Financial Mathematics

Ito’s lemma
\( f(x) := x^2 \)

\[
f(x + h) = f(x) + [f'(x)]h + [f''(x)][h^2/2]
\]

\[
(f(x + h)) - (f(x)) = \frac{y - x}{y - x} [f'(x)]h + [f''(x)][h^2/2]
\]

\[
(f(y)) - (f(x)) = [f'(x)][y - x] + [f''(x)][(y - x)^2/2]
\]
\[ f(x) := x^2 \quad f'(x) = 2x \quad f''(x) = 2 \]

\[
(f(y)) - (f(x)) = [f'(x)][y - x] + [f''(x)][(y - x)^2 / 2]
\]
\[ f(x) := x^2 \quad f'(x) = 2x \quad f''(x) = 2 \]

\[ (f(y)) - (f(x)) = [f'(x)][y - x] + [f''(x)][(y - x)^2/2] \]

\[ y^2 - x^2 = 2x[y - x] + 2[(y - x)^2/2] \]

\[ = 2x(y - x) + (y - x)^2 \]

\[ y^2 - x^2 = 2x(y - x) + (y - x)^2 \]
\[ f(x) := x^3 \]

\[
f(x + h) = f(x) + [f'(x)]h + [f''(x)][h^2/2] + [f'''(x)][h^3/6]
\]

\[
(f(x + h)) - (f(x)) = [f'(x)]h + [f''(x)][h^2/2] + [f'''(x)][h^3/6]
\]

\[
(f(y)) - (f(x)) = [f'(x)][y - x] + [f''(x)][(y - x)^2/2] + [f'''(x)][(y - x)^3/6]
\]

\[
y^2 - x^2 = 2x(y - x) + (y - x)^2
\]
\[ f(x) := x^3 \quad f'(x) = 3x^2 \quad f''(x) = 6x \quad f'''(x) = 6 \]

\[
(f(y)) - (f(x)) = [f'(x)][y - x] + [f''(x)][(y - x)^2/2] + [f'''(x)][(y - x)^3/6]
\]

\[
(f(y)) - (f(x)) = [f'(x)][y - x] + [f''(x)][(y - x)^2/2] + [f'''(x)][(y - x)^3/6]
\]

\[
y^2 - x^2 = 2x(y - x) + (y - x)^2
\]
\[ f(x) := x^3 \quad f'(x) = 3x^2 \quad f''(x) = 6x \quad f'''(x) = 6 \]

\[ (f(y)) - (f(x)) = [f'(x)][y - x] + [f''(x)][(y - x)^2/2] + [f'''(x)][(y - x)^3/6] \]

\[ y^3 - x^3 = 3x^2[y - x] + 6x[(y - x)^2/2] + 6[(y - x)^3/6] \]

\[ = 3x^2(y - x) + 3x(y - x)^2 + (y - x)^3 \]

\[ y^2 - x^2 = 2x(y - x) + (y - x)^2 \]
Fix an integer \( k \geq 1 \).

Define \( B^k \) by:

\[
B^k_{n/k} = (C_1 + \cdots + C_n) / \sqrt{k}
\]

\( \forall t \in [n/k, (n + 1)/k) \), \( B^k_t = B^k_{n/k} \)

Let \( X^2 := (B^k)^2 \).

**Goal:** Find an SDE that \( X^2 \) solves.

Let \( \Delta t := 1/k \). \( \forall t \geq 0 \),

\[
X_{t+\Delta t} - X_t = (B^k_{t+\Delta t})^2 - (B^k_t)^2
\]

\[
= 2B^k_t (B^k_{t+\Delta t} - B^k_t) + (B^k_{t+\Delta t} - B^k_t)^2
\]

\[
y^3 - x^3 = 3x^2(y - x) + 3x(y - x)^2 + (y - x)^3
\]

\[
y^2 - x^2 = 2x(y - x) + (y - x)^2
\]
Fix an integer \( k \geq 1 \).

Define \( B^k_\bullet \) by:
\[
B^k_{n/k} = \frac{C_1 + \cdots + C_n}{\sqrt{k}}
\]
\[
\forall t \in \left[\frac{n}{k}, \frac{n + 1}{k}\right), \quad B^k_t = B^k_{n/k}
\]

Let \( X_\bullet := (B^k_\bullet)^2 \).

Goal: Find an SDE that \( X_\bullet \) solves.

Let \( \Delta t := \frac{1}{k} \). \( \forall t \geq 0 \),
\[
X_{t+\Delta t} - X_t = (B^k_{t+\Delta t})^2 - (B^k_t)^2
\]
\[
= 2B^k_t (B^k_{t+\Delta t} - B^k_t) + (B^k_{t+\Delta t} - B^k_t)^2
\]
\[
= 2B^k_t (B^k_{t+\Delta t} - B^k_t) + \Delta t
\]

\[
y^3 - x^3 = 3x^2(y - x) + 3x(y - x)^2 + (y - x)^3
\]

\[
B^k_{t+\Delta t} - B^k_t \in \{ \pm 1/\sqrt{k} \} = \{ \pm (\Delta t)^{1/2} \}^9
\]
Fix an integer $k \geq 1$.

Define $B^k_n$ by:

$$B^k_{n/k} = \frac{(C_1 + \cdots + C_n)}{\sqrt{k}} \quad \forall t \in [n/k, (n + 1)/k), \quad B^k_t = B^k_{n/k}$$

Let $X^2 := (B^k_n)^2$.

**Goal:** Find an SΔE that $X^2$ solves.

Let $\Delta t := 1/k$. $\forall t \geq 0$,

$$X_{t+\Delta t} - X_t - X_t = 2B^k_t (B^k_{t+\Delta t} - B^k_t) + \Delta t$$

**SΔE:** $\Delta X_t = [2B^k_t][\Delta B^k_t] + [1][\Delta t]$

$$= 2B^k_t (B^k_{t+\Delta t} - B^k_t) + \Delta t$$

$$y^3 - x^3 = 3x^2(y - x) + 3x(y - x)^2 + (y - x)^3$$

$$B^k_{t+\Delta t} - B^k_t \in \left\{ \pm \frac{1}{\sqrt{k}} \right\} = \left\{ \pm (\Delta t)^{1/2} \right\}$$
Fix an integer $k \geq 1$.

Define $B^k_\bullet$ by: $B^k_{n/k} = (C_1 + \cdots + C_n)/\sqrt{k}$

\[ \forall t \in [n/k, (n + 1)/k), \quad B^k_t = B^k_{n/k} \]

Let $X_\bullet := (B^k_\bullet)^2$.

Goal: Find an $S\Delta E$ that $X_\bullet$ solves.

Sol’n: $\triangle X_t = [2B^k_t][\triangle B^k_t] + [1][\triangle t]$
Fix an integer \( k \geq 1 \).

Define \( B^k \) by:
\[
B^k_t = (C_1 + \cdots + C_n)/\sqrt{k}
\]
\[\forall t \in [n/k, (n + 1)/k), \quad B^k_t = B^k_{n/k}\]

Let \( X^k := (B^k)^2 \).

Goal: Find an SDE that \( X^k \) solves.

Sol’n: \[
\triangle X_t = [2B^k_t][\triangle B^k_t] + [1][\triangle t]
\]
\[k \to \infty \]

Let \( W^k \) be a Brownian motion.

Let \( U^k := W^k \).

Next: Rate of change of \( W^k \)

\( U^k \) is a solution to:
\[
dU_t = [2W_t][dW_t] + [1][dt], \quad U_0 = 0
\]

\[
y^3 - x^3 = 3x^2(y - x) + 3x(y - x)^2 + (y - x)^3
\]

\[
B^k_{t+\triangle t} - B^k_t \in \{\pm 1/\sqrt{k}\} = \{\pm (\triangle t)^{1/2}\}\]
Fix an integer $k \geq 1$.

Define $B_k^k$ by: $B_{n/k}^k = (C_1 + \cdots + C_n)/\sqrt{k}$

$\forall t \in [n/k, (n + 1)/k)$, $B_t^k = B_{n/k}^k$

Let $X_k := (B_k^k)^3$. Rate of change?

Goal: Find an SΔE that $X_k$ solves.

Let $\Delta t := 1/k$. $\forall t \geq 0$,

$X_{t+\Delta t} - X_t = (B_{t+\Delta t}^k)^3 - (B_t^k)^3$

$= 3(B_t^k)^2(B_{t+\Delta t}^k - B_t^k) + 3B_t^k(B_{t+\Delta t}^k - B_t^k)^2 + (B_{t+\Delta t}^k - B_t^k)^3$

$y^3 - x^3 = 3x^2(y - x) + 3x(y - x)^2 + (y - x)^3$

$B_{t+\Delta t}^k - B_t^k \in \{\pm 1/\sqrt{k}\} = \{\pm (\Delta t)^{1/2}\}$
Fix an integer $k \geq 1$.

Define $B^k_\bullet$ by: $B^k_{n/k} = (C_1 + \cdots + C_n) / \sqrt{k}$ for all $t \in [n/k, (n + 1)/k)$, \( B^k_t = B^k_{n/k} \)

Let $X_\bullet := (B^k_\bullet)^3$.

Goal: Find an SDE that $X_\bullet$ solves.

Let $\triangle t := 1/k$. For all $t \geq 0$,

\[
X_{t+\triangle t} - X_t = (B^k_{t+\triangle t})^3 - (B^k_t)^3
\]

\[
= 3(B^k_t)^2(B^k_{t+\triangle t} - B^k_t) \triangle t + 3B^k_t(B^k_{t+\triangle t} - B^k_t)^2 + (B^k_{t+\triangle t} - B^k_t)^3
\]

\[
\in \{3(B^k_t)^2(B^k_{t+\triangle t} - B^k_t) + 3B^k_t(\triangle t) \pm (\triangle t)^{3/2}\}
\]

\[
B^k_{t+\triangle t} - B^k_t \in \{\pm 1/\sqrt{k}\} = \{\pm(\triangle t)^{1/2}\}
\]
Fix an integer $k \geq 1$.

Define $B^k_\bullet$ by: $B^k_{n/k} = (C_1 + \cdots + C_n)/\sqrt{k}$

$\forall t \in [n/k, (n + 1)/k), \quad B^k_t = B^k_{n/k}$

Let $X_\bullet := (B^k_\bullet)^3$.

Goal: Find an SDE that $X_\bullet$ solves.

Let $\Delta t := 1/k$. $\forall t \geq 0,$

$$X_{t+\Delta t} - X_t \in \{3(B^k_t)^2(B^k_t + \Delta t - B^k_t) + 3B^k_t(\Delta t) \pm (\Delta t)^{3/2}\}$$

$$\in \{3(B^k_t)^2(B^k_t + \Delta t - B^k_t) + 3B^k_t(\Delta t) \pm (\Delta t)^{3/2}\}$$
Fix an integer \( k \geq 1 \).

Define \( B^k_\bullet \) by:
\[
B^k_{n/k} = \left( C_1 + \cdots + C_n \right) / \sqrt{k} \\
\forall t \in \left[ n/k, (n + 1)/k \right), \quad B^k_t = B^k_{n/k}
\]

Let \( X_\bullet := (B^k_\bullet)^3 \).

Goal: Find an S\( \Delta \)E that \( X_\bullet \) solves.

Let \( \Delta t := 1/k \). \( \forall t \geq 0 \),
\[
X_{t+\Delta t} - X_t \\
\in \{ 3(B^k_t)^2(B^k_{t+\Delta t} - B^k_t) + 3B^k_t(\Delta t) \pm (\Delta t)^{3/2} \}
\]

\( X_\bullet \) approximates a solution \( Y_\bullet \) to:
\[
\Delta Y_t = 3[(B^k_t)^2][\Delta B^k_t] + [3B^k_t][\Delta t], \quad Y_0 = 0
\]

Goal: Estimate \( |X_t - Y_t| \).
Fix an integer $k \geq 1$.

$$X_{t+\Delta t} - X_t \in \{3(B_t^k)^2(B_{t+\Delta t}^k - B_t^k) + 3B_t^k(\Delta t) \pm (\Delta t)^{3/2}\}$$

Let $X_\bullet := (B_\bullet^k)^3$. $\Delta t := 1/k$

$X_\bullet$ approximates a solution $Y_\bullet$ to:

$$\Delta Y_t = \frac{1}{k^{1.2}}[\Delta B_t^k] + [3B_t^k][\Delta t], \quad Y_0 = 0$$

**Goal:** Estimate $|X_t - Y_t|$.  
Let $t_0 := 175 + \Delta t - B_t^k + 3B_t^k(\Delta t) \pm (\Delta t)^{3/2}$

$X_\bullet$ approximates a solution $Y_\bullet$ to:

$$\Delta Y_t = 3[(B_t^k)^2][\Delta B_t^k] + [3B_t^k][\Delta t], \quad Y_0 = 0$$

**Goal:** Estimate $|X_t - Y_t|$.  

Fix an integer $k \geq 1$.

\begin{align*}
X_{t+\Delta t} - X_t \\
\in \{3(B_t^k)^2(B_t^k + \Delta t - B_t^k) + 3B_t^k(\Delta t) \pm (\Delta t)^{3/2}\}
\end{align*}

Let $X_\bullet := (B_\bullet^k)^3$. $\Delta t := 1/k$

$X_\bullet$ approximates a solution $Y_\bullet$ to:

$\Delta Y_t = 3[(B_t^k)^2][\Delta B_t^k] + [3B_t^k][\Delta t]$, $Y_0 = 0$

Goal: Estimate $|X_t - Y_t|$.

Let $t_0 := 175$. Goal: Estimate $|X_{t_0} - Y_{t_0}|$.

Add over $t \in \{0, 1/k, \ldots, 175 - (1/k)\}$: $\Delta X_t - \Delta Y_t = \pm (1/k)^{3/2}$

$-175k[(1/k)^{3/2}] \leq (X_{t_0} - X_0) - (Y_{t_0} - Y_0) \leq 175k[(1/k)^{3/2}]$

$-b \leq a \leq b \iff |a| \leq b$
Fix an integer $k \geq 1$. Let $k$ vary.

$$X_t + \triangle t - X_t \in \{3(B_t^k)^2(B_t^k + \triangle t - B_t^k) + 3B_t^k(\triangle t) \pm (\triangle t)^{3/2}\}$$

Let $X_\bullet := (B_\bullet^k)^3$. $\triangle t := 1/k$

$X_\bullet$ approximates a solution $Y_\bullet$ to:

$$\triangle Y_t = 3[(B_t^k)^2][\triangle B_t^k] + [3B_t^k][\triangle t], \quad Y_0 = 0$$

Goal: Estimate $|X_t - Y_t|$.

Let $t_0 := 175$. Goal: Estimate $|X_{t_0} - Y_{t_0}|$.

Add over $t \in \{0, 1/k, \ldots, 175 - (1/k)\}$: $\triangle X_t - \triangle Y_t = \pm (1/k)^{3/2}$

$$-175k[(1/k)^{3/2}] \leq (X_{t_0} - X_0) - (Y_{t_0} - Y_0) \leq 175k[(1/k)^{3/2}]$$

$$|X_{t_0} - Y_{t_0}| \leq 175k[(1/k)^{3/2}]$$

Key pt: $175k[(1/k)^{3/2}] \to 0$, as $k \to \infty$
Let \( X^k_\bullet := (B^k_\bullet)^3 \). \( \triangle t := 1/k \)

\( X^k_\bullet \) approximates a solution \( Y^k_\bullet \) to:
\[
\triangle Y^k_t = 3[(B^k_t)^2][\triangle B^k_t] + [3B^k_t][\triangle t], \quad Y^k_0 = 0
\]

Let \( \forall t_0 \geq 0.75. \quad |X^k_{t_0} - Y^k_{t_0}| \rightarrow 0 \) surely

\[
|X^k_{t_0} - Y^k_{t_0}| \leq 175k[(1/k)^{3/2}]
\]

Key pt: \( 175k[(1/k)^{3/2}] \rightarrow 0, \) as \( k \rightarrow \infty \)
Let $X^k := (B^k)^3$. $\Delta t := 1/k$

$X^k$ approximates a solution $Y^k$ to:
$\Delta Y^k_t = 3[(B^k_t)^2][\Delta B^k_t] + [3B^k_t][\Delta t], \quad Y^k_0 = 0$

$\forall t_0 \geq 0, \quad |X^k_{t_0} - Y^k_{t_0}| \to 0 \quad \text{in probability}$

Get “surely” because:

Third derivative of $(\bullet)^3$ is constant, so $|\text{error}|$ was independent of $\omega$.

For $(\bullet)^4$, we’d get “in probability”.

175k additions negligible?
$\pm (\Delta t)^{3/2}$
Let \( X^k_s := (B^k_s)^3 \). \( \Delta t := 1/k \)

\( X^k_s \) approximates a solution \( Y^k_s \) to:

\[
\Delta Y^k_t = 3[(B^k_t)^2][\Delta B^k_t] + [3B^k_t][\Delta t], \quad Y^k_0 = 0
\]

\[\forall t_0 \geq 0, \quad |X^k_{t_0} - Y^k_{t_0}| \to 0 \quad \text{in probability}\]

Let \( W_s \) be a Brownian motion.

Let \( U_s := W^3_s \). \( U_s \) is the solution to:

\[
dU_t = [3W^2_t][dW_t] + [3W_t][dt], \quad U_0 = 0
\]

1st deriv. of \((\cdot)^3\), evaluated at \( W_t \)

2nd deriv. of \((\cdot)^3\), divided by 2!, then evaluated at \( W_t \)
Intuition:

\[(\Delta B_t)^2 = \Delta t\] not negligible

\[175k(\Delta t) = 175 \not\to 0\]

Let \(X^k_\bullet := (B^k_\bullet)^3\). \(\Delta t := 1/k\)

\(X^k_\bullet\) approximates a solution \(Y^k_\bullet\) to:

\[\Delta Y^k_t = 3[(B^k_t)^2][\Delta B^k_t] + [3B^k_t][\Delta t], \quad Y^k_0 = 0\]

\[\forall t_0 \geq 0, \quad |X^k_{t_0} - Y^k_{t_0}| \to 0\] in probability

Let \(W_\bullet\) be a Brownian motion.

Let \(U_\bullet := W^3_\bullet\). \(U_\bullet\) is the solution to:

\[dU_t = [3W^2_t][dW_t] + [3W_t][dt], \quad U_0 = 0\]

1st deriv. of \((\bullet)^3\), evaluated at \(W_t\)

2nd deriv. of \((\bullet)^3\), divided by 2!, then evaluated at \(W_t\)
Intuition:

\[(\triangle B_t)^2 = \triangle t \text{ not negligible}\]

\[175k(\triangle t) = 175 \not\to 0\]

\[(\triangle B_t)^3 = \pm (\triangle t)^{3/2}\]

\[175k(\triangle t)^{3/2} = 175/\sqrt{k}\]

\[\triangle t \coloneqq 1/k\]

Let \(W_\bullet\) be a Brownian motion.

Let \(U_\bullet := W_\bullet^3\). \(U_\bullet\) is the solution to:

\[dU_t = [3W_t^2][dW_t] + [3W_t][dt], \quad U_0 = 0\]

1st deriv. of \((\bullet)^3\), evaluated at \(W_t\)

2nd deriv. of \((\bullet)^3\), divided by 2!, then evaluated at \(W_t\)
Intuition:

\((\Delta B_t)^2 = \Delta t\) not negligible

\[175k(\Delta t) = 175 \not\to 0\]

\((\Delta B_t)^3 = \pm (\Delta t)^{3/2}\) negligible

\[175k(\Delta t)^{3/2} = 175/\sqrt{k} \to 0\]

Taylor’s formula

\[\Delta [(B_t)^3] = 3(B_t)^2[\Delta B_t] + 3B_t[(\Delta B_t)^2] + (\Delta B_t)^3\]

\[d[(W_t)^3] = 3(W_t)^2[dW_t] + 3W_t[(dW_t)^2] + (dW_t)^3\]

Let \(W_\bullet\) be a Brownian motion.

Let \(U_\bullet := W_\bullet^3\). \(U_\bullet\) is the solution to:

\[dU_t = [3W_t^2][dW_t] + [3W_t][dt], \quad U_0 = 0\]

1st deriv. of \((\bullet)^3\), evaluated at \(W_t\)

2nd deriv. of \((\bullet)^3\), divided by 2!, then evaluated at \(W_t\)
**Intuition:**

\[(dW_t)^2 = dt \quad \text{and} \quad (dW_t)(dt) = (dW_t)^3 = 0\]

and\[ (dt)^2 = (dW_t)^4 = (dW_t)^3(dW_t) = 0\]

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**Taylor's formula**

\[
\Delta [(B_t)^3] = 3(B_t)^2[\Delta B_t] + 3B_t[\Delta (B_t)^2] + (\Delta B_t)^3
\]

\[
d[(W_t)^3] = 3(W_t)^2[dW_t] + 3W_t[(dW_t)^2] + (dW_t)^3
\]

---

**Let** \( W \) **be a Brownian motion.**

**Let** \( U := W^3 \). \( U \) **is the solution to:**

\[
dU_t = [3W_t^2][dW_t] + [3W_t][dt], \quad U_0 = 0
\]

1st deriv. of \((\cdot)^3\), evaluated at \( W_t \)

2nd deriv. of \((\cdot)^3\), divided by 2!, then evaluated at \( W_t \)
Intuition:
\[(dW_t)^2 = dt \quad \text{and} \quad (dW_t)(dt) = 0 \quad \text{and} \quad \Rightarrow 0^2 = 0 = 0\]

First Version of Itô’s Lemma:
Let \( W_\bullet \) be a Brownian motion.
Assume \( f : \mathbb{R} \to \mathbb{R} \) is \( C^2 \).

Let \( W_\bullet \) be a Brownian motion. Let \( U_\bullet := W_\bullet^3 \). \( U_\bullet \) is the solution to:
\[dU_t = \boxed{3W_t^2}[dW_t] + \boxed{3W_t}[dt], \quad U_0 = 0\]
1st deriv. of \((\bullet)^3\), evaluated at \( W_t \)
2nd deriv. of \((\bullet)^3\), divided by 2!, then evaluated at \( W_t \)
Intuition:
\[(dW_t)^2 = dt \text{ and } (dW_t)(dt) = 0 \text{ and } (dt)^2 = 0\]

First Version of Itô’s Lemma:

Let \( W \) be a Brownian motion.
Assume \( f : \mathbb{R} \to \mathbb{R} \) is \( C^2 \). Let \( U := f(W) \).

Then \( U \) solves:
\[dU_t = [f'(W_t)][dW_t] + [(f''(W_t))/2][dt],\]
\[U_0 = f(0)\]

Let \( W \) be a Brownian motion.
Let \( U := W^3 \). \( U \) is the solution to:
\[dU_t = [3W_t^2][dW_t] + [3W_t][dt], \quad U_0 = 0\]

1st deriv. of \((\cdot)^3\), evaluated at \( W_t \)
2nd deriv. of \((\cdot)^3\), divided by 2!, then evaluated at \( W_t \)
Intuition:
\[(dW_t)^2 = dt \text{ and } (dW_t)(dt) = 0 \text{ and } (dt)^2 = 0\]

First Version of Itô’s Lemma:
Let \( W_\bullet \) be a Brownian motion.
Assume \( f : \mathbb{R} \rightarrow \mathbb{R} \) is \( C^2 \). Let \( U_\bullet := f(W_\bullet) \).
Then \( U_\bullet \) solves:
\[dU_t = [f'(W_t)][dW_t] + [(f''(W_t))/2][dt],\]
\[U_0 = f(0)\]

Second Version of Itô’s Lemma:
Let \( W_\bullet \) be a Brownian motion.
Assume \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) is \( C^2 \). Let \( U_t := f(W_t, t) \).
Intuition:

\[(dW_t)^2 = dt \quad \text{and} \quad (dW_t)(dt) = 0 \quad \text{and} \quad (dt)^2 = 0\]

Second Version of Itô’s Lemma:

Let \( W_\bullet \) be a Brownian motion.
Assume \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) is \( C^2 \). Let \( U_t := f(W_t, t) \).
Then \( U_\bullet \) solves:

\[
dU_t = \left[ (\partial_1 f)(W_t, t) \right][dW_t] + \left[ (\partial_1^2 f)(W_t, t) \right][dt]
+ \left[ (\partial_2 f)(W_t, t) \right][dt],
\]

Second Version of Itô’s Lemma:

Let \( W_\bullet \) be a Brownian motion.
Assume \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) is \( C^2 \). Let \( U_t := f(W_t, t) \).
Intuition:
\[(dW_t)^2 = dt \text{ and } (dW_t)(dt) = 0 \text{ and } (dt)^2 = 0\]

Second Version of Itô’s Lemma:

Let \(W\) be a Brownian motion.
Assume \(f : \mathbb{R}^2 \to \mathbb{R}\) is \(C^2\). Let \(U_t := f(W_t, t)\).
Then \(U\) solves:
\[
dU_t = \left[(\partial_1 f)(W_t, t)\right][dW_t] + \left[(\partial_2^2 f)(W_t, t)\right]/2[dt] + \left[(\partial_2 f)(W_t, t)\right][dt],
\]
\(U_0 = f(0, 0)\)

“Pf”:
\[
[f(s + \Delta s, t + \Delta t)] - [f(s, t)] \approx L_{f'(s,t)}(\Delta s, \Delta t) + \left[1/(2!)\right][Q_{f''(s,t)}(\Delta s, \Delta t)]
\]
Intuition:

\[(dW_t)^2 = dt \text{ and } (dW_t)(dt) = 0 \text{ and } (dt)^2 = 0\]

Second Version of Itô's Lemma:

Let \( W_t \) be a Brownian motion.
Assume \( f : \mathbb{R}^2 \to \mathbb{R} \) is \( C^2 \). Let \( U_t := f(W_t, t) \).
Then \( U_t \) solves:

\[
dU_t = \left[ (\partial_1 f)(W_t, t) \right][dW_t] + \left[ \frac{(\partial_2^2 f)(W_t, t)}{2} \right][dt] \\
+ \left[ (\partial_2 f)(W_t, t) \right][dt],
\]

\( U_0 = f(0, 0) \)

"Pf": \[
[f(s + \Delta s, t + \Delta t)] - [f(s, t)] \approx \\
[(\partial_1 f)(s, t)][\Delta s] + [(\partial_2 f)(s, t)][\Delta t] \\
+ \frac{1}{2!} [Q_{f''}(s, t)(\Delta s, \Delta t)]
\]
Intuition:
\[(dW_t)^2 = dt \text{ and } (dW_t)(dt) = 0 \text{ and } (dt)^2 = 0\]

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Let \(W_t\) be a Brownian motion.
Assume \(f : \mathbb{R}^2 \rightarrow \mathbb{R}\) is \(C^2\). Let \(U_t := f(W_t, t)\).

Then \(U_t\) solves:
\[dU_t = [(\partial_1 f)(W_t, t)]dW_t + [(\partial_1^2 f)(W_t, t))/2]dt + [(\partial_2 f)(W_t, t)]dt,\]
\[U_0 = f(0, 0)\]

"Pf": \([f(s + \Delta s, t + \Delta t)] - [f(s, t)] \sim\]
\[\[(\partial_1 f)(s, t)[\Delta s] + [(\partial_2 f)(s, t)][\Delta t] + [(\partial_1^2 f)(s, t)][\Delta s]^2/2 + [(\partial_2^2 f)(s, t)][\Delta t]^2/2 + [(\partial_1 \partial_2 f)(s, t)][\Delta s][\Delta t]\]
Intuition:

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\[+ \left[ (\partial_2 f)(W_t, t) \right] [dt],\]
\[U_0 = f(0, 0)\]

“Pf”:

\[
[f(W_t + dW_t, t + dt)] - [f(W_t, t)] = \]
\[
[ (\partial_1 f)(W_t, t) ][dW_t] + \left[ (\partial_2 f)(W_t, t) \right] [dt] + \left[ (\partial_1^2 f)(W_t, t) \right] [dW_t]^2 / 2 + \left[ (\partial_2^2 f)(W_t, t) \right] [dt]^2 / 2 \]

\[+ \left[ (\partial_1 \partial_2 f)(W_t, t) \right] [dW_t][dt] = 0\]
Intuition:
\[(dW_t)^2 = dt \text{ and } (dW_t)(dt) = 0 \text{ and } (dt)^2 = 0\]

Second Version of Itô’s Lemma:
Let \( W \) be a Brownian motion.
Assume \( f : \mathbb{R}^2 \to \mathbb{R} \) is \( C^2 \). Let \( U_t := f(W_t, t) \).
Then \( U \) solves:
\[dU_t = \left( \partial_1 f(W_t, t) \right)[dW_t] + \left( \frac{\partial_2^2 f(W_t, t)}{2} \right)[dt] \]
\[+ \left( \partial_2 f(W_t, t) \right)[dt], \]
\[U_0 = f(0, 0)\]

“Pf”:
\[\left[ f(W_t + dW_t, t + dt) \right] - \left[ f(W_t, t) \right] = \]
\[\left( \partial_1 f(W_t, t) \right)[dW_t] + \left( \partial_2 f(W_t, t) \right)[dt] \]
\[+ \left( \frac{\partial_2^2 f(W_t, t)}{2} \right)[dt/2]\]
“QED”
Let $W_t$ be a Brownian motion.
Let $V_t$ be the solution to the system.
Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is $C^2$. Let $U_t := f(W_t, t)$.
Then $U_t$ solves: $dU_t = \left[ (\partial_1 f)(W_t, t) [dW_t] + \left( \frac{\partial^2 f}{\partial_1^2} \right)(W_t, t) [dt] \right] + \left[ (\partial_2 f)(W_t, t) [dt] \right]$
Assume $U_t$ solves: $dU_t = \left[ (\partial_1 f)(W_t, t) [dW_t] + \left( \frac{\partial^2 f}{\partial_1^2} \right)(W_t, t) [dt] \right] + \left[ (\partial_2 f)(W_t, t) [dt] \right]$
\begin{align*}
U_0 &= f(0, 0) \\
U_0 &= f(0, 0)
\end{align*}
Let \( W \) be a Brownian motion.

Let \( V \) be the solution to \[ (dV_t)^2 = [\sigma(V_t, t)]^2 [dt] \]

\[ dV_t = [\sigma(V_t, t)][dW_t] + [\mu(V_t, t)][dt] \]

Assume \( f : \mathbb{R}^2 \to \mathbb{R} \) is \( C^2 \). Let \( U_t := f(V_t, t) \).

Then \( U \) solves:

\[ dU_t = [(\partial_1 f)(V_t, t)][dV_t] + [(\partial_1^2 f)(V_t, t))/2][(dV_t)^2] + [(\partial_2 f)(V_t, t)][dt], \]

\[ U_0 = f(V_0, 0) \]
Let $W_\bullet$ be a Brownian motion.

Let $V_\bullet$ be the solution to

$$dV_t = [\sigma(V_t, t)][dW_t] + [\mu(V_t, t)][dt].$$

Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is $C^2$. Let $U_t := f(V_t, t)$.

Then $U_\bullet$ solves:

$$dU_t = [(\partial_1 f)(V_t, t)][dV_t] + [(\partial_2^2 f)(V_t, t))/2][(dV_t)^2] + [\partial_2 f(V_t, t)][dt]$$

$$= [(\partial_1 f)(V_t, t)][\sigma(V_t, t)][dW_t] + [(\partial_1 f)(V_t, t)][\mu(V_t, t)][dt]$$

$$+ [(\partial_2^2 f)(V_t, t))/2][\sigma(V_t, t)]^2[dt] + [(\partial_2 f)(V_t, t)][dt]$$

$$U_0 = f(V_0, 0).$$
Third version of Itô’s Lemma:

Let $W_\bullet$ be a Brownian motion.

Let $V_\bullet$ be the solution to

$$dV_t = [\sigma(V_t, t)][dW_t] + [\mu(V_t, t)][dt]$$

Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is $C^2$. Let $U_t := f(V_t, t)$.

Then $U_\bullet$ solves:

$$dU_t U_t = [((\partial_1 f)(V_t, t))[\sigma(V_t, t)][dW_t] + [(\partial_1 f)(V_t, t)][\mu(V_t, t)][dt]$$

$$+ [(\partial_2 f)(V_t, t)][\sigma(V_t, t)][dW_t] + [(\partial_2 f)(V_t, t)][\mu(V_t, t)][dt]$$

$$+ [(\partial_2^2 f)(V_t, t))/2][\sigma(V_t, t)]^2[dt] + [(\partial_2 f)(V_t, t)][dt],$$

$$U_0 = f(V_0, 0).$$
Third version of Itô’s Lemma:

Let $W_\bullet$ be a Brownian motion.

Let $V_\bullet$ be the solution to

$$dV_t = [\sigma(V_t, t)][dW_t] + [\mu(V_t, t)][dt]$$

Assume $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is $C^2$. Let $U_t := f(V_t, t)$.

Then $U_\bullet$ solves:

$$dU_t = [(\partial_1 f)(V_t, t)][\sigma(V_t, t)][dW_t] + [(\partial_1 f)(V_t, t)][\mu(V_t, t)][dt] + [(\partial_2^2 f)(V_t, t))/2][\sigma(V_t, t)]^2[dt] + [(\partial_2 f)(V_t, t)][dt],$$

careful proof omitted

$$U_0 = f(V_0, 0)$$
Stochastic Differential Eq’ns (SDEs)

\[ dX_t = [(0.01)X_t] \, dW_t + [(0.05)X_t] \, dt, \]
\[ X_0 = 1 \]

Goal: Solve this SDE.

Problem: Find an expression \( X_t \) of \( t \) such that

\[ \frac{dX_t}{dt} = (0.05)X_t, \quad X_0 = 1. \]

Compute \( X_t \).

\[ Y_t := \ln X_t \quad \text{CHAIN RULE!} \]

\[ \frac{dY_t}{dt} = \frac{1}{X_t} \frac{dX_t}{dt} = \frac{1}{X_t} (0.05)X_t = 0.05 \]

\[ Y_t = (0.05)t \]

\[ X_t = e^{Y_t} = e^{(0.05)t} \]
Stochastic Differential Eq’ns (SDEs)

\[ dX_t = [(0.01)X_t]\,dW_t + [(0.05)X_t]\,dt, \]

\[ X_0 = 1 \]

Goal: Solve this SDE.

\[ Y_t := \ln X_t \]

\[ dY_t = \frac{1}{X_t}[dX_t] + \frac{1}{2!} \left[ - \left( \frac{1}{X_t} \right)^2 \right] [dX_t]^2 \]

\[ \ln' = \frac{1}{\bullet} \]

\[ \ln'' = \frac{-1}{(\bullet)^2} \]

\[ Y_t := \ln X_t \]

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Stochastic Differential Eq’ns (SDEs)

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\[ Y_t := \ln X_t \]

\[ dY_t = \frac{1}{X_t} [dX_t] + \frac{1}{2!} \left[ - \left( \frac{1}{X_t} \right)^2 \right] [dX_t]^2 \]

\[ = \frac{dX_t}{X_t} - \frac{1}{2} \left( \frac{dX_t}{X_t} \right)^2 \]

\[ \ln' = \frac{1}{\bullet} \]

\[ \ln'' = \frac{-1}{(\bullet)^2} \]
Stochastic Differential Eq’ns (SDEs)

\[ dX_t = [(0.01)X_t] dW_t + [(0.05)X_t] dt, \]
\[ X_0 = 1 \]

Goal: Solve this SDE.

\[ Y_t := \ln X_t \quad \text{LOG-ITÔ!} \]

\[
dY_t = \frac{dX_t}{X_t} - \frac{1}{2} \left( \frac{dX_t}{X_t} \right)^2
\]

\[
= \frac{dX_t}{X_t} - \frac{1}{2} \left( \frac{dX_t}{X_t} \right)^2
\]

\[ \ln' = \frac{1}{\bullet} \]
\[ \ln'' = \frac{-1}{(\bullet)^2} \]

Exercise: EXP-ITÔ
Stochastic Differential Eq’ns (SDEs)

\[ dX_t = [(0.01)X_t] \, dW_t + [(0.05)X_t] \, dt, \]

\[ X_0 = 1 \]

Goal: Solve this SDE.

\[ Y_t := \ln X_t \]

\[
\begin{align*}
    dY_t &= \frac{dX_t}{X_t} - \frac{1}{2} \left( \frac{dX_t}{X_t} \right)^2 \\
    &= (0.01) \, dW_t + (0.05) \, dt \\
    &\quad - \left[ 1/2 \right] [(0.01) \, dW_t + (0.05) \, dt]^2 \\
    &= (0.01) \, dW_t + (0.05) \, dt \\
    &\quad - \left[ 1/2 \right] [(0.01)^2 \, dt] \\
    Y_t &= (0.01)W_t + [(0.05) - (1/2)(0.01)^2]_t
\end{align*}
\]

\[ Y_0 = 0 \]
Stochastic Differential Eq’ns (SDEs)

\[ dX_t = [(0.01)X_t] \, dW_t + [(0.05)X_t] \, dt, \]
\[ X_0 = 1 \]

Goal: Solve this SDE.

\[ Y_t := \ln X_t \]
\[ Y_t = (0.01)W_t + [(0.05) - (1/2)(0.01)^2]t \]
\[ X_t = \exp((0.01)W_t + [(0.05) - (1/2)(0.01)^2]t) \]
\[ = e^{(0.01)W_t + [(0.05) - (1/2)(0.01)^2]t} \]

Discussion: Compute \( \mathbb{E}[X_5], \text{Var}[X_5] \).

Discussion: Compute \( \mathbb{E}[(X_5 - 7)_+] \)

\[ Y_t = (0.01)W_t + [(0.05) - (1/2)(0.01)^2]t \]
**Stochastic Differential Eq’ns (SDEs)**

**SKILL:** Solve \( dX_t = \sigma dW_t + \mu \, dt \) when \( \sigma \) and \( \mu \) are constant.

**Sol’n:** \( X_t = X_0 + \sigma W_t + \mu t \)

**SKILL:** Solve \( dX_t = \sigma X_t \, dW_t + \mu X_t \, dt \) when \( \sigma \) and \( \mu \) are constant.

**Sol’n:** \( Y_t := \ln X_t \) \text{ LOG-ITÔ!} \\
\[
dY_t = \frac{dX_t}{X_t} - \frac{1}{2} \left( \frac{dX_t}{X_t} \right)^2 \\
= (\sigma \, dW_t + \mu \, dt) - (\sigma \, dW_t + \mu \, dt)^2/2 \\
= \sigma \, dW_t + [\mu - (\sigma^2/2)] \, dt \\
Y_t = Y_0 + \sigma \, W_t + [\mu - (\sigma^2/2)] \, t \\
X_t = X_0 \, e^{\sigma \, W_t + [\mu - (\sigma^2/2)] \, t} \]
Ornstein-Uhlenbeck SDE

Suppose \( X_t \) solves
\[
dX_t = 0.4 \, dW_t - 0.05 \, X_t \, dt, \quad X_0 = 3.
\]
Can we compute \( X_6 \)?
Ornstein-Uhlenbeck SDE

Suppose \( X_t \) solves
\[
dX_t = 0.4 \, dW_t - 0.05 \, X_t \, dt, \quad X_0 = 3.
\]
Can we compute \( X_6 \)?

Let \( U_t := e^{0.05t} \, X_t \).

\[
dU_t = [d(e^{0.05t})][X_t] + [e^{0.05t}][dX_t] + [d(e^{0.05t})][dX_t]
\]
\[
= [d(e^{0.05t})][X_t] + [e^{0.05t}][0.4 \, dW_t - 0.05 \, X_t \, dt]
\]
\[
= 0.4 \, e^{0.05t} \, dW_t
\]
Suppose $X_t$ solves
\[ dX_t = 0.4 \, dW_t - 0.05 X_t \, dt, \quad X_0 = 3. \]
Can we compute $X_6$? $X_6 := e^{-0.3} U_6$

Let $U_t := e^{(0.05)t} X_t$.
\[ dU_t = 0.4 \, e^{(0.05)t} \, dW_t \]
\[ U_6 = e^{0.3} X_6 \]
\[ = 0.4 \, e^{(0.05)t} \, dW_t \]
Suppose $X_\bullet$ solves
\[ dX_t = 0.4 \, dW_t - 0.05 \, X_t \, dt, \quad X_0 = 3. \]
Can we compute $X_6$?
\[ X_6 := e^{-0.3} U_6 \]
\[ X_6 = 3e^{-0.3} + (0.4)e^{-0.3} \int_0^6 e^{(0.05)t} \, dW_t \]
Let $U_t := e^{(0.05)t} X_t$.
\[ dU_t = 0.4 \, e^{(0.05)t} \, dW_t \]
\[ U_6 - U_0 = \int_0^6 0.4 \, e^{(0.05)t} \, dW_t \]
\[ e^{-0.3} \times \left( U_6 = 3 + (0.4) \int_0^6 e^{(0.05)t} \, dW_t \right) \]
Ornstein-Uhlenbeck SDE

Suppose $X_\bullet$ solves

$$dX_t = 0.4 \, dW_t - 0.05 \, X_t \, dt, \quad X_0 = 3.$$ 

Can we compute $X_6$?

$$X_6 = 3e^{-0.3} + (0.4)e^{-0.3} \int_0^6 e(0.05)t \, dW_t$$

$$N \left( 0, \int_0^6 e(0.1)t \, dt \right)$$

$$10(e^{0.6} - 1)$$

Fact: $\int_0^T f(t) \, dW_t$ is normal, with mean $= 0$ and variance $= \int_0^T [f(t)]^2 \, dt$. 
Ornstein-Uhlenbeck SDE

Suppose $X_t$ solves

$$dX_t = 0.4 \, dW_t - 0.05 \, X_t \, dt, \quad X_0 = 3.$$

Can we compute $X_6$?

$$X_6 = 3e^{-0.3} + (0.4)e^{-0.3} \int_0^6 e^{(0.05)t} \, dW_t$$

$$N \left(0, \int_0^6 e^{(0.1)t} \, dt\right)$$

$$10(e^{0.6} - 1)$$

$$N(0, [(0.4)e^{-0.3}]^2[10(e^{0.6} - 1)])$$
Ornstein-Uhlenbeck SDE

Suppose \( X_t \) solves
\[
dX_t = 0.4\, dW_t - 0.05\, X_t\, dt, \quad X_0 = 3.\]
Can we compute \( X_6 \)?

\[
X_6 = 3e^{-0.3} + (0.4)e^{-0.3} \int_0^6 e^{(0.05)t} \, dW_t
\]

\[
N(0, [(0.4)e^{-0.3}]^2[10(e^{0.6} - 1)])
\]

\( X_6 \) is \( N(3e^{-0.3}, [(0.4)e^{-0.3}]^2[10(e^{0.6} - 1)]) \)

\[
N(0, [(0.4)e^{-0.3}]^2[10(e^{0.6} - 1)])
\]
Suppose $X_t$ solves
$$dX_t = 0.4 \, dW_t - 0.05 \, X_t \, dt, \quad X_0 = 3.$$ Can we compute $X_6$?

$$X_6 = 3e^{-0.3} + (0.4)e^{-0.3} \int_0^6 e^{(0.05)t} \, dW_t$$

$$N(0, [(0.4)e^{-0.3}]^2[10(e^{0.6} - 1)])$$

$X_6$ is $N(3e^{-0.3}, [(0.4)e^{-0.3}]^2[10(e^{0.6} - 1)])$