Risk-neutral modelling with exponential Levy processes

For a martingale measure $Q$

$$e^{-rt}S_t = \hat{S}_t = E^Q[\hat{S}_T|\mathcal{F}_t]$$

An option with terminal payoff $H_T$ has a value at time $t$ given by the discounted expectation

$$\Pi_t(H_T) = e^{-r(T-t)} E^Q[H_T|\mathcal{F}_t]$$

Generalize geometric Brownian motion with exponential Levy process

$$S_t = S_0 \exp((r - \sigma^2/2)t + \sigma W_t) \rightarrow S_t = S_0 \exp(rt + X_t)$$

with the restrictions on exponential moments

$$\int_{|x|\geq 1} e^x \nu(dx) < \infty$$

and the martingale condition

$$\gamma + \sigma^2/2 + \int (e^x - 1 - x1_{|x|\leq 1}) \nu(dx) = \psi(-i) = 0$$

European Call options form a basis for other European payoffs

$$C_t(t, S_t; T, K) = e^{-r(T-t)} E^Q[(S_T - K)^+|\mathcal{F}_t] = Se^{-rt} \int_k^{\infty} dx \rho_T(x) (e^{r\tau + x} - e^k)$$

where

$$\tau = T - t$$

and $k = \ln(K/S)$

Implied volatility can be computed for each strike and maturity as an ”in-
consistency check” for Black-Scholes. $C_{BS}(T, K, \sigma^{imp}(T, K)) = C^{market}(T, K)$

Geometric Brownian motion alone constricts $\sigma^{imp}(T, K)$ to a 2-d flat plane. Here are a few cross sections of $\sigma^{imp}(T, K)$ vs. $K/S$ (moneyness) for a given $T$ for a few different commodity futures markets.

**Implied Vol vs. Moneyness CZ2006 2/8/2006**

- $\text{vol} = 0.260$ skew $= 0.678$ kurt $= 0.799$

\[ \text{vol} = 0.319 \quad \text{skew} = -0.47 \quad \text{kurt} = 0.957 \]

\[ \text{vol} = 0.608 \quad \text{skew} = 0.094 \quad \text{kurt} = 0.771 \]
General statements about implied volatility surfaces of exponential Levy processes

- Addition of jumps allows an explanation of implied volatility surfaces seen in financial market. Skew (slope) and Smile (curvature) of these surfaces can be fit to model parameters with good precision.
- IV surfaces are constant with time to expiration (floating smile) and
independent of $S_t$ (sticky delta or moneyness). The second feature is opposed sticky strike (generally stochastic volatility models. cf Chap 15) which have a correlation between $S_t$ and $\sigma^{imp}(T - t, K)$

- Short term skew is well represented by the jumps of levy processes
- Flattening of the skew with option maturity. This occurs in accord with the central limit theorem $c_3 \propto 1/\sqrt{T}$ and $c_4 \propto 1/T$ as shown in Chap 3. However Additive processes of Chap 14 can modify this moment decay.

So how do we compute $C_T(k) = Se^{-rt} \int_k^\infty dx \rho_T(x)(e^{r\tau + x} - e^k)$?

Given the characteristic function $\Phi_T(\nu)$ we can use an assortment of Fourier Transform techniques that all have a similar smell to them.

1. Method of Scott, Chen, Heston, Bates, et. al.: Calculate delta ($\Pi_1$) and probability option expires in the money ($\Pi_2$) with Fourier variable $k = \ln(K)$

$$C(k) = S\Pi_1 - e^{-rt} K \Pi_2$$

$$\Pi_1 = \frac{1}{2} + \frac{1}{2\pi} \int d\nu e^{-ik\nu} e^{i\nu rT} \frac{\Phi_T(\nu - i)}{i\nu \Phi_T(-i)}$$

$$\Pi_2 = \frac{1}{2} + \frac{1}{2\pi} \int d\nu e^{-ik\nu} e^{i\nu rT} \frac{\Phi_T(\nu)}{i\nu}$$

Although delta (not necessarily hedge ratio) is computed for free, convergence is slow and FFT difficult

2. Method A of Carr and Madan: Since $C(k)$ is not integrable, subtract time value = $(e^{s+r\tau} - e^k)^+$

$$C(k) = (e^{s+r\tau} - e^k)^+ + \frac{1}{2\pi} \int d\nu e^{-ik\nu} e^{i\nu rT} \frac{\Phi_T(\nu - i) - 1}{i\nu(1 + i\nu)}$$
For better convergence, one can replace time value with a Black-Scholes option value \( C^*_BS(k) \) to yield a smooth function:

\[
C(k) = C^*_BS + \frac{1}{2\pi} \int d\nu e^{-i\nu k} e^{i\nu r\tau} \frac{\Phi_\tau(\nu - i) - \Phi^*_BS(\nu - i)}{i\nu(1 + i\nu)}
\]

Still converges slowly but FFT can be utilized.

3. Method B of Carr and Madan: Dampened Call price \( e^{\alpha k}C(k) \) is integrable with \( \alpha > 0 \)

\[
\Xi(\nu) = \int d\nu e^{i\nu k} e^{\alpha k} C(k) = \frac{e^{-r\tau\Phi_\tau(\nu - i(\alpha + 1))}}{\alpha^2 + \alpha - \nu^2 + i(2\alpha + 1)\nu}
\]

\[
C(k) = \frac{e^{-\alpha k}}{2\pi} \int d\nu e^{-i\nu k} \Xi(\nu)
\]

Converges much quicker, but there might be additional conditions on \( \alpha \) to insure only imaginary roots of \( \Xi(\nu) \) and thus integrability along \( \text{Re}(\nu) \)

4. Method of Lewis using Generalized FT: \( Fg(z) = \int e^{izx} g(x) dx \) with \( z \) complex and \( x = \ln(S_\tau/S_0) \) as Fourier variable.

\[
FC(z) = \frac{\Phi_\tau(-z)e^{(1+iz)(k-r\tau)}}{iz(i z + 1)}
\]

\[
1 < Iz = \mu < 1 + \alpha
\]

\[
C(x) = \frac{e^{\mu x + (1-\mu)(k-r\tau)}}{2\pi} \int_{-\infty}^{\infty} du \frac{e^{iu(k-r\tau-x)}\Phi_\tau(-i\mu - u)}{(iu - \mu)(1 + iu - \mu)}
\]

Similar convergence and \( \alpha \) choice issues as (3)

From limited experience, (3) seems to work quite well except when calibration takes model parameters into integrand into delinquent territory.