Pricing and hedging in incomplete markets

Chapter 10
From Chapter 9:

- Pricing Rules: Market complete + nonarbitrage $\implies$ Asset prices

- The idea is based on perfect hedge:

$$H = V_0 + \int_0^T \phi_t dS_t + \int_0^T \phi_0^t dS_t^0$$

- With completeness, any contingent claim can be perfectly hedged.

- With nonarbitrage, $V_0$ could pin down.
Also From Chapter 9:

▶ Market completeness breaks down when there are even small jumps

▶ So without perfect hedges, the risk to do hedging can’t be completely ruled out, we have to find ways out.
In this chapter:

- Merton’s approach (10.1): ignore the extra risks $\rightarrow$ pin down pricing and hedging

- Superhedging (10.2): leads to a bound for prices (preference-free, but the bound is too wide)

- Expected utility max (10.3): choosing hedge by min some measure of hedging errors $\rightarrow$ utility indifference price

- Special case of the above where the loss function is quadratic (10.4)
Merton’s Approach:

- In Merton:
  \[ S_t = S_0 \exp \left[ \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i \right] \]
  
  \( W_t \): SBM; \( N_t \): Poisson process with \( \lambda \); \( Y_i \sim N(m, \delta^2) \)

- He assigns a choice from many risk-neutral measures:
  \[ Q_M : S_t = S_0 \exp \left[ \mu^M t + \sigma W^M_t + \sum_{i=1}^{N_t} Y_i \right] \]
Merton’s Approach:

- $Q_M$ just shift the drift of the BM, and left the jumps unchanged

- Rationale: jump risks are diversifiable, so no risk premium/no change of measure upon it.

- Application: Euro option with $H(S_T)$ has price process:

$$\Pi^M_t = e^{-r(T-t)} E^{Q_M}[H(S_T)|\mathcal{F}_t]$$
Merton’s Approach:

- Furthermore, since $S_t$ is a Markov process (under $\mathbb{Q}_M$), so $\mathcal{F}_t$ contains as much info as $S_t$, thus:
  \[
  \Pi^M_t = e^{-r(T-t)} E^{\mathbb{Q}_M} [(S_T - K)^+ | S_t = S]
  \]

- Then by conditioning on the $\#$ of jumps $N_t$, we can express $\Pi^M_t$ as a weighted sum of B-S prices, finally, we get (set $\tau = T - t$):
  \[
  \Pi(\tau, S; \sigma) = e^{-r\tau} E[H(Se^{(r-\sigma^2/2)\tau + \sigma W_\tau})]
  \]
Merton’s Approach:

▶ For call and put options, apply Ito to

\[ e^{-rt} C(t, S_t). \]

\[ \hat{\Pi}_t^M = e^{-rt} \Pi_t^M = E^{Q_M}[e^{-rT}(S_T - K)^+|\mathcal{F}_t] \]

▶ the discounted value is a martingale under \( Q_M \), so

\[ \hat{\Pi}_T^M - \hat{\Pi}_0^M = \hat{H}(S_T) - E^{Q_M}[H(S_T)] \]

▶ Merton gives the hedging portfolio \((\phi_t^0, \phi_t)\):

\[ \phi_t = \frac{\partial \Pi_t^M}{\partial S}(t, S_t) \] and \[ \phi_t^0 = \phi_t S_t - \int_0^t \phi dS \]
Merton’s Approach:

- From this self-financing strategy, the risk from the diffusion part is hedged, but the discounted hedging error is:

\[
\hat{H} - e^{-rT} V_T(\phi) = \hat{\Pi}^M_T - \hat{\Pi}^M_0 - \int_0^t \frac{\partial \Pi^M}{\partial S}(u, S_u^-) d\hat{S}_u
\]

- Go back to Merton’s rational, how could we hedge jump risk: he assumes the jumps across the stocks are independent, so in a large market a diversified portfolios such as market index would not have jumps, ‘coz they cancel out each other.
Superhedging:

- A conservative approach to hedge:
  \[ \mathbb{P}(V_T(\phi) = V_0 + \int_0^t \phi dS \geq H) = 1 \]
  Here \( \phi \) is said to superhedge against the claim \( H \).

- Defn: The cost of superhedging: the cheapest superhedging strategy,
  \[ \Pi^{sup}(H) = \inf \{ V_0, \exists \phi \in S, \mathbb{P}(V_0 + \int_0^T \phi dS \geq H) = 1 \} \]
Superhedging:

- Intuition: When some option writer/seller is willing to take the risk at some certain price, it means he can at least partially hedge the option with a cheaper cost, thus the this price represents an upper bound for the option.

- Similarly, the cost of superhedging a short position in \( H \), given by \(-\Pi^{sup}(-H)\) gives a lower bound on the price.

- Henceforth, we pin down an interval:

\[
[-\Pi^{sup}(-H), \Pi^{sup}(H)]
\]
Superhedging:

Prop 10.1 Cost of superhedging:

Consider a European option with a positive payoff $H$ on an underlying asset described by a semimartingale $(S_t)_{t \in [0,T]}$ and assume that

$$\sup_{Q \in \mathcal{M}(S)} E^Q[H] < \infty$$

Then the following duality relation holds:

$$\inf_{\phi \in \mathcal{S}} \{ \hat{V}_t(\phi), \mathbb{P}(V_T(\phi) \geq H) = 1 \} = \text{esssup} E^Q[\hat{H}|\mathcal{F}_t]$$
Superhedging:

Prop 10.1 Cost of superhedging (con’d):

- In particular, the cost of the cheapest superhedging strategy for $H$ is given by

$$\Pi^{\text{sup}}(H) = \text{esssup}_{Q \in M_a(S)} E^Q[\hat{H}]$$

where $M_a(S)$ is the set of martingale measure absolutely continuous wrt to $\mathbb{P}$
Superhedging:

Prop 10.1 Cost of superhedging (comments):

- preference-free method: no subjective risk aversion parameter nor ad hoc choice of a martingale measure

- in terms of equivalent martingale measures, superhedging cost corresponds to the value of the option under the least favorable martingale measure
Superhedging:

Application of Prop 10.1: Superhedging in exponential-Levy processes: Prop10.2

So we have $S_t = S_0 \exp X_t$ where $(X_t)$ is a Levy process, if $X$ has infinite variation, no Brownian component, negative jumps of arbitrary size and Levy measure $\nu$: $\int_0^1 \nu(dy) = +\infty$ and $\int_{-1}^0 \nu(dy) = +\infty$ then the range of prices is:

$$\left[ \inf_{Q \in M(S)} E^Q[(S_T - K)^+] , \sup_{Q \in M(S)} E^Q[(S_T - K)^+] \right]$$
Superhedging:

Application of Prop 10.1: Superhedging in exponential-Levy processes: Prop10.2

If \( X \) is a jump-diffusion process with diffusion coefficient \( \sigma \) and compound Poisson jumps then the price range for a call option is:

\[
\left[ C^{BS}(0, S_0; T, K; \sigma), S_0 \right]
\]
Superhedging: Comments

- From the above, the superhedging cost is too high. Consider \( S_t = S_0 \exp(\sigma W_t + aN_t) \), apply prop10.1, we find that the superhedging cost is given by \( S_0 \), so however small the jump is, the cheapest superhedging strategy for a call option is a complete hedge.
Utility Maximization

- “As if” method: the agent is picking some strategy to max utility level:

\[
\max_Z E^P[U(Z)]
\]

usually, \( U : \mathbb{R} \rightarrow \mathbb{R} \) is concave, increasing, and \( P \) could be seen either as a prob distribution objectively or subjectively describe future events.

- The concavity of \( U \) is related to risk aversion of the agent. say \( U(x) = \ln(x) \), \( U(x) = \frac{x^{1-\alpha}}{1-\alpha} \).
Utility Maximization: Certainty equivalent

Another way to measure risk aversion: $c(x, H)$

$U(x + c(x, H)) = E[U(x + H)] \implies c(x, H) = U^{-1}(E[U(x + H)]) - x$

Intuition: at the same level $x$, faced with the same $H$, the higher compensation you require, the more risk averse you are

Notice: $c$ is not linear in $H$, $c$ depends on $x$
Utility Maximization: Utility indifference price

- The agent wants to max his final wealth:

\[ V_T = x + \int_0^T \phi_t dS_t: \]

\[ u(x, 0) = \sup_{\phi \in S} \mathbb{E}^\mathbb{P}[U(x + \int_0^T \phi_t dS_t)] \]

- Suppose now it buys an option, with terminal payoff \( H \), at price \( p \), then

\[ u(x - p, H) = \sup_{\phi \in S} \mathbb{E}^\mathbb{P}[U(x - p + H + \int_0^T \phi_t dS_t)] \]
The utility indifference price is defined as price $\pi_U(x, H)$:

$$u(x, 0) = u(x - \pi_U(x, H), H)$$

Notice:
1. $\pi_U$ is not linear in $H$
2. $\pi_U$ depends on initial wealth, except for special utility like: $U(x) = 1 - e^{-\alpha x}$
3. To same $U$, same $x$, same $H$, buying and selling derives different price:

$$u(x, 0) = u(x + p, -H)$$
Utility Maximization: More comments

► The “As if” method: from vNM, Savage

► Hard to identify U and $\mathbb{P}$, and there is homogeneity among agents

► Attack to nonlinearity: remedies–quadratic hedging(where the utility is: $U(x) = -x^2$)
Utility Maximization: Quadratic hedging

- As if the agent is choosing so to min the hedging error in a mean square sense.

- Different criterion to be min in a least squares sense can be:
  1. hedging error at maturity $\Rightarrow$ “Mean-variance hedging”;
  2. hedging error measure locally in time $\Rightarrow$ local risk min.

- The two approaches are equivalent if the discounted price is a martingale measure.
Going Further: “Optimal” martingale measures

- By fund theorem, choosing an arbitrage-free pricing is choosing a martingale measure \( Q \sim \mathbb{P} \)

- More general, we’re choosing prob measures according to:

\[
J_f(Q) = E^\mathbb{P} \left[ f\left( \frac{dQ}{d\mathbb{P}} \right) \right]
\]

where \( f : [0, \infty) \rightarrow \mathbb{R} \) is str convex, \( J_f \) a measure of deviation from the prior \( \mathbb{P} \)
Going Further: “Optimal” martingale measures

- Some example: relative entropy:

\[ H(Q, P) = E^P \left[ \frac{dQ}{dP} \ln \frac{dQ}{dP} \right] \]

- quadratic distance:

\[ E \left[ \left( \frac{dQ}{dP} \right)^2 \right] \]
Going Further: “Optimal” martingale measures

- More on relative entropy: here $f = x \ln x$

$$H(Q, P) = E^P \left[ \frac{dQ}{dP} \ln \frac{dQ}{dP} \right] = E^Q \left[ \ln \frac{dQ}{dP} \right]$$

- So given $(S_t)$ the minimal entropy martingale model is defined as a martingale $(S^*_t)$ such that the $Q^*$ of $S^*$ minimizes the relative entropy wrt $P$ among all martingale process:

$$\inf_{Q \in \mathcal{M}^a(S)} H(Q, P)$$
Going Further: “Optimal” martingale measures

- Interpretation for min entropy martingale model: minimizing relative entropy corresponds to choosing a martingale measure by adding the least amount of info to the prior model.

- Existence: ? But for exp-Levy, nice result (analytic computable) in Prop10.7