Financial Mathematics
Basics of measures
Q: How far can length be ext’d, maintaining \( \sigma \)-additivity and translation invariance?

Def’n: \( \mathcal{I} := \{ \text{intervals} \} \subseteq 2^\mathbb{R} \).

Fact: \( \mathcal{I} \) is a near algebra on \( \mathbb{R} \), length \( \sigma \)-a., \( \sigma \)-f.

Def’ns:

A subset of \( \mathbb{R} \) is **Borel** if it’s an elt of \( \langle \mathcal{I} \rangle_\sigma \).

The unique extension of length to \{Borel sets in \( \mathbb{R} \}\} \langle \text{length} \rangle_\sigma \)

is called **Lebesgue measure** on \( \mathbb{R} \).

Let \( \mathcal{A} \) be a \( \sigma \)-alg. on a set \( M \).

Let \( \mu : \mathcal{A} \to [0, \infty] \) be \( \sigma \)-additive.

A subset \( Z \subseteq M \) is **null** (or \( \mu \)-null) if \( \exists A \in \mathcal{A} \) s.t. \( \mu(A) = 0 \) and \( Z \subseteq A \).

\[ \exists \text{Borel set of msr zero with a non-Borel subset.} \]

Currently, that subset **cannot** be measured.

The fix: “completion”…
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s.t. $\mu(A) = 0$ and $Z \subseteq A$.

$P, Q \subseteq M$ are **essentially equal** (written $P \cong Q$)
if $\exists$ null sets $Z, Z' \subseteq M$ s.t. $(P \setminus Z) \cup Z' = Q$.

A subset $C \subseteq M$ is **conull in** $M$ (or $\mu$-conull)
if $M \setminus C$ is null.
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Fact: \( A, B \in A, A \cong B \implies \mu(A) = \mu(B) \)

Def’n: A subset \( P \subseteq M \) is **measurable** (or \( \mu \)-measurable) if \( \exists A \in A \) s.t. \( P \cong A \).
Q: How far can length be ext’d, maintaining σ-additivity and translation invariance?

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**Fact:** \( A, B \in \mathcal{A}, \quad A \overset{\circ}{=} B \quad \Rightarrow \quad \mu(A) = \mu(B) \)

**Def’n:** A subset \( P \subseteq M \) is **measurable** (or \( \mu\)-**measurable**) if \( \exists A \in \mathcal{A} \) s.t. \( P \overset{\circ}{=} A \).

**Def’n:** The **completion** of \( \mathcal{A} \) \( \textbf{(w.r.t.} \ \mu) \) is \( \sqcup \mathcal{A} := \{ \mu\text{-measurable sets} \} \)
Q: How far can length be ext’d, maintaining \( \sigma \)-additivity and translation invariance?

\[ P, Q \subseteq M \text{ are } \textbf{essentially equal} \text{ (written } P \equiv Q \text{)} \text{ if } \exists \text{ null sets } Z, Z' \subseteq M \text{ s.t. } (P \setminus Z) \cup Z' = Q. \]

Fact: \( A, B \in \mathcal{A}, \quad A \equiv B \quad \Rightarrow \quad \mu(A) = \mu(B) \)

Def’n: A subset \( P \subseteq M \) is \textbf{measurable} (or \( \mu \)-measurable) if \( \exists A \in \mathcal{A} \text{ s.t. } P \equiv A. \)

Def’n: The \textbf{completion of} \( \mathcal{A} \) (w.r.t. \( \mu \)) is \( \overline{\mathcal{A}} := \{ \mu \text{-measurable sets} \} =: \overline{\mathcal{A}}^\mu \)

Fact: \( \overline{\mathcal{A}} \) is a \( \sigma \)-algebra.

Def’n: The \textbf{completion of} \( \mu \) is the function \( \overline{\mu} : \overline{\mathcal{A}} \rightarrow [0, \infty] \) “well-defined”

defined by: \( \overline{\mu}(P) = \mu(A), \quad \forall A \in \mathcal{A} \text{ s.t. } A \equiv P. \)

Note: \( A, B \in \mathcal{A}, \quad A \equiv P, \ B \equiv P \quad \Rightarrow \quad \mu(A) = \mu(B) \)

Fact: \( \overline{\mu} : \overline{\mathcal{A}} \rightarrow [0, \infty] \) is \( \sigma \)-additive.
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Def’ns:
A subset of \( \mathbb{R} \) is **measurable** if it’s an elt of the completion of \( \{ \text{Borel sets in } \mathbb{R} \} \) w.r.t. Lebesgue measure.

Note: It’s hard to make non-measurable sets, or even non-Borel sets.

Def’n: The **completion of** \( A \) (w.r.t. \( \mu \)) is

\[
\overline{A} := \{ \mu\text{-measurable sets} \} =: \overline{A}^\mu
\]

Fact: \( \overline{A} \) is a \( \sigma \)-algebra.

Def’n: The **completion of** \( \mu \) is the function

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\overline{\mu} : \overline{A} \to [0, \infty]
\]

defined by:

\[
\overline{\mu}(P) = \mu(A), \quad \forall A \in A \text{ s.t. } A \cong P.
\]

Note: \( A, B \in A, \quad A \cong P, \ B \cong P \quad \Rightarrow \quad \mu(A) = \mu(B)\)

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**Note:** It’s hard to make non-measurable sets, or even non-Borel sets.

Fact: \{Borel sets in \( \mathbb{R} \}\} is countably generated.

\[ \langle \{(a, b) \mid a, b \in \mathbb{Q}, a < b\} \rangle_{\sigma} \]

Def’n:
A \( \sigma \)-algebra \( \mathcal{A} \) on \( M \) is **countably generated** if \( \exists \) a countable set \( \mathcal{C} \subseteq \mathcal{A} \) s.t. \( \mathcal{A} = \langle \mathcal{C} \rangle_{\sigma} \).
Q: How far can length be ext’d, maintaining $\sigma$-additivity and translation invariance? 😊

Def’ns:
A subset of $\mathbb{R}$ is **measurable** if it’s an elt of the completion of $\{\text{Borel sets in } \mathbb{R}\}$ w.r.t. Lebesgue measure.

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A $\sigma$-algebra $\mathcal{A}$ on $\mathcal{M}$ is **countably generated** if $\exists$ a countable set $\mathcal{C} \subseteq \mathcal{A}$ s.t. $\mathcal{A} = \langle \mathcal{C} \rangle_\sigma$.

Traditional terminology:
Def’n: A countably generated $\sigma$-algebra on $\mathcal{M}$, if $\exists$ a countable set $\mathcal{C} \subseteq \mathcal{A}$ s.t. $\mathcal{A} = \langle \mathcal{C} \rangle_\sigma$. 
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A subset of \( \mathbb{R} \) is **measurable** if it’s an elt of the completion of \( \{ \text{Borel sets in } \mathbb{R} \} \) w.r.t. Lebesgue measure.

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A \( \sigma \)-algebra \( \mathcal{A} \) on \( M \) is **countably generated** if \( \exists \) a countable set \( \mathcal{C} \subseteq \mathcal{A} \) s.t. \( \mathcal{A} = \langle \mathcal{C} \rangle_\sigma \).

**Traditional terminology:**
If \( \mathcal{A} \) is a countably generated \( \sigma \)-algebra on \( M \), then the elements of \( \mathcal{A} \) are called **Borel sets**.
If, furthermore, \( \mu : \mathcal{A} \to [0, \infty] \) is \( \sigma \)-finite, then the els of the completion of \( \mathcal{A} \) w.r.t. \( \mu \) are called **measurable** sets.
Def’ns: A **Borel space** is a set with a countably generated $\sigma$-algebra on it. A **measure on a Borel space** $(M, \mathcal{B})$ is a $\sigma$-additive function $\mu : \mathcal{B} \rightarrow [0, \infty]$. A **measure space** is a Borel space with a $\sigma$-finite measure on it. A measure $\mu$ on a Borel space $(M, \mathcal{B})$ is a **probability measure** if $\mu(M) = 1$.

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If $\mathcal{A}$ is a countably generated $\sigma$-algebra on $M$, then the elements of $\mathcal{A}$ are called **Borel sets**. If, furthermore, $\mu : \mathcal{A} \rightarrow [0, \infty]$ is $\sigma$-finite, then the elts of the completion of $\mathcal{A}$ w.r.t. $\mu$ are called **measurable sets**.
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Def’n: A msr $\mu$ on $M$ is **finite** if $\mu(M) < \infty$. Fact: $\mu$ is finite iff, $\forall A \in \mathcal{A}, \mu(A) < \infty$.

**monotonicity**: $\forall A, B \in \mathcal{A}, A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$

Fact: measures are monotone.

$$B = A \bigcup (B \setminus A), \mu(B \setminus A) \geq 0 \text{ QED}$$
Def’ns: A **Borel space** is a set with a countably generated $\sigma$-algebra on it. A **measure** on a Borel space $(M, \mathcal{B})$ is a $\sigma$-additive function $\mu : \mathcal{B} \to [0, \infty]$. A **measure space** is a Borel space with a $\sigma$-finite measure on it. A measure $\mu$ on a Borel space $(M, \mathcal{B})$ is a **probability measure** if $\mu(M) = 1$.

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Def’n: For any countable set $M$, the **counting measure** on $M$ is the measure $\mu : 2^M \to [0, \infty]$ def’d by $\mu(S) = \#S$.

**monotonicity**: $\forall A, B \in \mathcal{A}$, $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$

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Def’ns: A **Borel space** is a set with a countably generated $\sigma$-algebra on it. A **measure on a Borel space** $(M, \mathcal{B})$ is a $\sigma$-additive function $\mu : \mathcal{B} \to [0, \infty]$. A **measure space** is a Borel space with a $\sigma$-finite measure on it. A measure $\mu$ on a Borel space $(M, \mathcal{B})$ is a **probability measure** if $\mu(M) = 1$. 

Def’n: A msr $\mu$ on $M$ is **finite** if $\mu(M) < \infty$. 

Fact: $\mu$ is finite iff, $\forall A \in \mathcal{A}$, $\mu(A) < \infty$. 

Def’n: For any countable set $M$, the **counting measure** on $M$ is the measure $\mu : 2^M \to [0, \infty]$ def’d by $\mu(S) = \#S$. It is a measure on the (countably generated) Borel space $(M, 2^M)$. $\mu$ is finite iff $M$ is finite. 

*more examples soon...*
Def’n: For any countable set $M$, 

**counting measure** on $M$ is the measure 

$$\mu : 2^M \to [0, \infty]$$ 

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It is a measure on the (countably generated) Borel space $(M, 2^M)$.

$$\mu$$ is finite iff $M$ is finite.
Def’n: \( \forall \text{set } M, \)
\[ 2^M \text{ is the discrete } \sigma\text{-algebra on } M \]
and \( \{\emptyset, M\} \text{ is the indiscrete } \sigma\text{-algebra on } M. \)

Note: If \( M \) is uncountable,
then \( 2^M \) is not countably generated,
so \((M, 2^M)\) is not a Borel space.

Note: If \( \#M > 1, \)
then \( \{\emptyset, M\} \) is pathological
but in a different way...

Def’n: For any countable set \( M, \)
\textbf{counting measure on } M \text{ is the measure}
\[ \mu : 2^M \to [0, \infty] \text{ def’d by } \mu(S) = \#S. \]
It is a measure on the (countably generated) Borel space \((M, 2^M)\).
\[ \mu \text{ is finite iff } M \text{ is finite. } \]
Def’n: \( \forall \text{set } M, \)
\[ 2^M \text{ is the discrete } \sigma\text{-algebra on } M \]
and \( \{\emptyset, M\} \) is the \textit{indiscrete} \( \sigma\text{-algebra on } M. \)

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Note: If \( \#M > 1, \)
Def’n: A Borel space \( (M, \mathcal{A}) \) is \textit{standard} if
\textit{but} in a different way.

\textit{countably generated}
Def'n: \( \forall \text{set } M, \)

\[ 2^M \] is the **discrete** \( \sigma \)-algebra on \( M \)
and \( \{\emptyset, M\} \) is the **indiscrete** \( \sigma \)-algebra on \( M \).

Note: If \( \# M > 1 \),
then \( \{\emptyset, M\} \) is pathological
but in a different way...

Def'n: A Borel space \( (M, \mathcal{A}) \) is **standard** if

\[ \forall x, y \in M, \quad x \neq y \implies \exists A \in \mathcal{A} \quad \text{s.t. } x \in A \quad \text{and } y \notin A. \]


![Diagram showing separation of points in a Borel space](image-url)
Def’n: \( \forall M, \)
\[ 2^M \] is the **discrete** \( \sigma \)-algebra on \( M \)
and \( \{\emptyset, M\} \) is the **indiscrete** \( \sigma \)-algebra on \( M \).

**Note:** If \( \#M > 1 \),
then \( \{\emptyset, M\} \) does not separate points,
so \((M, \{\emptyset, M\})\) is not standard.

Def’n: A Borel space \((M, \mathcal{A})\) is **standard** if
\[ \forall x, y \in M, \]
\[ x \neq y \Rightarrow \exists A \in \mathcal{A} \quad \text{s.t.} \quad x \in A \quad \text{and} \quad y \notin A. \]

The \( \sigma \)-algebra “separates points”
Def’n: \( \forall \text{set } M, \)
\[ 2^M \text{ is the discrete } \sigma\text{-algebra on } M \text{ and } \{\emptyset, M\} \text{ is the indiscrete } \sigma\text{-algebra on } M. \]

Note: If \( \#M > 1, \)
then \( \{\emptyset, M\} \) does not separate points,
so \( (M, \{\emptyset, M\}) \) is not standard.

Completions are usually not ctbly gen’d:
\[ M := \mathbb{R}, \quad \mathcal{B} := \{\emptyset, M\}, \quad \lambda := \text{Lebesgue measure} \]
\[ \overline{\mathcal{B}} = \{\text{null sets}\} \cup \{\text{conull sets}\} \]
\[ \forall x \in \mathbb{R}, \quad \{x\} \in \overline{\mathcal{B}}, \quad \text{so } \overline{\mathcal{B}} \text{ separates points.} \]

However, \( \overline{\mathcal{B}} \) is not countably generated,
so \( (\mathbb{R}, \overline{\mathcal{B}}) \) is not a Borel space.