Financial Mathematics
Risk-neutrality and delta-hedging
\begin{align*}
1.05 \, x - 1.01 \, y &= 5 \\
0.95 \, x - 1.01 \, y &= 0 \\
x - y &= ?
\end{align*}

Next goal:

Describe a method to compute ? without solving a system of equations.

Key point to remember:

? does not depend on the probability of an uptick or downtick.

Is probability theory then useless? \textbf{NO}:
The trick is to imagine another universe in which the probabilities somehow make the computation of ? easy.

More on this in a moment . . .
According to the selected model,
probability uptick = 70%,
probability downtick = 30%

Problem:
Find the expected value and return, in our world, after one month, of
(a) $1 invested in the bank; and
(b) $1 invested in Euros.

Assume that the bank pays 1% per month on savings accounts.
(Same as on loans.)
According to the selected model,

probability up tick = 70%,

probability down tick = 30%

Assume that the bank pays 1% per month on savings accounts. (Same as on loans.)
According to the selected model,
probability uptick = 70%,
probability downtick = 30%

Assume that the bank pays
1% per month on savings accounts.
(Same as on loans.)
(a)

Bank: $1

70% → $1.01

30% → $1.01

expected bank value:

\[(70\%)(1.01) + (30\%)(1.01) = 1.01\]

(guaranteed – risk-free) expected bank return: 1%

(b)

Euro: $1

70% → $1.05

30% → $0.95

According to this model, \( S \in \{1.05, 0.95\} \) a.s.
(a) Bank: $1

70% → $1.01
30% → $1.01

expected bank value:

$1 = (70\%)(1.01) + (30\%)(1.01) = 1.01$

(expected guaranteed - risk-free) expected bank return: 1%

(b) Euro: $1

70% → $1.05
30% → $0.95

expected Euro value:

$1 = (70\%)(1.05) + (30\%)(0.95) = 1.02$

(expected Euro return: 2%)
$1\% < 2\%$

expected bank return $< \text{expected Euro return}$

Bank is “risk-free”. Euros are “risky”.

expected bank return $1\%$

expected Euro return $2\%$
1% < 2%
expected bank return < expected Euro return
Bank is “risk-free”. Euros are “risky”.

Economics: Investors are “risk-averse”. So risky investments must have a higher expected rate of return than risk-free investments, or they won’t sell.

Imagine a world in which bank and Euros have the same expected return. This is the “risk-neutral” world.

risk-free ≠ risk-neutral
(60\%) (1.05) + (40\%) (0.95) = 1.01

(b) 60\%

70\%  $1.05$

30\%  $0.95$

(70\%) (1.05) + (30\%) (0.95) = 1.02
(60\%)(1.05) + (40\%)(0.95) = 1.01

("60\%-40\% world")
(a) Bank: $1

$1 \times 70\% = $1.01

$1 \times 30\% = $1.01

\((70\%) \times (1.01) + (30\%) \times (1.01) = 1.01\)

Imagine a 60-40 world

does not change

---

Euro: $1

$1 \times 60\% = $1.05

$1 \times 40\% = $0.95

\((60\%) \times (1.05) + (40\%) \times (0.95) = 1.01\)

imagine a 60-40 world
$1 in bank:

3 ×

Bank: $1

60% → $1.01

40% → $1.01

$1 in Euros:

Euro: $1

60% → $1.05

40% → $0.95

$ (60%)(1.01) + (40%)(1.01) = 1.01

expected bank return: 1%

$ (60%)(1.05) + (40%)(0.95) = 1.01

expected Euro return: 1%

imagine a 60-40 world

does not change

We have achieved risk-neutrality!
$3$ in bank:

\[
(60\%)(3.03) + (40\%)(3.03) = 3.03
\]

expected bank return: 1%

$1$ in Euros:

\[
(60\%)(1.05) + (40\%)(0.95) = 1.01
\]

expected Euro return: 1%
$3 in bank:

\[ (60\%) \times 3 = 1.80 \]
\[ (40\%) \times 3 = 1.20 \]
\[ 1.80 + 1.20 = 3.00 \]

$2 in Euros:

\[ (60\%) \times 2 = 1.20 \]
\[ (40\%) \times 2 = 0.80 \]
\[ 1.20 + 0.80 = 2.00 \]

\( (60\%)(2.10) + (40\%)(1.90) = 2.02 \)

expected Euro return: 1%
$3$ in bank
and
$2$ in Euros:

The same logic will work
on any portfolio.

$5$

$\text{60\%} \rightarrow \$5.13$

$\text{40\%} \rightarrow \$4.93$

$(\text{60\%})(5.13) + (\text{40\%})(4.93) = 5.05$

Expected Portfolio Return: $1\%$

$(1.01)^5 = 5.05$

In this risk-neutral world, the expected return on any bank-Euro portfolio is $1\%$ per month.
−$y$ in bank

and

$x$ in Euros:

```
(60\%)\times (5) + (40\%)\times (0) = 3
```

Expected Portfolio Return: 1%

```
(1.01)^n = 3
```

```
? = \frac{3}{1.01} = 2.97
```

“Change of Measure”
Change from the “real” or “physical” world to the “risk-neutral” world.

(70–30) (60–40)
Coin-flippers got price!

How can we figure out the hedging strategy, without solving a system of equations?
\[
\begin{align*}
\text{Want: } x & \quad (\text{We want to “get hedge”.)} \\
x - y &= ? = 2.97 \quad (\text{We’re pricers.})
\end{align*}
\]
\[
\begin{align*}
x & \times \begin{pmatrix} $1 & \overset{\text{naive volatility}}{\downarrow \text{difference: 0.1}} & $1.05 \\ & \overset{\text{difference}}{\downarrow \text{difference: 0.1}} & $0.95 \end{pmatrix} \\
- y & \times \begin{pmatrix} $1 & \overset{\text{naive volatility}}{\downarrow \text{difference: 0}} & $1.01 \\ & \overset{\text{difference}}{\downarrow \text{difference: 0}} & $1.01 \end{pmatrix} \\
& = \begin{pmatrix} ? & \overset{\text{naive volatility}}{\downarrow \text{difference: 5}} & $5 \\ & \overset{\text{difference}}{\downarrow \text{difference: 5}} & $0 \end{pmatrix}
\end{align*}
\]

\[
x - y = ? = 2.97 \quad (\text{We're pricers.})
\]

Want: \( x \) (We want to “get hedge”.)
\[ 40 \times \begin{pmatrix} \$1 & \$1.05 \\ \$0.95 \end{pmatrix} \text{ naïve volatility difference: 0.1} \]

\[ -y \times \begin{pmatrix} \$1 & \$1.01 \\ \$1.01 \end{pmatrix} \text{ naïve volatility difference: 0} \]

\[ \neq \begin{pmatrix} \$ \, ? & \$5 \\ \$0 \end{pmatrix} \text{ naïve volatility difference: 5} \]

\[ x - y = ? = 2.97 \text{ (We’re pricers.)} \]

Want: \[ x \text{ (We want to “get hedge”.)} \]
\[ x - y = ? = 2.97 \text{ (We're pricers.)} \]

Want: \[ x \text{ (We want to “get hedge”.)} \]

\[ x = 50 = \frac{5}{0.1} = \text{option naïve volatility} \]

Euro naïve volatility
\[
\begin{align*}
50 \times \begin{pmatrix} \$1 \\ \$0.95 \end{pmatrix} & \text{ naïve volatility difference: 0.1} \\
- y \times \begin{pmatrix} \$1 \\ \$1.01 \end{pmatrix} & \text{ naïve volatility difference: 0} \\
\end{align*}
\]

\[
\begin{align*}
= \begin{pmatrix} \$5 \\ \$0 \end{pmatrix} \text{ naïve volatility difference: 5} \\
\end{align*}
\]

\[
x - y = ? = 2.97 \quad \text{(We’re pricers.)}
\]

Pricers got hedge

This is called the **Delta of the option**.

\[
x = 50 = \frac{5}{0.1} = \frac{\text{option naïve volatility}}{\text{Euro naïve volatility}}
\]
Pricers got hedge!