Financial Mathematics

Testing
the Black-Scholes formula
inputs: $T, \sigma_*, r_*, S_0, K$  

Let $K' := \frac{K}{e^{r_* T}}$.  

Let $d_{\pm} := \frac{\ln(S_0/K')}{\sigma_* \sqrt{T}} \pm \frac{\sigma_* \sqrt{T}}{2}$.  

output: $S_0[\Phi(d_{+})] - K'[\Phi(d_{-})]$  

first version of Black-Scholes

inputs: $\sigma, r, S_0, K$  

Let $K' := \frac{K}{e^{r}}$.  

Let $d_{\pm} := \frac{\ln(S_0/K')}{\sigma} \pm \frac{\sigma}{2}$.  

output: $S_0[\Phi(d_{+})] - K'[\Phi(d_{-})]$  

version zero of Black-Scholes

Let $d_{\pm} := \frac{\ln(S_0/K)}{\sigma_* \sqrt{T}} + \frac{\sigma_* \sqrt{T}}{2}$.  

output: $S_0[\Phi(d_{+})] - [Ke^{-r_* T}][\Phi(d_{-})]$  

second version of Black-Scholes

forward price on stock  

Let $F := S e^{r_* T}$.  

Let $d_{\pm} := \frac{\ln(F/K)}{\sigma_* \sqrt{T}} \pm \frac{\sigma_* \sqrt{T}}{2}$.  

output: $e^{-r_* T} \left( F[\Phi(d_{+})] - K[\Phi(d_{-})] \right)$  

forward price on option  

third version of Black-Scholes

Do these formulas really approximate the CRR price?
Kyle wants right, but not obligation, to buy 5000 shares of ABC for $5000, 30 days from now.

\[ N := \text{number of seconds in 30 days} \]

Gail selects:

\[ N\text{-subperiod 50.001-49.999 CRR model} \]

\[ S_0[\Phi(d_+)] - K'[\Phi(d_-)] \]

Do these formulas really approximate the CRR price?
Kyle wants right, but not obligation, to buy 5000 shares of ABC for $5000, 30 days from now.

\[ N := \text{number of seconds in 30 days} \]

Gail selects:

\[ N\text{-subperiod 50.001-49.999} \]

CRR model

\[ V = e^{-rN}E = 120.7994402 \]

PRESENT FORMULA

TIME NORMALIZED

\[ 5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \]

Do these formulas really approximate the CRR price?
Kyle wants right, but not obligation, to buy 5000 shares of ABC for $5000, Gail, seller 30 days from now.

\[ N := \text{number of seconds in 30 days} \]

Gail selects:

- \[ N\text{-subperiod} \ 50.001-49.999 \] CRR model

Market analyst: (ann) vol = 0.200002881086

Banker: \[ V = 120.7994402 \]

(annual) continuous compounding nominal rate

\[ = 0.05000 \left( S_0 [\Phi(d_+)] - K' [\Phi(d_-)] \right) \]

\[ V = 120.7994402 \]

5000\left( S_0 [\Phi(d_+)] - K' [\Phi(d_-)] \right) \]
Kyle wants right, **but not** obligation, to buy 5000 shares of ABC for $5000, 30 days from now. \( K = 1 \) \( T = 30/365 \)

\( N := \text{number of seconds in 30 days} \)

Gail selects:

\( N \)-subperiod 50.001-49.999 CRR model

**Market analyst:** (ann) vol = 0.200002881086

**Banker:**

(annual) continuous compounding nominal rate

\[ = 0.0315359998802 \]

\[ V = 120.7994402 \]

5000\( (S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \) close?
\[ T = \frac{30}{365} \quad \text{Assume: Initial price} = \$1/\text{share}. \]
\[ r = r_T T \]
\[ \sigma = \sigma_T \sqrt{T} \]
\[ K = 1 \]
\[ K' = \frac{K}{e^r} \]
\[ d_+ = \frac{\ln(S_0/K')}{-\frac{\sigma}{2}} + \frac{\sigma \text{nn vol}}{2} = 0.200002881086 \]
\[ d_- = \frac{\ln(S_0/K')}{-\frac{\sigma}{2}} - \frac{\sigma \text{s compounding nominal rate}}{2} = 0.315359998802 \]

\underline{Market analyst:} \ (\text{ann}) \ \text{vol} = 0.200002881086

\underline{Banker:} \ 
(\text{annual}) \ \text{continuous compounding nominal rate} = 0.0315359998802

\[ V = 120.7994402 \]

\[ 5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \]

\[ \text{close?} \]
\[ T = \frac{30}{365} \]
\[ r = r_T = 0.00259199999014 \]
\[ \sigma = \sigma_T = 0.057339043865 \]
\[ K = 1 \]
\[ K' = \frac{K}{e^r} \]
\[ d_+ = \frac{\ln(S_0/K')}{\sigma} + \frac{\sigma}{2} \]
\[ d_- = \frac{\ln(S_0/K')}{\sigma} - \frac{\sigma}{2} \]

Market analyst: (ann) vol = 0.200002881086

Banker: (annual) continuous compounding nominal rate = 0.0315359998802

\[ V = 120.7994402 \]

\[ 5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \]
\[ T = \frac{30}{365} \]
\[ r = r^* T = 0.00259199999014 \]
\[ \sigma = \sigma^* \sqrt{T} = 0.057339043865 \]
\[ K = 1 \]
\[ K' = e^{rT} = 0.997411356345 \]
\[ d_+ = \frac{\ln(S_0/K')}{\sigma} + \frac{\sigma}{2} \]
\[ d_- = \frac{\ln(S_0/K')}{\sigma} - \frac{\sigma}{2} \]

Market analyst: (ann) vol = 0.2000002881086
Banker: (annual) continuous compounding nominal rate = 0.0315359998802

\[ V = 120.7994402 \]

\[ 5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \]
\[ T = \frac{30}{365} \]
\[ r = r_* T = 0.002591999999014 \]
\[ \sigma = \sigma_* \sqrt{T} = 0.057339043865 \]
\[ K = 1 \]
\[ K' = \frac{K}{e^{r}} = 0.997411356345 \]
\[ d_+ = \frac{\ln(\frac{S_0}{K'}) + \frac{\sigma}{2}}{\sigma} \]
\[ d_- = \frac{\ln(\frac{S_0}{K'}) - \frac{\sigma}{2}}{\sigma} \]
\[ \ln(\frac{S_0}{K'}) = 0.002591999999014 \]

The option is (bogus) “at the money”.

Whenever \( S_0 = K \),
\[ \ln(\frac{S_0}{K'}) = r \]

\[ V = 120.7994402 \]

\[ 5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \]
\[ T = \frac{30}{365} \]
\[ r = r_* T = 0.00259199999014 \]
\[ \sigma = \sigma_* \sqrt{T} = 0.057339043865 \]
\[ K = 1 \]

\[ K' = \frac{K}{e^r} = 0.997411356345 \]

\[ d_+ = \frac{\ln(S_0/K')}{\sigma} + \frac{\sigma}{2} = \left( \begin{array}{c} 0.0452047996532 \\ +0.0286695219325 \end{array} \right) \]

\[ d_- = \frac{\ln(S_0/K')}{\sigma} - \frac{\sigma}{2} = \left( \begin{array}{c} 0.0452047996532 \\ -0.0286695219325 \end{array} \right) \]

\[ \ln(S_0/K') = 0.002591999999014 \]

\[ V = 120.7994402 \]

\[ 5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \]

close?
\[ T = \frac{30}{365} \]
\[ r = r \ast T = 0.00259199999014 \]
\[ \sigma = \sigma \ast \sqrt{T} = 0.057339043865 \]
\[ K = 1 \]
\[ K' = \frac{K}{e^r} = 0.997411356345 \]
\[ d_+ = \left( 0.04520479965320, 0.04520479965320 \right) \]
\[ + 0.02866952193 \left( 0.0738743215857, 0.0738743215857 \right) \]
\[ d_- = \left( 0.04520479965320, 0.04520479965320 \right) \]
\[ - 0.02866952193 \left( 0.0165352777207, 0.0165352777207 \right) \]
\[ V = 120.7994402 \]
\[ 5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \]
\[ T = \frac{30}{365} \]
\[ r = r_0 T = 0.00259199999014 \]
\[ \sigma = \sigma_0 \sqrt{T} = 0.057339043865 \]
\[ K = 1 \]
\[ K' = \frac{K}{e^r} = 0.997411356345 \]
\[ d_+ = \left( 0.0452047996532 + 0.0286695219325 \right) = 0.0738743215857 \]
\[ d_- = \left( 0.0452047996532 - 0.0286695219325 \right) = 0.0165352777207 \]
\[ \Phi(d_+) = 0.52944 \]
\[ \Phi(d_-) = 0.50660 \]
\[ V = 120.7994402 \]
\[ 5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \]
\[ T = \frac{30}{365} \]
\[ r = r_* T = 0.00259199999014 \]
\[ \sigma = \sigma_* \sqrt{T} = 0.057339043865 \]
\[ K = 1 \]
\[ K' = \frac{K}{e^r} = 0.997411356345 \]
\[ \Phi(d_) = 0.52944 \]
\[ \Phi(d-) = 0.50660 \]
\[ S_0[\Phi(d_)] - K'[\Phi(d-)] = 0.024151406898 \]
\[ V = 120.7994402 \]
\[ 5000(S_0[\Phi(d_)] - K'[\Phi(d-)]) \]
\[ T = \frac{30}{365} \]
\[ r = r_\ast T = 0.002591999999014 \]
\[ \sigma = \sigma_\ast \sqrt{T} = 0.057339043865 \]
\[ K = 1 \]
\[ K' = \frac{K}{e^r} = 0.997411356345 \]
\[ \Phi(d_+) = 0.52944 \]
\[ \Phi(d-) = 0.50660 \]
\[ S_0[\Phi(d_+)] - K'[\Phi(d_-)] = 0.024151406898 \]
\[ V = 120.7994402 \]

\[ 5000 \left( S_0[\Phi(d_+)] - K'[\Phi(d_-)] \right) \]
\[ = 120.7570345 \]

\[ 5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \]
inputs: $T, \sigma_*, r_*, S_0, K$

Let $K' := \frac{K}{e^{r_*T}}$.

Let $d_\pm := \frac{\ln(S_0/K')}{\sigma_*\sqrt{T}} \pm \frac{\sigma_*\sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

first version of Black-Scholes

Let $d_\pm := \frac{\ln(S_0/K) + r_*T}{\sigma_*\sqrt{T}} \pm \frac{\sigma_*\sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - [Ke^{-r_*T}][\Phi(d_-)]$

second version of Black-Scholes

forward price on stock

Let $F := Se^{r_*T}$.

Let $d_\pm := \frac{\ln(F/K)}{\sigma_*\sqrt{T}} \pm \frac{\sigma_*\sqrt{T}}{2}$.

output: $e^{-r_*T}(F[\Phi(d_+)] - K[\Phi(d_-)])$

third version of Black-Scholes

inputs: $\sigma, r, S_0, K$

Let $K' := \frac{K}{e^{r}}$.

Let $d_\pm := \frac{\ln(S_0/K')}{\sigma} \pm \frac{\sigma}{2}$.

output: $S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

version zero of Black-Scholes

Do these formulas really approximate the CRR price?

YES

YES

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inputs: $T, \sigma_* , r_*$
inputs: $T, \sigma_* , r_* , S_0 , K$

Let $d_\pm := \frac{\left[\ln (S_0 / K)\right] + r_* T}{\sigma_* \sqrt{T}} \pm \frac{\sigma_* \sqrt{T}}{2}$.

output: $S_0 \left[ \Phi (d_+) \right] - \left[ Ke^{-r_* T} \right] \left[ \Phi (d_-) \right]$

second version of Black-Scholes
inputs: $T, \sigma_*, r_*, S_0, K$

Let $d_\pm := \frac{[\ln(S_0/K)] + r_*T}{\sigma_*\sqrt{T}} \pm \frac{\sigma_*\sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - [Ke^{-r_*T}][\Phi(d_-)]$

second version of Black-Scholes

$\text{BlSch}(T, \sigma_*, r_*, S_0, K) :=$

$S_0 \left[ \Phi \left( \frac{[\ln(S_0/K)] + r_*T}{\sigma_*\sqrt{T}} + \frac{\sigma_*\sqrt{T}}{2} \right) \right]$

$- \left[ Ke^{-r_*T} \right] \left[ \Phi \left( \frac{[\ln(S_0/K)] + r_*T}{\sigma_*\sqrt{T}} - \frac{\sigma_*\sqrt{T}}{2} \right) \right]$

Fact: For all $T > 0, r_* > 0, S_0 > 0$ and $K > 0$, $\sigma_* \mapsto \text{BlSch}(T, \sigma_*, r_*, S_0, K) : (0, \infty) \to (0, \infty)$

Exercise: Prove this. is increasing.
Definition:
For all $V > 0$, $T > 0$, $r_\ast > 0$, $S_0 > 0$ and $K > 0$, if $\exists \sigma_\ast > 0$ such that

$$V = \text{BlSch}(T, \sigma_\ast, r_\ast, S_0, K)$$

then this solution $\sigma_\ast$ is unique and is called the implied volatility associated to

$V$, $T$, $r_\ast$, $S_0$ and $K$.

$\text{BlSch}(T, \sigma_\ast, r_\ast, S_0, K) :=$

$$S_0 \left[ \Phi \left( \frac{[\ln(S_0/K)] + r_\ast T}{\sigma_\ast \sqrt{T}} + \frac{\sigma_\ast \sqrt{T}}{2} \right) \right]$$

$$- \left[ Ke^{-r_\ast T} \right] \left[ \Phi \left( \frac{[\ln(S_0/K)] + r_\ast T}{\sigma_\ast \sqrt{T}} - \frac{\sigma_\ast \sqrt{T}}{2} \right) \right]$$

Fact: For all $T > 0$, $r_\ast > 0$, $S_0 > 0$ and $K > 0$, $\sigma_\ast \mapsto \text{BlSch}(T, \sigma_\ast, r_\ast, S_0, K) : (0, \infty) \rightarrow (0, \infty)$

Exercise: Prove this. is increasing.
Definition:
For all \( V > 0, \ T > 0, \ r_* > 0, \ S_0 > 0 \) and \( K > 0 \), if \( \exists \sigma_* > 0 \) such that
\[
V = \text{BlSch}(T, \sigma_*, r_*, S_0, K)
\]
then this solution \( \sigma_* \) is unique and is called the **implied volatility** associated to \( V, \ T, \ r_*, \ S_0 \) and \( K \).

Fiction: Black-Scholes works, So why teach BS??
i.e., volatility, drift and risk-free rates are constant.

Fiction: Home mortgage interest rates stay constant over thirty year periods.
Nevertheless: They’re useful, because . . .
they give a way of comparing mortgages.

**dimensionless**

Similar for Black-Scholes.

Next subtopic: Volatility smiles and skews and volatility surfaces
Definition:
For all $V > 0$, $T > 0$, $r_* > 0$, $S_0 > 0$ and $K > 0$, if $\exists \sigma_* > 0$ such that
$$V = \text{BlSch}(T, \sigma_*, r_*, S_0, K)$$
then this solution $\sigma_*$ is unique and is called the **implied volatility** associated to $V$, $T$, $r_*$, $S_0$ and $K$.

Fiction: Black-Scholes works, i.e., volatility, drift and risk-free rates are constant.

Pick a financial instrument (e.g., a stock).
Look up $S_0$. Look up $r_*$. Fix $T$.
For various choices of $K$,
look up $V$, compute $\sigma_*$ and plot $(K, \sigma_*)$.
The result is called the **volatility smile** or the **volatility skew**, depending on whether it’s concave up or concave down.
Definition:
For all \( V > 0, T > 0, r_* > 0, S_0 > 0 \) and \( K > 0 \),
if \( \exists \sigma_* > 0 \) such that
\[
V = \text{BlSch}(T, \sigma_*, r_*, S_0, K)
\]
then this solution \( \sigma_* \) is unique and is called
the **implied volatility** associated to \( V, T, r_*, S_0 \) and \( K \).

Fiction: Black-Scholes works, i.e., volatility, drift and risk-free rates are constant.

Pick a financial instrument (e.g., a stock).
Look up \( S_0 \). Look up \( r_* \).
For various choices of \( K \) and \( T \),
look up \( V \), compute \( \sigma_* \) and plot \( (K, T, \sigma_*) \).
The result is called
the **volatility surface**.