Financial Mathematics
One period pricing and hedging
Dan wants 100 Euros one month from now. He’ll receive $100 one month from now from some source, but only has $3 right now. Current price is $1/Euro. **Worry**: Rises to $1.

Take out a loan? **Loan rate**: 1% per month!

Dan has poor credit . . . No loans for Dan!

Dollar price of a Euro a month from now is unknown. **Call** it $S$. Dan wants a contract that will pay him $100(S - 1)$, if $S > 1$.

Alice agrees to sell Dan a contract of this form. **What if** $S \leq 1$?

(Money burns a hole in Dan’s pocket, and he knows he’ll spend the $3 by the end of the month.

if he doesn’t spend it now.

*So he can’t count on having more than $100 at the end of the month.*)
Dan wants 100 Euros one month from now.
He’ll receive $100 one month from now from some source, but only has $3 right now. 
Current price is $1/Euro. Worry: Rises to > $1.
Take out a loan? Loan rate: 1% per month!
Dan has poor credit . . . No loans for Dan!
Dollar price of a Euro a month from now is unknown.
Call it $S$. Dan wants a contract that will pay him
100($S - 1$), if $S > 1$.

Alice agrees to sell Dan a contract of this form.
What if $S \leq 1$?

Futures or forward: 100($S - 1$), if $S \leq 1$.
i.e., Dan pays Alice 100(1 – $S$), if $S \leq 1$.

Knowing Dan is irresponsible,
Alice refuses to agree to this.
Option: 0, if $S \leq 1$. 

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Dan wants 100 Euros one month from now. He’ll receive $100 one month from now
from some source, but only has $3 right now. Current price is $1/Euro. Worry: Rises to \(> 1\).
Take out a loan? Loan rate: 1% per month!

Dan has poor credit . . . No loans for Dan!
Dollar price of a Euro a month from now is unknown. Call it \(S\). Dan and Alice agree on an option that will
pay him \(\begin{cases} 100(S-1), & \text{if } S > 1 \\ 0, & \text{if } S \leq 1 \end{cases}\) one month from now.

\[4\]
Dan wants 100 Euros one month from now. He’ll receive $100 one month from now from some source, but only has $3 right now. Current price is $1/Euro. Worry: Rises to $ > $1. Take out a loan? Loan rate: 1% per month! Dan has poor credit . . . No loans for Dan! Dollar price of a Euro a month from now is unknown. Call it $S$. Dan and Alice agree on an option that will pay him \[
\begin{cases}
100(S - 1), & \text{if } S > 1 \\
0, & \text{if } S \leq 1
\end{cases}
\] one month from now.

This is the payoff or claim. The claim is contingent!
Dan wants 100 Euros one month from now. He’ll receive $100 one month from now from some source, but only has $3 right now. Current price is $1/Euro. Worry: Rises to $1. Take out a loan? Loan rate: 1% per month!

Dan has poor credit . . . No loans for Dan! Dollar price of a Euro a month from now is unknown. Call it $S$. Dan and Alice agree on an option that will pay him

\[
\begin{cases}
100(S - 1), & \text{if } S > 1 \\
0, & \text{if } S \leq 1
\end{cases}
\]

one month from now.

\[
\begin{cases}
100(S - 1), & \text{if } S - 1 > 0 \\
0, & \text{if } S - 1 \leq 0
\end{cases}
\]

\[
100 \begin{cases}
S - 1, & \text{if } S - 1 > 0 \\
0, & \text{if } S - 1 \leq 0
\end{cases}
\]
Dan wants 100 Euros one month from now. He’ll receive $100 one month from now from some source, but only has $3 right now. Current price is $1/Euro. **Worry:** Rises to $>1. Take out a loan? **Loan rate:** 1% per month!

Dan has poor credit . . . No loans for Dan!

Dollar price of a Euro a month from now is unknown. **Call it** \( S \). Dan and Alice agree on an option that will pay him

\[
\begin{cases} 
100(S - 1), & \text{if } S > 1 \\
0, & \text{if } S \leq 1 
\end{cases}
\]

one month from now.

\[
100 \begin{cases} 
S - 1, & \text{if } S - 1 > 0 \\
0, & \text{if } S - 1 \leq 0 
\end{cases}
\]

\[
100 \begin{cases} 
S - 1, & \text{if } S - 1 > 0 \\
0, & \text{if } S - 1 \leq 0 
\end{cases}
\]

\[
x + \begin{cases} 
x, & \text{if } x > 0 \\
0, & \text{if } x \leq 0 
\end{cases}
\]
Dan wants 100 Euros one month from now. He’ll receive $100 one month from now from some source, but only has $3 right now. Current price is $1/Euro. Worry: Rises to > $1. Take out a loan? Loan rate: 1% per month! Dan has poor credit ... No loans for Dan! Dollar price of a Euro a month from now is unknown. Call it $S$. Dan and Alice agree on an option that will pay him \[
\begin{cases}
100(S - 1), & \text{if } S > 1 \\
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\end{cases}
\]

\[
100 \begin{cases}
S - 1, & \text{if } S - 1 > 0 \\
0, & \text{if } S - 1 \leq 0
\end{cases}
\]

\[
100(S - 1) + \begin{cases}
x, & \text{if } x > 0 \\
0, & \text{if } x \leq 0
\end{cases}
\]
Dan wants 100 Euros one month from now. He’ll receive $100 one month from now from some source, but only has $3 right now. Current price is $1/Euro. **Worry:** Rises to > $1. Take out a loan? **Loan rate:** 1% per month! Dan has poor credit . . . No loans for Dan! Dollar price of a Euro a month from now is unknown. **Call it** $S$. Dan and Alice agree on an option that will pay him $100(S - 1) +$ one month from now.

What price does she charge?

\[
x + := \begin{cases} 
    x, & \text{if } x > 0 \\
    0, & \text{if } x \leq 0
\end{cases}
\]
Dan wants 100 Euros one month from now. He’ll receive $100 one month from now from some source, but only has $3 right now. Current price is $1/Euro. Worry: Rises to $1. Take out a loan? Loan rate: 1% per month! Dan has poor credit . . . No loans for Dan! Dollar price of a Euro a month from now is unknown. Call it $S$. Dan and Alice agree on an option that will pay him $100(S - 1)_+ \text{ one month from now.}$

What price does she charge? More or less than $3? Step 1: Model “the underlying”, i.e., the Euro, i.e., $S$. Alice selects: A 1-subperiod 70 – 30 CRR model,
Dan wants 100 Euros one month from now. He’ll receive $100 one month from now from some source, but only has $3 right now. Current price is $1/Euro. **Worry:** Rises to $>$ $1. Take out a loan? **Loan rate:** 1% per month! Dan has poor credit . . . No loans for Dan! Dollar price of a Euro a month from now is unknown. **Call it** $S$. Dan and Alice agree on an option that will pay him $100(S - 1)_+$ one month from now. **What** price does she charge? More or less than $3? **Step 1:** Model “the underlying”, i.e., the Euro, i.e., $S$. Alice selects: A 1-subperiod 70 – 30 CRR model, in which one **ASSUMES** that $\exists d, u \in \mathbb{R}$ s.t. $d < u$ and s.t. the dollar price of one Euro has a 70% chance of changing from 1 to $1 \times e^u$ and a 30% chance of changing from 1 to $1 \times e^d$.
Dollar price of a Euro a month from now is $S$.

**Step 1:** Model “the underlying”, i.e., the Euro, i.e., $S$.

Alice selects: A 1-subperiod $70 - 30$ CRR model, in which one ASSUMES that $\exists d, u \in \mathbb{R}$ s.t. $d < u$ and s.t. the dollar price of one Euro has a 70% chance of changing from 1 to $1 \times e^u$

and a 30% chance of changing from 1 to $1 \times e^d$.

**NOTE:** $S$ is a binary random variable, whose distribution is described by:

$$\Pr[S = e^u] = 0.7 \quad \text{and} \quad \Pr[S = e^d] = 0.3.$$
Dollar price of a Euro a month from now is $S$.

Step 1: Model “the underlying”, i.e., the Euro, i.e., $S$. Alice selects: A 1-subperiod $70 - 30$ CRR model, in which one ASSUMES that $\exists d, u \in \mathbb{R}$ s.t. $d < u$ and s.t. the dollar price of one Euro has a 70% chance of changing from 1 to $1 \times e^u$ and a 30% chance of changing from 1 to $1 \times e^d$.

NOTE: $S$ is a binary random variable, whose distribution is described by:

$$\Pr[S = e^u] = 0.7 \quad \text{and} \quad \Pr[S = e^d] = 0.3.$$  

Step 2: Calibrate the model. Alice asks her market analyst for the (one-month) drift := $E[\ln S]$ and volatility := $SD[\ln S]$. She gets this answer:

\[
\begin{align*}
\text{drift} &= 0.018765126 \quad \text{and} \quad \text{vol} = 0.045864002 \\
\text{unrealistically high} &\gg \text{CRR assumes independence} \gg \text{low} \quad 0.225181512/12
\end{align*}
\]
Step 2: Calibrate the model.
Alice asks her market analyst for the one-year drift \( \text{drift} := E[\ln S] \) and volatility \( \text{volatility} := \text{SD}[\ln S] \).
She gets this answer:
\[
drift = 0.018765126 \quad \text{and} \quad \text{vol} = 0.045864002
\]

**NOTE:** \( S \) is a binary random variable, whose distribution is described by:
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\Pr[S = e^u] = 0.7 \quad \text{and} \quad \Pr[S = e^d] = 0.3.
\]

**NOTE:** \( S \) is a binary random variable, whose distribution is described by:
\[
\Pr[S = e^u] = 0.7 \quad \text{and} \quad \Pr[S = e^d] = 0.3.
\]

**NOTE:** \( S \) is a binary random variable, whose distribution is described by:
\[
\Pr[\ln S = u] = 0.7 \quad \text{and} \quad \Pr[\ln S = d] = 0.3.
\]

Step 2: Calibrate the model.
Alice asks her market analyst for the one-year drift \( \text{drift} := E[\ln S] \) and volatility \( \text{volatility} := \text{SD}[\ln S] \).
She gets this answer:
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drift = 0.018765126 \quad \text{and} \quad \text{vol} = 0.045864002
\]
Step 2: Calibrate the model.

Alice asks her market analyst for the one-year drift := E[ln S] and volatility := SD[ln S].

She gets this answer:

\[ \text{drift} = 0.018765126 \quad \text{and} \quad \text{vol} = 0.045864002 \]

**NOTE:** \( S \) is a binary random variable, whose distribution is described by:

\[ \Pr[S = e^u] = 0.7 \quad \text{and} \quad \Pr[S = e^d] = 0.3. \]

**NOTE:** \( \ln S \) is a binary random variable, whose distribution is described by:

\[ \Pr[\ln S = u] = 0.7 \quad \text{and} \quad \Pr[\ln S = d] = 0.3. \]

\[ E[\ln S] = (0.7)u + (0.3)d \]

\[ \text{SD}[\ln S] = \sqrt{(0.7)(0.3)(u - d)} \]
Step 2: Calibrate the model. Alice asks her market analyst for the one-year drift $\beta := \mathbb{E}[\ln S]$ and volatility $\sigma := \text{SD}[\ln S]$. She gets this answer:

$\beta = 0.018765126$ and $\sigma = 0.045864002$

**NOTE:** $S$ is a binary random variable, whose distribution is described by:

$\Pr[S = e^u] = 0.7$ and $\Pr[S = e^d] = 0.3$.

**NOTE:** $\ln S$ is a binary random variable, whose distribution is described by:

$\Pr[\ln S = u] = 0.7$ and $\Pr[\ln S = d] = 0.3$.

$0.018765126 = (0.7)u + (0.3)d \implies \begin{cases} u = 0.0487902 \\
d = -0.0512933 \end{cases}$
Step 2: Calibrate the model.

Alice asks her market analyst for the one-year drift $:= \mathbb{E}[\ln S]$ and volatility $:= \text{SD}[\ln S]$.

She gets this answer:

$$\text{drift} = 0.018765126 \quad \text{and} \quad \text{vol} = 0.045864002$$

**NOTE:** $S$ is a binary random variable, whose distribution is described by:

$$\Pr[S = e^u] = 0.7 \quad \text{and} \quad \Pr[S = e^d] = 0.3.$$

\[
\begin{align*}
  u &= 0.0487902 & \Rightarrow & & e^u &= 1.0500000 \\
  d &= -0.0512933 & \Rightarrow & & e^d &= 0.9500000 \\
\end{align*}
\]

$$u = 0.0487902 \quad \text{and} \quad d = -0.0512933$$
Step 2: Calibrate the model.

Alice asks her market analyst for the one-year drift \( \text{drift} := \mathbb{E} \{ \ln S \} \) and volatility \( \text{volatility} := \text{SD} \{ \ln S \} \).

She gets this answer:

\[
\text{drift} = 0.018765126 \quad \text{and} \quad \text{vol} = 0.045864002
\]

**NOTE:** \( S \) is a binary random variable, whose distribution is described by:

\[
\Pr[S = e^u] = 0.7 \quad \text{and} \quad \Pr[S = e^d] = 0.3.
\]

\[
\begin{align*}
u &= 0.0487902 \quad \Rightarrow \quad e^u &= 1.0500000 \\
d &= -0.0512933 \quad \Rightarrow \quad e^d &= 0.9500000
\end{align*}
\]

According to this model, \( S \in \{1.05, 0.95\} \) a.s.

**Recall:** Dollar price of a Euro a month from now is \( S \).

Step 3: Find a perfect hedging strategy.
According to this model, \( S \in \{1.05, 0.95\} \) a.s.  
Recall: Dollar price of a Euro a month from now is \( S \).

Step 3: Find a perfect hedging strategy.

Alice sets up a hedging portfolio:

- \( x \) Euros and a \( y \) dollar bank loan.

\[
\begin{align*}
  x & \times \begin{pmatrix} $1 & $1.05 \\ $0.95 & \end{pmatrix} \\
\end{align*}
\]

NOTE: Alice does not have access to a bank that holds Euros. Her Euros all go “under the mattress”.

According to this model, \( S \in \{1.05, 0.95\} \) a.s.

Recall: Dollar price of a Euro a month from now is \( S \).

Step 3: Find a perfect hedging strategy.
According to this model, $S \in \{1.05, 0.95\}$ a.s.

Recall: Dollar price of a Euro a month from now is $S$.

**Step 3: Find a perfect hedging strategy.**

Alice sets up a **hedging portfolio**:

- $x$ Euros and a $y$ dollar bank loan.

\[
\begin{align*}
 x \times & \begin{pmatrix} \$ 1 & \$ 1.05 \\ \$ 0.95 & \$ 1 \end{pmatrix} \\
-y \times & \begin{pmatrix} \$ 1 & \$ 1.01 \\ \$ 1.01 & \$ 1 \end{pmatrix}
\end{align*}
\]

**Loan rate: 1% per month!**
According to this model, $S \in \{1.05, 0.95\}$ a.s. Recall: Dollar price of a Euro a month from now is $S$.

**Step 3: Find a perfect hedging strategy.**

Alice sets up a **hedging portfolio**:

- $x$ Euros and a $y$ dollar bank loan.

\[
\begin{align*}
\begin{pmatrix}
 x \\
 -y
\end{pmatrix} & \times \begin{pmatrix}
 1 \\
 1
\end{pmatrix} = \begin{pmatrix}
 x + y \\
 y
\end{pmatrix} \\
\begin{pmatrix}
 1.05 \\
 0.95
\end{pmatrix} & \times \begin{pmatrix}
 1 \\
 1
\end{pmatrix} = \begin{pmatrix}
 1.05 \\
 0.95
\end{pmatrix} \\
\begin{pmatrix}
 1.01 \\
 1.01
\end{pmatrix} & \times \begin{pmatrix}
 1 \\
 1
\end{pmatrix} = \begin{pmatrix}
 1.01 \\
 1.01
\end{pmatrix}
\end{align*}
\]

Dan and Alice agree on an option that will pay him $100(S - 1)^+$ one month from now.
$1.05 \times x - 1.01 \times y = 5$

Alice sets up a hedging portfolio: $x$ Euros and a $y$ dollar bank loan.

\[
\begin{pmatrix}
  x \\ -y
\end{pmatrix} \times \begin{pmatrix}
  1 & $1.05 \\
  1 & $0.95
\end{pmatrix} = \begin{pmatrix}
  $? \\
  $0
\end{pmatrix} + \begin{pmatrix}
  y \\
  y
\end{pmatrix} \times \begin{pmatrix}
  1 & $1.01 \\
  1 & $1.01
\end{pmatrix}
\]

Alice sets up a hedging portfolio: $x$ Euros and a $y$ dollar bank loan.
\[1.05 \, x - 1.01 \, y = 5\]
\[0.95 \, x - 1.01 \, y = 0\]

Alice sets up a hedging portfolio: 
\(x\) Euros and \(y\) dollar bank loan.
\[1.05 x - 1.01 y = 5\]
\[0.95 x - 1.01 y = 0\]
\[x - y = \, ?\]

Alice sets up a **hedging portfolio**: 
\[x \text{ Euros} \quad \text{and} \quad y \text{ dollar bank loan.}\]
1.05 \( x - 1.01 \) \( y \) = 5
0.95 \( x - 1.01 \) \( y \) = 0
\[ x - y = ? \]

\[ x = 50 \]
\[ y = 47.03 \]
\[ ? = 2.97 \]

Step 3: Find a perfect hedging strategy.
Ans: Alice charges Dan $2.97
   borrows 47.03 from the bank
   and buys 50 Euros.

What price does she charge? More or less than $3?
Ans: $2.97
Ans: less

Alice sets up a hedging portfolio:
\[ x \] Euros and a \( y \) dollar bank loan.