Pricing in Incomplete Markets:

Relating actuarial and financial paradigms.

Abstract:

Many schemes have been devised to value securities and insurance contracts. We relate the underlying financial economic and actuarial principles to reconcile these apparently ad hoc incomplete market pricing techniques.

Multiperiod model of securities markets [Pliska]

- $T+1$ trading dates : $t = 0, 1, \ldots, T$
- finite sample space $\Omega = \{\omega_1, \omega_2, \ldots, \omega_K\}$
- a probability measure P on Ω with $P(\omega) > 0 \ \forall \omega \in \Omega$
- a filtration $\mathbb{F} = \{\mathcal{F}_t \mid t = 0, 1, \ldots T\}$ describing how information about security prices is revealed to investors (a sequence of finer partitions)
- a bank account process $B = \{B_t \mid t = 0, 1, \ldots T\}$

  $B_0 = 1$ and $B_t(\omega) > 0$ non-decreasing

  interest rate $r_t \triangleq \dfrac{(B_t - B_{t-1})}{B_{t-1}}$
- N risky security processes $S_n = \{S_n(t) \geq 0 \mid t = 0, 1, \ldots T\}$

  $n = 1, \ldots, N$

<u>Defn</u> A <u>trading strategy</u> $\overrightarrow{H} = (H_0, H_1, \ldots, H_N)$ is a vector of stochastic processes (denoting the holdings in each of the securities at each point in time and in each state of the world) which is predictable wrt $\mathbb{F}$
ie $H_n(t)$ is measurable wrt $\mathcal{F}_{t-1}$, $t = 1, 2, \ldots T$

So investors can base their trading position only on currently available information and nothing more. There is no prescience.

<u>value process</u>

$$V_t = \begin{cases} H_0(1) B_0 + \sum_{n=1}^{N} H_n(1) S_n(0) & t = 0 \\ H_0(t) B_t + \sum_{n=1}^{N} H_n(t) S_n(t) & t \geq 1 \end{cases}$$

<u>Defn</u> A trading strategy H is stb <u>self-financing</u> if

$$V_t = H_0(t+1)B_t + \sum_{n=1}^{N} H_n(t+1)S_n(t) \qquad t=1,\dots,T-1$$

LHS = time t value of the portfolio just <u>before</u> any transactions

RHS = time t value of the portfolio just <u>after</u> any time t transactions ie just before the portfolio is carried forward to time $t+1$

So no money is added or withdrawn from the portfolio.

<u>Defn</u> discounted price process $S_n^*(t) \triangleq \dfrac{S_n(t)}{B_t} \qquad t=0,\dots T \quad n=1,\dots N$

<u>Defn</u> An <u>arbitrage opportunity</u> is a trading strategy H st

i) $V_0 = 0$

ii) $V_T \geqslant 0$

iii) $E[V_T] > 0$

iv) H is self-financing

<u>Defn</u> A <u>risk neutral measure</u> (or martingale measure) is a probability measure Q st

i) $Q(\omega) > 0 \quad \forall \omega \in \Omega$

ii) S_n^* is a martingale under Q

ie $E_Q[S_n^*(t+s) \mid \mathcal{F}_t] = S_n^*(t)$, $t, s \geqslant 0$

Let $\mathcal{M} = \{ Q \mid Q \text{ is a martingale measure} \}$

__Prop__ There are no arbitrage opportunities iff there exists a martingale meas Q.

__Defn__ A __contingent claim__ is a random variable X that represents the time T payoff from a 'seller' to a 'buyer'.

__Defn__ A contingent claim is __attainable__ if there exists a self-financing trading strategy st $V_T(\omega) = X(\omega)\ \forall \omega \in \Omega$.

__Prop__ (Risk neutral valuation)
If there are no arbitrage opportunities then for an attainable X, $V_t^* = \frac{V_t}{B_t} = E_Q\left[\frac{X}{B_T} \mid \mathcal{F}_t\right]$ $t=0,..T$

For __all__ martingale measures Q

Assume our multiperiod securities market has no arbitrage opportunities ie $\mathbb{M} \neq \emptyset$

__Defn__ A market is stb __complete__ if every contingent claim is attainable otherwise it's stb __incomplete__.

__Prop__ The multiperiod model is complete iff if every underlying single period model is complete.

__Prop__ The multiperiod model is complete iff $\mathbb{M}$ is a singleton.

__Prop__ contingent claim X is attainable iff

$E_Q\left[\frac{X}{B_T}\right]$ takes the same value for every $Q \in \mathbb{M}$

Constraints on asset prices (Duffie) $\begin{cases} \text{absence of arbitrage} \\ \text{single agent optimality} \\ \text{market equilibrium} \end{cases}$

Dfn Let $S \subset \mathbb{R}$, a function $u: S \to \mathbb{R}$ is called a utility function if it is strictly concave, strictly increasing and continuous on S.

Consider the single period model again in which $d+1$ assets are priced at time 0 and time 1.

Prices at time 0 $\quad \pi = (\pi_0, \pi_1, \ldots \pi_d)$

" ______ " 1 $\quad S = (S_0, S_1, \ldots S_d)$

At time 0, the investor chooses a portfolio $\xi = (\xi_0, \ldots \xi_d)$ denoting the shares of each asset.

Prop [Follmer & Schied] Consider a risk averse agent with utility function $u: \mathbb{R} \to \mathbb{R}$ bounded from above with amount w to invest.

$\max_{\xi} E[u(\xi \cdot S)]$ subject to $\pi \cdot \xi \leq w$ has a solution

iff the market is arbitrage-free.

Cor Suppose the market is arbitrage-free with maximizer ξ^*

Then $\dfrac{dP^*}{dP} = \dfrac{u'(\xi^* \cdot Y)}{E[u'(\xi^* \cdot Y)]}$ defines a risk-neutral measure

where $Y_i = \dfrac{S_i}{1+r} - \pi_i$, $i = 1, \ldots d$, the discounted net gain.

Note that this is a specific choice of an equiv mart meas in terms of marginal utility.

<u>Example</u> exponential utility $u(x) = 1 - e^{-\alpha x}$

turns out P^* is an Esscher transform of P which minimizes relative entropy wrt P.

Tying things together [Le Roy & Werner]

<u>State Prices</u> (Single period valuation)

$\mathcal{S} = \{1, 2, \ldots S\}$ finite set of states

$\mathbb{R}^{\mathcal{S}}$ = set of payoffs at end of period

J securities with payoffs $\underset{J \times S}{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_J \end{bmatrix}$ and prices $p = \begin{bmatrix} p_1 \\ \vdots \\ p_J \end{bmatrix}$

asset span $\mathcal{M} = \{ z \in \mathbb{R}^{\mathcal{S}} \mid z = hX \text{ for some } h \in \mathbb{R}^J \}$

Given security prices p at date 0, define the <u>payoff pricing functional</u>

$$q : \mathcal{M} \longrightarrow \mathcal{P}(\mathbb{R})$$

$$z \longmapsto q(z) = \{ w \in \mathbb{R} \mid w = ph \text{ for some } h \text{ st } z = hX \}$$

payoff $\longmapsto$ { prices of portfolios that generate that payoff }

If the law of one price holds then q is single-valued.

$$\begin{tikzcd} \mathcal{M} \arrow[r, hook] \arrow[d, "q"'] & \mathbb{R}^S \arrow[dl, dotted, "Q"] \\ \mathbb{R} & \end{tikzcd}$$

Security prices exclude arbitrage iff $\exists$ strictly positive Q extending q

let e_s denote the payoff with 1 in the sth state and 0's elsewhere and $q_s = Q(e_s)$ the <u>state price</u> of state s.

$$\pi_s^* = \frac{q_s}{\Sigma q_s} = \text{risk-neutral prob}$$

$$r = \frac{1}{\Sigma q_s} = \text{risk-free return}$$

<u>Prop</u> $Q(z) = \frac{1}{r} E^*[z] \quad \forall z \in \mathbb{R}^S$

'Valuation using risk-neutral probabilities and discounting at the risk-free rate.'

Now $\mathbb{R}^S$ is a Hilbert space under the expectations inner product $x \cdot y = E[xy] = \sum_s \pi_s x_s y_s$ for a prob meas π on S

By the Riesz-Frechet thm $\exists!\ k_q \in \mathcal{M}$ (the <u>pricing kernel</u>) st

$$q(z) = k_q \cdot z = E_\pi[k_q z] \quad \forall z \in \mathcal{M}$$

Once again, if there is no arbitrage $\exists$ strictly positive state price vector $q = (q_1, \ldots, q_S)$

st $q(z) = \sum_s q_s z_s = E_\pi\left[\frac{q}{\pi} z\right]$

state price deflator

'Valuation using real-world probabilities and deflators.'

<u>NB</u> i) the pricing kernel k_q exists and is unique (under no arbitrage) regardless of whether the market is complete or not.

ii) $r k_q = \dfrac{\pi^*}{\pi}$ Radon-Nikodym deriv

'Deflators as a change of measure.'

Equilibrium pricing - single period [Panjer]

Consider a generic agent who consumes c_0 at time 0 and $c_1(\omega)$ at time 1 in state ω. Let security j have price x_j at time 0 and payoff $X_j(\omega)$ at time 1 in state ω. Assume agents have homogeneous beliefs and agent i has utility $u_{i0}(c_{i0}) + u_{i1}(c_{i1}(\omega))$ where u_{i0} and u_{i1} are increasing concave and twice differentiable. Assume agents make decisions to maximize their individual expected utilities $u_{i0}(c_{i0}) + \sum_\omega \pi(\omega)\, u_{i1}(c_{i1}(\omega))$ where π is the physical measure (nature's).

In equilibrium, prices x_j and consumption allocations (to produce c_1) are such that each agent's expected utility is maximized ie there is no incentive for any agent to trade at these prices. (A Nash equilibrium) Denote the optimal consumption process in equilibrium by $\{c_{i0}^*, c_{i1}^*(\omega)\}$. Now consider security j with current price x_j. If any agent is offered a choice of buying any amount α of this security at time 0, the optimal choice is $\alpha = 0$.

Suppose an agent purchases α units of x_j at time 0. Then this agents expected utility becomes

$$u_{i0}(c_{i0}^* - \alpha x_j) + \sum_\omega \pi(\omega)\, u_{i1}^*(c_{i1}^*(\omega) + \alpha X_j(\omega))$$

First order conditions imply

$$x_j = \sum_\omega \pi(\omega) \frac{u_{i1}'(c_{i1}^*(\omega))}{u_{i0}'(c_{i0}^*)} X_j(\omega)$$

$$= E_\pi[Z_i X_j]$$

a pricing kernel!

So in the context of a single period model of state prices we see how the absence of arbitrage, single agent optimality, and market equilibrium are related. Of course there are more general elaborations. The point is in this simple setting we are given clues for pricing in incomplete markets!

eg
- generalize the notion of trading strategy
  - drop self-financing
  - $V_T(\omega)$ 'close' to $X(\omega)$
  - optimize mean-variance of V_t
  - minimize 'risk' of V_t

    ⋮
- define a criteria to 'choose' an EMM Q

- use a consumption-based model (Cochrane)

$$P_t = E[m_{t+1} x_{t+1}] \quad \text{asset price}$$

$$m_{t+1} = f(\text{data, parameters}) \quad \text{stochastic discount factor}$$

x_{t+1} asset payoff

Three themes emerge risk $\begin{cases} \text{preference} \\ \text{aversion} \\ \text{premia} \end{cases}$

'Fair' deal bounds

Let $\mathcal{P}$ be a set of random variables, the payoffs and $\pi : \mathcal{P} \longrightarrow \mathbb{R}$ a function giving a price and $A \subset \mathcal{P}$ acceptable payoffs.

The buy price for payoff $X \in \mathcal{P}$ is

$$b(X) \triangleq \sup_{Y \in \mathcal{P}} \{-\pi(Y) \mid Y + X \in A\}$$

The sell price for payoff $X \in \mathcal{P}$ is

$$s(X) \triangleq \inf_{Y \in \mathcal{P}} \{\pi(Y) \mid Y - X \in A\} = -b(-X)$$

ie to sell X or to buy $-X$ is considered equivalent.

The interpretation of $-b(X)$ as the cost of rendering X acceptable gives rise to a correspondence between coherent or convex risk measures $-b$ and acceptance sets A.

Moreover instead of using acceptance sets we could have used a preference relation $\succcurlyeq$ between payoffs.

$$Y - X \in A \quad \text{iff} \quad Y \succcurlyeq X$$

Aside:

- Appropriate formulation of $S(X)$ leads to certainty equivalent of X.
- Appropriate formulation of $b(X)$ leads to indifference price of X.

(See Follmer & Schied)

Premium principles

Consider an insurance contract over a time period $[0,T]$
Let $(\Omega, \mathcal{F}, \pi)$ be a probability space
An insurance contract is described by a non negative bounded random variable $X : \Omega \to \mathbb{R}$ where $X(\omega)$ represents the payoff at time T if state ω occurs.

Let L be a set of non-negative r.v. st

$$aX \ , \ (X-a)_+ \ , \ (X-(X-a)_+) \in L \quad \forall X \in L$$
$$a \in [0,\infty)$$

Just as we considered constraints like absence of arbitrage, single agent optimality and market equilibrium on asset prices, actuaries consider premium principles which impose structure on insurance pricing.

Insurance prices of the contracts of L are a functional $H : L \Longrightarrow \mathbb{R}$

P1) $H(X) \geq 0 \quad \forall X \in L$

P2) If $c \in [0,\infty)$ then $H(c) = c$

(When there is no uncertainty, there is no safety loading)

P3) $H(X) \leq \sup_{\omega \in \Omega} X(\omega) \quad \forall X \in L$

P4) $H(aX+b) = a\,H(X)+b \quad \forall X \in L$ st $aX+b \in L$, $a,b \in [0,\infty)$

P5) $H(X) = \underbrace{H(X-(X-a)_+)}_{\text{premium}} + \underbrace{H((X-a_+))}_{\text{reinsurance premium}} \quad \forall X \in L$, $a \in [0,\infty)$

P6) If $X(\omega) \leq Y(\omega) \ \forall \omega \in \Omega$ then $H(X) \leq H(Y)$

P7) If $E(X-a)_+ \leq E(Y-a)_+ \quad \forall a \in [0, \infty)$

then $H(X) \leq H(Y)$

(ie H preserves stop-loss order)

P8) $H(X+Y) \leq H(X) + H(Y) \quad \forall X, Y \in L$ st $X+Y \in L$

(Diversification)

P9) $H(aX + (1-a)Y) \leq aH(X) + (1-a)H(Y)$

$\forall X, Y \in L$ and $a \in [0,1]$ st $aX + (1-a)Y \in L$

P10) The price $H(X)$ of the insurance contract X depends only its distribution F_X

P11) $\lim_{n \to +\infty} H(X - (X-n)_+) = H(X)$

<u>defn</u> A function $\nu : 2^{\Omega} \longrightarrow \mathbb{R}^+$ is called a <u>capacity</u> if

i) $\nu(\emptyset) = 0, \quad \nu(\Omega) = 1$

ii) If $A, B \in 2^{\Omega}$ and $A \subseteq B$ then $\nu(A) \leq \nu(B)$

A capacity is stb <u>convex</u> if $\nu(A) + \nu(B) \leq \nu(A \cup B) + \nu(A \cap B)$

<u>defn</u> Let $\nu : 2^{\Omega} \to \mathbb{R}^+$ be a capacity and X a r.v. on $(\Omega, \mathcal{F})$ then the <u>Choquet integral of X wrt ν</u> is

$$\int_{\Omega} X \, d\nu \triangleq \int_0^{+\infty} \nu(\{\omega \mid X(\omega) > x\}) \, dx$$

Modified Greco thm

If $H : L \to \mathbb{R}$ satisfies (P1), (P2), (P5), (P6), (P8) and (P11) then $\exists$ convex capacity $\nu : 2^{\Omega} \to \mathbb{R}$ st

$$\forall X \in L \quad H(X) = \int_{\Omega} X \, d\nu = \int_{0}^{+\infty} \nu\{\omega \mid X(\omega) > x\} \, dx$$

Let X be a non-neg r.v. and f increasing with $f(0) = 0$ and $f(1) = 1$ then we can define an insurance premium functional

$$H(X) = \int_{0}^{+\infty} (1 - f(F_X(t))) \, dt$$

$$= \int_{0}^{+\infty} g(S_X(t)) \, dt = \int_{\Omega} X \, d\nu$$

with $g(x) = 1 - f(1-x)$ and $\nu = f \circ \Pi$

Such H are called <u>distortion</u> operators [Wang et al]

Examples of premium principles

Net Premium $\quad H[X] = EX$

Expected Value Premium $\quad H[X] = (1+\theta) EX, \quad \theta > 0$

Variance Premium $\quad H[X] = EX + \alpha \operatorname{Var} X, \quad \alpha > 0$

Standard Deviation Premium $\quad H[X] = EX + \beta\sqrt{\text{Var}X}\ , \ \beta > 0$

Exponential Premium $\quad H[X] = \frac{1}{\alpha}\ln E[\,e^{\alpha X}]\ , \ \alpha > 0$

Esscher Premium
$$H[X] = \frac{E[X e^{Z}]}{E[e^{Z}]}$$
for some r.v. Z
eg $Z = \alpha X$, $\alpha > 0$

Proportional Hazards Premium
$$H[X] = \int_0^\infty [S_X(t)]^c\,dt \qquad 0 < c < 1$$
where $S_X(t) = P\{\omega \in \Omega \mid X(\omega) > t\}$

Equivalent Utility $\quad H[X]$ solves the equation
$$u(w) = E[u(w - X + H)]$$
where u is an increasing, concave utility of wealth of the insurer and w is the initial wealth of the insurer.

Wang's Premium
$$H[X] = \int_0^\infty g(S_X(t))\,dt$$
where g is increasing, concave : $[0,1] \to [0,1]$

Swiss Premium $\quad H$ solves $E[u(X-\rho H)] = u((1-\rho)H)$

for some $\rho \in [0,1]$ and some u incr, convex

Dutch Premium

$$H[X] = EX + \theta E[(X-\alpha EX)_+]$$

$$\alpha \geqslant 1 \quad 0 < \theta \leq 1$$

Premium principles vs risk measures [Jarrow]

Artzner, Delbaen, Eber & Heath [ADEH] consider risk measures over a time period $[0,T]$
Let the states of nature be Ω, $|\Omega| = n$
and $P: 2^\Omega \to \mathbb{R}$ the physical probability measure
$P(\omega) > 0 \ \forall \omega \in \Omega$. Random variable $X: \Omega \to \mathbb{R}$
represent possible risks at time T and let
$G = \{ X: \Omega \to \mathbb{R} \text{ r.v.} \}$ be the set of all risks.

$L_+ = \{ X \in G \mid X(\omega) \geqslant 0 \ \forall \omega \in \Omega \}$

$L_- = -L_+$

$L_{--} = \{ X \in G \mid X(\omega) < 0 \ \forall \omega \in \Omega \}$

Let r denote the dollar return over $[0,T]$
(one plus a percent) to a riskless asset

Note that $r \in G$

A <u>risk measure</u> is a mapping $\rho : \mathcal{G} \to \mathbb{R}$

[ADEH] say a risk measure is <u>coherent</u> if it satisfies:

Axiom T (Translation Invariance) $\forall X \in \mathcal{G}$, $\forall \alpha \in \mathbb{R}$

$$\rho(X + \alpha r) = \rho(X) - \alpha$$

Axiom S (Subadditivity) $\forall X, Y \in \mathcal{G}$

$$\rho(X + Y) \leq \rho(X) + \rho(Y)$$

Axiom PH (Positive Homogeneity) $\forall X \in \mathcal{G}$, $\lambda \geq 0$

$$\rho(\lambda X) = \lambda \rho(X)$$

Axiom M (Monotonicity) $\forall X \leq Y$

$$\rho(Y) \leq \rho(X).$$

Jarrow introduces:

Axiom BR (Bounded Relevance)

$\forall X \in \mathcal{G}$ with $X \leq 0$, $X \neq 0$ and $\{\omega \mid X(\omega) = 0\} \neq \emptyset$, $\rho(X) > 0$

Axiom TM (Translation Monotonicity)

$\forall \alpha > 0$ and $\forall X \in \mathcal{G}$ with $X \leq 0$ and $X \neq 0$,

$$\rho(X + \alpha r) < \rho(X) < \rho(X - \alpha r)$$

Given a risk measure ρ define its <u>acceptance set</u>

as $A_\rho = \{X \in G \mid \rho(X) \leq 0\}$

[ADEH] argue all reasonable risk measures have acceptance sets that satisfy:

Ax 1. $A \supseteq L_+$

Ax 2. $A \cap L_{--} = \emptyset$

Ax 3. A is convex

$(a, b \in A \Rightarrow \lambda a + (1-\lambda) b \in A, \ 0 < \lambda < 1)$

Ax 4. A is a positive homogeneous cone

$(a \in A \Rightarrow \lambda a \in A, \ \lambda \geq 0)$

[ADEH] show every coherent risk measure's acceptance set is closed and satisfies Axioms 1–4.

Given an acceptance set $A \subset G$, define a risk measure

$$\rho_A(X) = \inf\{m \in \mathbb{R} \mid mr + X \in A\}$$

Jarrow introduces Axiom 2^* $A \cap L_- = \{0\}$

[ADEH] show $Ax.2^* \Rightarrow Ax.2$

Thm (Jarrow)

If a risk measure satisfies axioms TM, BR, S, PH and M then the acceptance set generated by the risk measure is closed and satisfies axioms 1, 2^*, 3 and 4.

So we see that insurance premium principles H and risk measures ρ can share properties. Moreover under the circumstances above [ADEH] would consider abiding insurance premium principles as reasonable risk measures.

Cramér - Lundberg model

$$X(N_t) = \sum_{k=1}^{N_t} X_k \qquad 0 \leq t \leq T \quad \text{where}$$

(X_k) are iid claims with common distribution fn F

(N_t) a homogeneous Poisson process with intensity $\lambda > 0$

so that $N_t = \sup\{n \in \mathbb{N} \mid T_1 + \dots + T_n \leq t\}$

where (T_k) are iid with $\mathrm{Exp}(\lambda)$ distn

r.v. T_k denotes the occurence time of the kth claim X_k

$(X_k) \perp\!\!\!\perp (T_k)$

So $X(N_t)$ is a compound Poisson process

$$P(X(N_t) \le x) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} F^{k*}(x), \quad x \ge 0$$

kth convolution of F, $F^{k*}(x) = P(X_1 + \cdots + X_k \le x)$

Consider the set of prob meas Q equivalent to P that preserve the compound structure of X ie X is a compound Poisson process <u>under Q</u>

This set has been characterized by Delbaen & Haezendonck [DH] and is parameterized by a pair (k, ν) where $k \ge 0$ and $\nu : \mathbb{R}_+ \to \mathbb{R}$ is a non-neg measurable fn with $E_P[\nu(X_1)] = 1$

The density process

$$\xi_t = E_P[\xi_T \mid \mathcal{F}_t] \qquad 0 \le t \le T$$

of the Radon-Nikodym deriv $\xi_T = \frac{dQ}{dP}$

is given by
$$\xi_t = \exp\left(\sum_{j=1}^{N_t} \ln(k\nu(X_j)) + \lambda(1-k)t\right)$$

Denote the equiv meas Q corresponding to (k, ν) by $P^{k,\nu}$

We can interpret k as the market price of frequency risk and ν as the market price of claim size risk.

Suppose that insurance company liabilities are of the form

$$L_t = X(N_t) + P_t \quad \text{for } 0 \le t \le T \text{ where}$$

$X(N_t)$ = accumulated claims up to time t

P_t = the premium paid by the insurer to a reinsurer to cover the remaining risk $X(N_T) - X(N_t)$ over $(t, T]$

Assume the reinsurance market for the take-over of insurance policies is arbitrage-free ie $\exists$ an equiv martingale measure Q st L_t is a Q-martingale. If one further assumes P_t under Q is linear ie $P_t = p(Q)(T-t)$ then [DH] show $X(N_t)$ remains a compound Poisson process under Q, so $\exists\,(k,\nu)$ st $Q = P^{k,\nu}$

Now L_t a Q-martingale implies

$$p(Q) = E_Q[X(N_1)] = E_Q[N_1]\,E_Q[X_1]$$
$$= \lambda k\, E_P[X_1\, \nu(X_1)]$$

Appropriate choices of (k,ν) lead to expected value, variance, and Esscher premium principles!

So arbitrage-free reinsurance markets lead to some premium principles.