Var} \left[ \frac{D_n}{\sqrt{n}} \right] = 1, \]
i.e., \( \frac{D_n}{\sqrt{n}} \) is standard. \( (D_n)_o = \frac{D_n}{\sqrt{n}} \)

the standardization of \( D_n \)
Definition: \( \forall \) integers \( n > 0 \),
\[
D_n := C_1 + \cdots + C_n
\]
models (\#heads) – (\#tails) after \( n \) flips of a fair coin.

Preview of the **Central Limit Theorem**:
\[
\frac{D_n}{\sqrt{n}} \xrightarrow{\text{in distribution}} Z, \quad \text{as} \ n \to \infty.
\]

- **Z**: Standard normal random variable
- **Definition?**

\[
\begin{align*}
E \left[ \frac{D_n}{\sqrt{n}} \right] &= 0 \quad \text{and} \quad \Var \left[ \frac{D_n}{\sqrt{n}} \right] = 1, \\
i.e., \quad \frac{D_n}{\sqrt{n}} \quad \text{is standard.} \quad (D_n)_{\circ} &= \frac{D_n}{\sqrt{n}}
\end{align*}
\]
Definition: \( \forall \text{ integers } n > 0, \) 
\[ D_n := C_1 + \cdots + C_n \] 
models (\#heads) − (\#tails) after \( n \) flips of a fair coin

Preview of the Central Limit Theorem:

\[ \frac{D_n}{\sqrt{n}} \to Z \] 
in distribution, as \( n \to \infty \).

\( \forall \) test functions \( \psi \), 
\[ E \left[ \psi \left( \frac{D_n}{\sqrt{n}} \right) \right] \to E[\psi(Z)] \] 
\( Z \) not yet def’d, so…
Definition: \( \forall \text{integers } n > 0, \) models \((\#\text{heads}) - (\#\text{tails})\) after \(n\) flips of a fair coin
\[ D_n := C_1 + \cdots + C_n \]

Preview of the Central Limit Theorem:
\[ \frac{D_n}{\sqrt{n}} \rightarrow Z \quad \text{in distribution}, \quad \text{as } n \rightarrow \infty. \]

\( \forall \text{test functions } \psi, \)
\[
E \left[ \psi \left( \frac{D_n}{\sqrt{n}} \right) \right] \rightarrow E[\psi(Z)]
\]
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}] \, dx
\]

Change every \(Z\) to \(x\) and then integrate against \(h(x) \, dx\), from \(-\infty\) to \(\infty\).

\( Z \) not yet def’d, so...

\( h(x) := \frac{e^{-x^2/2}}{\sqrt{2\pi}} \)
Definition: \( \forall \text{integers } n > 0, \)
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D_n := C_1 + \cdots + C_n
\]

Preview of the Central Limit Theorem:

\( \forall \) test functions \( \psi \),
\[
\mathbb{E} \left[ \psi \left( \frac{D_n}{\sqrt{n}} \right) \right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}] \, dx
\]

Relatively easy: “test function” = “continuous, compactly supported function”

\( \forall \) test functions \( \psi \),
\[
\mathbb{E} \left[ \psi \left( \frac{D_n}{\sqrt{n}} \right) \right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}] \, dx
\]
Definition: For all integers $n > 0$, let $D_n := C_1 + \cdots + C_n$, which models the difference between the number of heads and tails after $n$ flips of a fair coin.

Preview of the Central Limit Theorem:

For all test functions $\psi$,

$$\mathbb{E} \left[ \psi \left( \frac{D_n}{\sqrt{n}} \right) \right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}] \, dx$$

Relatively easy: “test function” = “continuous, compactly supported function”

Harder to prove: “test function” = “continuous, exponentially-bounded function”

A function $f$ is \textbf{exponentially bounded} if there exists $A, B$ such that $|f(x)| \leq Ae^{B|x|}$.
Definition: \( \forall \) integers \( n > 0 \),

\[ D_n := C_1 + \cdots + C_n \]

Preview of the Central Limit Theorem:

\( \forall \) continuous, exponentially-bounded \( \psi \),

\[ E \left[ \psi \left( \frac{D_n}{\sqrt{n}} \right) \right] \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) \left[ e^{-x^2/2} \right] dx \]

Exercise: Compute \( \lim_{n \to \infty} E \left[ \left( e^{D_n/\sqrt{n}} - 7 \right)_+ \right] \).

f exponentially bounded means:

\[ \exists A, B \text{ s.t. } |f(x)| \leq Ae^{B|x|} \]
Definition: \( \forall \) integers \( n > 0 \),
\[
D_n := C_1 + \cdots + C_n
\]

Preview of the **Central Limit Theorem**: 
\( \forall \) continuous, exponentially-bounded \( \psi \),
\[
\mathbb{E} \left[ \psi \left( \frac{D_n}{\sqrt{n}} \right) \right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}] \, dx
\]

Solution:
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [(e^x - 7)_+] [e^{-x^2/2}] \, dx = \ldots
\]

\( \text{exp-bdd} \rightarrow \psi(x) = (e^x - 7)_+ \)

Exercise: Compute \( \lim_{n \to \infty} \mathbb{E} \left[ \left( e^{D_n/\sqrt{n}} - 7 \right)_+ \right] \).

\( f \) exponentially bounded means:
\[
\exists A, B \text{ s.t. } |f(x)| \leq Ae^{B|x|}
\]
Definition: \( \forall \text{ integers } n > 0, \quad D_n := C_1 + \cdots + C_n \)

Preview of the Central Limit Theorem:
\( \forall \text{ continuous, exponentially-bounded } \psi, \)
\[
E \left[ \psi \left( \frac{D_n}{\sqrt{n}} \right) \right] \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}] \, dx
\]

Sophisticated solution: \( D_n/\sqrt{n} \rightarrow Z \) in distribution
\[
E \left[ (e^Z - 7)_+ \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [(e^x - 7)_+][e^{-x^2/2}] \, dx
\]

Change every \( Z \) to \( x \), etc.

Exercise: Compute \( \lim_{n \to \infty} E \left[ (e^{D_n/\sqrt{n}} - 7)_+ \right] \).

\textbf{f exponentially bounded means:}
\[ \exists A, B \text{ s.t. } |f(x)| \leq Ae^{B|x|} \]
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\( \forall \) continuous, exponentially-bounded \( \psi \),
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E \left[ \psi \left( \frac{D_n}{\sqrt{n}} \right) \right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}] \, dx
\]

Hint: \( \psi(x) := e^{ax+b} \)

Def’n: \( \forall X \), the **augmented expectation** of \( X \)
is defined by \( \text{AE}[X] := (E[X]) + \frac{1}{2}(\text{Var}[X]). \)

Fact: Fix \( a, b \in \mathbb{R} \). Let \( R_n := a \left( \frac{D_n}{\sqrt{n}} \right) + b \).

"E almost asymptotically commutes with \( e^{\cdot} \)"

Then
\[
\lim_{n \to \infty} E[e^{R_n}] = \lim_{n \to \infty} e^{\text{AE}[R_n]}
\]

Pf: \( \lim_{n \to \infty} E[e^{R_n}] \) by CLT, \( e^b e^{a^2/2} \) by CLT, exercise, exercise, \( \lim_{n \to \infty} e^{\text{AE}[R_n]} \).
Definition: \( \forall \) integers \( n > 0 \),
\[
D_n := C_1 + \cdots + C_n
\]

Preview of the Central Limit Theorem:
\( \forall \) continuous, exponentially-bounded \( \psi \),
\[
\mathbb{E} \left[ \psi \left( \frac{D_n}{\sqrt{n}} \right) \right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}] \, dx
\]

Hint: \( \psi(x) := (ax + b) \)

Def’n: \( \forall X \), the **augmented expectation of** \( X \)

is defined by \( \text{AE}[X] := (\mathbb{E}[X]) + \frac{1}{2}(\text{Var}[X]) \).

“asymptotically normal”

Fact: Fix \( a, b \in \mathbb{R} \). Let \( R_n := a \left( \frac{D_n}{\sqrt{n}} \right) + b \).

“\( \mathbb{E} \) nearly asymptotically commutes with \( e^{\bullet} \)”

Then
\[
\lim_{n \to \infty} \mathbb{E}[e^{R_n}] = \lim_{n \to \infty} e^{\text{AE}[R_n]}
\]

Pf:
\[
\lim_{n \to \infty} \mathbb{E}[e^{R_n}] \text{ \( \text{CLT} \)} = e^b e^{a^2/2} \text{ \( \text{CLT} \)} = \lim_{n \to \infty} e^{\text{AE}[R_n]}.
\]

QED
Sophisticated fact:

Fix $a, b \in \mathbb{R}$. Let $R := aZ + b$.

"E almost commutes with $e^\cdot$...

Then $\mathbb{E}[e^R] = e^{A\mathbb{E}[R]}$.

...but we need to go from the expectation to the augmented expectation”

Def’n: $\forall X$, the **augmented expectation** of $X$ is defined by $[A\mathbb{E}[X]] := (\mathbb{E}[X]) + \frac{1}{2}(\text{Var}[X])$.

"asymptotically normal"

Fact: Fix $a, b \in \mathbb{R}$. Let $R_n := a\left(\frac{D_n}{\sqrt{n}}\right) + b$.

"E almost asymptotically commutes with $e^\cdot$"

Then $\lim_{n \to \infty} \mathbb{E}[e^{R_n}] = \lim_{n \to \infty} e^{A\mathbb{E}[R_n]}$.

Pf: $\lim_{n \to \infty} \mathbb{E}[e^{R_n}] \overset{\text{CLT}}{=} e^b e^{a^2/2} \overset{\text{CLT}}{=} \lim_{n \to \infty} e^{A\mathbb{E}[R_n]}$. QED
Exercise: Let $n := 12$. Assume $X_1, \ldots, X_n$ iid.

\[
\mu := \mathbb{E}[X_1] = \cdots = \mathbb{E}[X_n]
\]
\[
\sigma := \text{SD}[X_1] = \cdots = \text{SD}[X_n]
\]

Let $S := X_1 + \cdots + X_n$.

Assume $\mathbb{E}[S] = 0.225181512$, $\text{SD}[S] = 0.158877565$. Find $\mu$ and $\sigma$.

Def’n: $\forall X$, the augmented expectation of $X$ is defined by $\text{AE}[X] := (\mathbb{E}[X]) + \frac{1}{2}(\text{Var}[X])$.

Fact: Fix $a, b \in \mathbb{R}$. Let $R_n := a \left( \frac{D_n}{\sqrt{n}} \right) + b$.

“E almost asymptotically commutes with $e$”

Then \[ \lim_{n \to \infty} \mathbb{E}[e^{R_n}] = \lim_{n \to \infty} e^{\text{AE}[R_n]} \]

Pf: $\lim_{n \to \infty} \mathbb{E}[e^{R_n}] \overset{\text{CLT}}{=} e^b e^{a^2/2} \overset{\text{CLT}}{=} \lim_{n \to \infty} e^{\text{AE}[R_n]}$. QED
Exercise: Let \( n := 12 \). Assume \( X_1, \ldots, X_n \) iid.

\[
\begin{align*}
\mu & := E[X_1] = \cdots = E[X_n] \\
\sigma & := SD[X_1] = \cdots = SD[X_n]
\end{align*}
\]

Let \( S := X_1 + \cdots + X_n \).

Assume \( E[S] = 0.225181512 \),
\( SD[S] = 0.158877565 \). Find \( \mu \) and \( \sigma \).

Solution:

\( E[S] = E[X_1] + \cdots + E[X_n] = n\mu = (12)\mu \),

so \( \mu = 0.225181512/12 \)
Exercise: Let $n := 12$. Assume $X_1, \ldots, X_n$ iid.

$$\mu := E[X_1] = \cdots = E[X_n]$$

$$\sigma := SD[X_1] = \cdots = SD[X_n]$$

Let $S := X_1 + \cdots + X_n$.

Assume $E[S] = 0.225181512$, $SD[S] = 0.158877565$. Find $\mu$ and $\sigma$.

Solution: $\mu = 0.225181512/12$

$$\operatorname{Var}[S] = \operatorname{Var}[X_1] + \cdots + \operatorname{Var}[X_n]$$

$$\mu = 0.225181512/12$$
Exercise: Let $n := 12$. Assume $X_1, \ldots, X_n$ iid.

$$
\mu := \mathbb{E}[X_1] = \cdots = \mathbb{E}[X_n]
$$

$$
\sigma := \text{SD}[X_1] = \cdots = \text{SD}[X_n]
$$

Let $S := X_1 + \cdots + X_n$.

Assume $\mathbb{E}[S] = 0.225181512$, $\text{SD}[S] = 0.158877565$. Find $\mu$ and $\sigma$.

Solution:

$$
\mu = \frac{0.225181512}{12}
$$

$$
(0.158877565)^2
$$

$$
\downarrow
$$

$$
\text{Var}[S] = \text{Var}[X_1] + \cdots + \text{Var}[X_n]
$$

$$
= n\sigma^2 = (12)\sigma^2,
$$

so $\sigma^2 = \frac{(0.158877565)^2}{12}$

so $\sigma = \frac{0.158877565}{\sqrt{12}}$
Exercise: Let \( n := 12 \). Assume \( X_1, \ldots, X_n \) iid.

\[
\mu := \mathbb{E}[X_1] = \cdots = \mathbb{E}[X_n]
\]

\[
\sigma := \text{SD}[X_1] = \cdots = \text{SD}[X_n]
\]

Let \( S := X_1 + \cdots + X_n \).

Assume \( \mathbb{E}[S] = 0.225181512, \)

\( \text{SD}[S] = 0.158877565 \). Find \( \mu \) and \( \sigma \).

Solution: \[
\mu = \frac{0.225181512}{12}
\]

\[
\sigma = \frac{0.158877565}{\sqrt{12}}
\]
Exercise: Let \( n := 12 \). Assume \( X_1, \ldots, X_n \) independent, identically distributed.

\[
\mu := \mathbb{E}[X_1] = \cdots = \mathbb{E}[X_n] \\
\sigma := \text{SD}[X_1] = \cdots = \text{SD}[X_n]
\]

Let \( S := X_1 + \cdots + X_n \).

Assume \( \mathbb{E}[S] = 0.225181512 \), \( \text{SD}[S] = 0.158877565 \). Find \( \mu \) and \( \sigma \).

Solution: \[
\mu = \frac{0.225181512}{12} \\
\sigma = \frac{0.158877565}{\sqrt{12}}
\]

Mean and variance are cut by a factor of 12. Standard deviation is cut by a factor of \( \sqrt{12} \).

Conversely, on adding \( n \) uncorrelated PCRVs, SD increases by a factor of \( \sqrt{n} \), NOT \( n \). A portfolio of uncorrelated assets is better...

Later: “Volatility” is an example of standard deviation, NOT variance.
Def'n: Let \( S \) and \( T \) be PCRVs.
Let \( F := \{(a, b) \in \mathbb{R}^2 \mid \Pr[(S = a) \& (T = b)] > 0\} \).

The joint distribution of \((S, T)\)
associates, to each element \((a, b) \in F\),
the value \(\Pr[(S = a) \& (T = b)]\).

Remark: To compute the distribution of \( S + T \),
you need to know the JOINT distr. of \((S, T)\).
Knowing both the distribution of \( S \)
and the distribution of \( T \)
is insufficient. Same for \( ST \).

However, if \( S \) and \( T \) are independent,
then their joint distribution
is determined by
their individual distributions,
because
\[
\Pr[(S = a) \& (T = b)] = (\Pr[S = a])(\Pr[T = b]).
\]

All this generalizes to \( \geq 2 \) PCRVs.