# Arbitrage Pricing, BS, Skews, Smiles, and Tying it Together

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6 April 2006

We have priced contingent claims using risk-neutral probability (martingales, change of measure, etc.).

We may also use $arbitrage$ arguments. $Arbitrage$ is simply (risk-free) free money. And an *arbitrage argument* says that there should be no (risk-free) free money.

How do we 'use arbitrage' to price a claim? We try to replicate the claim with securities (stocks and bonds).

A contingent claim, $f$, is $replicable$ if we can construct a portfolio Π such that

- The values of Π and $f$ are the same under every circumstance.

- Π is self financing. As time goes on, we only shift money around within the portfolio, we don't put anymore in (or take any out).

We will call Π the replicating portfolio (of $f$).

Why does arbitrage work? Let's do an example with gold.

Suppose the price of gold today is \$200 and the risk-free interest rate is 3%.

You don't want gold today (because it's out of fashion), but you do want gold in 6 months (when, of course, it will be all the rage).

You therefore buy a *forward contract*.

How much should you pay for this wonderful opportunity?

Suppose the forward contract costs \$250. You should then go to the bank, and borrow \$200. Use this money to buy some gold right now. Then *short* (sell) the forward (to a sucker).

In six months, what happens?

- You sell your gold for \$250
- You pay back your loan with your newly received funds
- You are left with \$250-\$200$e^{.5(.03)}$=\$46.97

Which is a lot of free money.

What if the forward contract, $F_0$, is selling for less than $\$200e^{.5(.03)}$? Well, you have to be able to sell an ounce of gold today. (We'll later assume we can always short things).

Assuming you have gold lying around, you'll (because you know the trick) sell your gold today and get \$200. Next, you put this \$200 in the bank. Finally, you go *long* (buy) the forward contract.

So what happens at the end of 6 months?

- Take your money, $\$200e^{.5(.03)}$ out of the bank.

- Use it to buy your gold back for $\$F_0$.

You have your gold back, and $\$(200e^{.5(.03)}$- $F_0)$. Since this number is positive, you are very happy.

Arbitrage therefore sets the price of the forward contract to be \$200$e^{.5(.03)}$.

This is true of any forward contract on an asset with no storage costs and which does not pay dividends.

Even more generally, any replicable claim will have the same price as its replicating portfolio.

Forward contracts are simple(!) to price. This is due in large part to the linearity of the payoffs at maturity.

Options are not so easy. The payoff at maturity has a kink. This nonlinearity is a bugger.

So what do we do about it?

- Model stocks using Black-Scholes (which is equivalent to the binary tree model in a certain sense).

- Use this model to come up with a formula for what the price of an option should be.

- Then, as is par for the course, we'll complain about the shortcomings of the Black-Scholes model.

The Black-Scholes Model assumes that a stock $S$ follows the process

$$\frac{dS}{S} = \mu dt + \sigma dW. \tag{1}$$

Where $W$ is a Brownian Motion.

The assumptions in the Black-Scholes (BS) model (for us...so far) are

- The stock price follows (1).
- There are no transaction costs (!) or taxes (!!).
- Short selling of securities is allowed, with no restrictions on use of proceeds (!!!).
- The stock does not pay a dividend.
- The market is complete, and there are no arbitrage opportunities.
- Trading is continuous (!).
- The risk-free interest rate is fixed.

There really aren't generally accepted superior models to the BS model (yet). It's just that:

- There are transaction costs to consider and (gasp) taxes.
- There are restrictions on use of proceeds for short sales.
- There are nonreplicable contingent claims. We can never get rid of all the risk.
- Trading is not continuous.

We would like to find a price for a contingent claim. For a derivative $f$ with underlying $S$ (following (1))

$$df = \left(\frac{\partial f}{\partial S}\mu + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2\right) dt + \frac{\partial f}{\partial S}\sigma dW \quad (2)$$

We want to construct a portfolio $\Pi$ that hedges away the risk.

We set $\Pi$ to have

$$\begin{aligned} -1 &: \quad \text{derivative} \\ \Delta &: \quad \text{shares} \end{aligned}$$

where $\Delta = \frac{\partial f}{\partial S}$.

We get that for a small change in time, $\delta t$, the corresponding change in $\Pi$ is given

$$\delta \Pi = -\delta f + \Delta \delta S$$

From the discrete versions of (1) and (2), we get

$$\delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2\right)\delta t. \tag{3}$$

But this implies the change in the portfolio is riskless (no uncertainty), and so by arbitrage, we must have

$$\begin{aligned}
\delta\Pi &= r\Pi\delta t \\
\left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2\right)\delta t &= r(-f + \Delta S)\delta t \\
\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 + r\Delta S\right)\delta t &= rf\delta t \\
\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 + r\Delta S &= rf \qquad (4)
\end{aligned}$$

The pde in (4) is the Black-Scholes-Merton differential equation.

By using only (1) and arbitrage, we must have that

- Any function $f$ that satisfies (4) is the price of some theoretical contingent claim.

- Every contingent claim must satisfy (4).

When

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 + r\frac{\partial f}{\partial S}S = rf$$

is solved with boundary conditions depicting a European call option with strike $K$,

$$f(S,T) = \max(S-K,0),$$

we get the Black-Scholes price of the option.

The BS price of a European call, $c$, (on a stock with no dividend) is

$$c = c(K, r, S_t, t, T, \sigma) = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2) \quad (5)$$

$$d_1 = \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \quad (6)$$

$\Phi$ is the cumulative distribution function of standard normal random variable ($N(0,1)$)

Here are a few properties of the BS price of $c$ (a benchmark test, really)

- We would expect that if $S_t$ is very large, $c$ should be priced like a forward contract (why?).

$$c \approx S_t - Ke^{-r(T-t)}$$

- When $\sigma$ is extremely small, we would expect that the payoff would be

$$c \approx \max(S_t e^{r(T-t)} - K, 0) \qquad (7)$$

(why?).

We also have

- $c$ is an increasing function of $\sigma$.

- $\frac{\partial c}{\partial S} = N(d_1)$.

From the last point, we can estimate the $\Delta$ to use in the replicating portfolio of $c$.

Prices are not set by the BS options price. Rather, markets set prices.

We may therefore go to the market to observe $K$, $r$, $S_0$, $T$. We can't observe $\sigma$.

We solve for $\sigma$ using (5). The number we get is called the implied volatility.

If we check market data for different strike prices, $K$, with all else being equal, we get different implied volatilities.

In fact we get what is called a volatility smile, or a volatility skew depending on the shape.

We have assumed that $\sigma$ is some intrinsic property of the underlying. It shouldn't vary with $K$.

The following data are from the website

http://www.fenews.comfen37/teach_notes/

teaching_notes.htm

Below are the prices for (European) call and put options on the QQQ (a NASDAQ 100 composite) for January 9, 2004. Expiration dates are January 16, and February 20.

| | *Calls* | | *Puts* | |
|---|---|---|---|---|
| *Strike* | *January* | *February* | *January* | *February* |
| 34 | 3.9 | 4.1 | 0.05 | 0.25 |
| 35 | 2.8 | 3.2 | 0.05 | 0.35 |
| 36 | 1.85 | 2.35 | 0.1 | 0.55 |
| 37 | 1 | 1.65 | 0.25 | 0.85 |
| 38 | 0.35 | 1.05 | 0.6 | 1.25 |
| 39 | 0.1 | 0.6 | 1.4 | 1.9 |
| 40 | 0.05 | 0.35 | 2.35 | 2.6 |

As we have seen, BS depends on $(K, r, S_t, t, T, q, \sigma)$, and the only unobservable quantity is $\sigma$. In the present case, for the February options, the data give

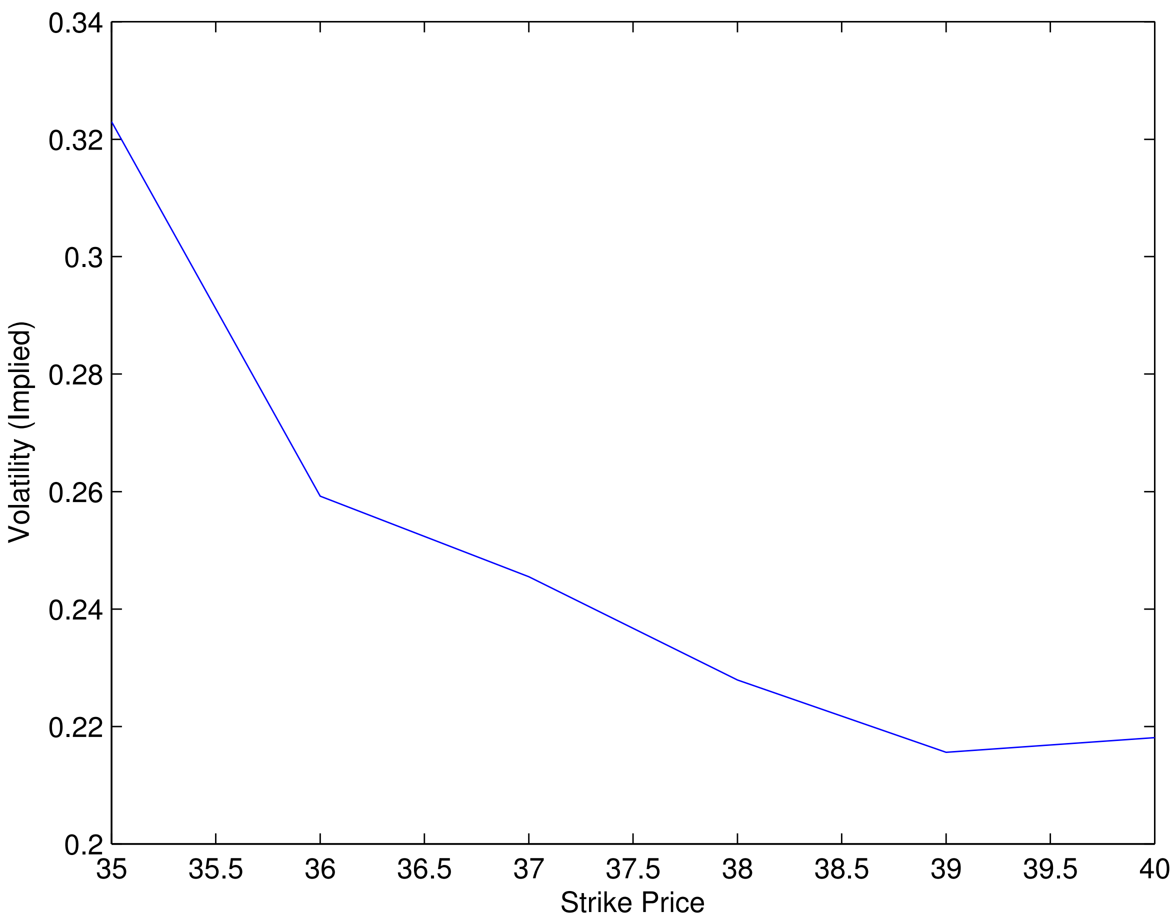
$$\begin{aligned} S_0 &= 37.73 \text{ (the price at closing Jan. 9, 2004)} \\ T - t &= 42/365 = .1151 \\ r &= .83 \\ q &= .18 \end{aligned}$$

Which gives

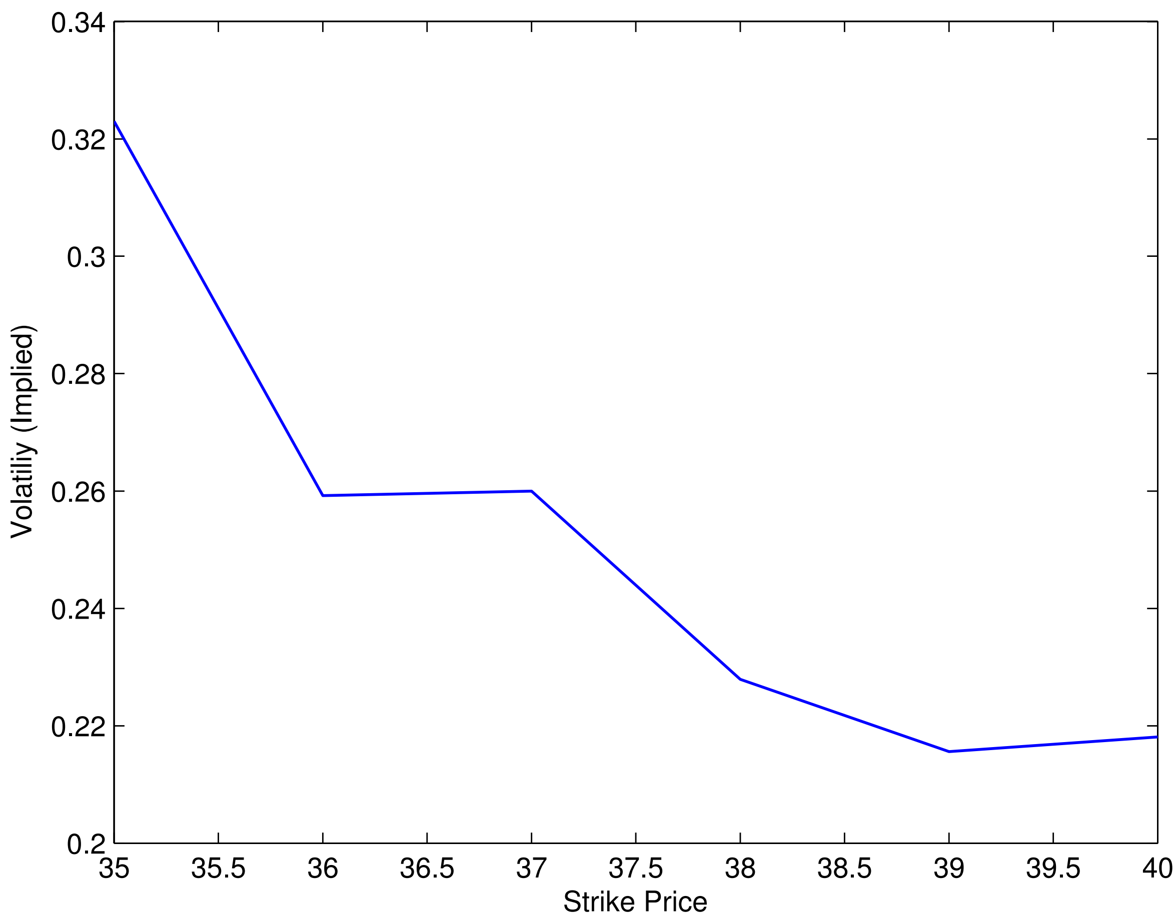
Implied Volatility

| February Call | February Put |
|---|---|
| 0.323 | 0.29 |
| 0.2592 | 0.2493 |
| 0.2455 | 0.2369 |
| 0.2279 | 0.2198 |
| 0.2156 | 0.2279 |
| 0.2181 | 0.2206 |

Graphically, plotting strike prices on the $x$-axis and implied volatility on the $y$-axis, we have:

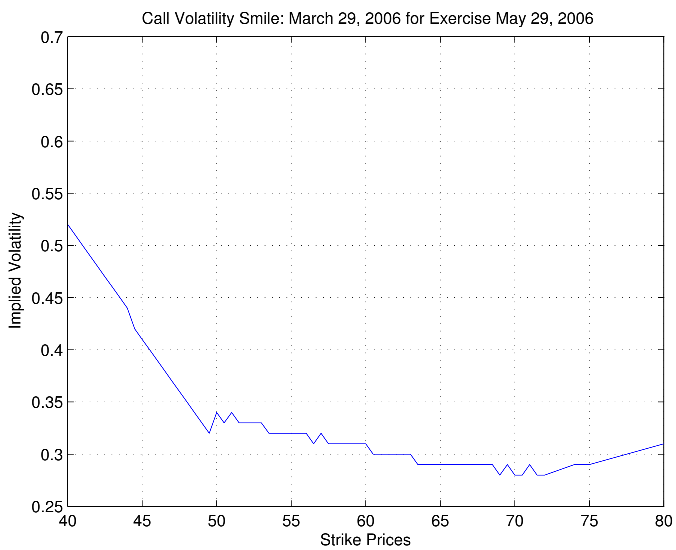


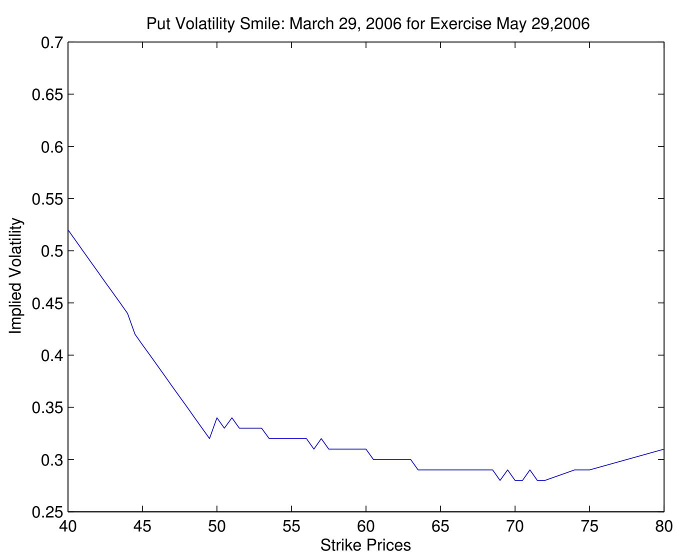
Suppose the volatility smile we observe looked more like:




We would likely think that the market was overpricing the call for one of the strike prices (which one?), and take a position.

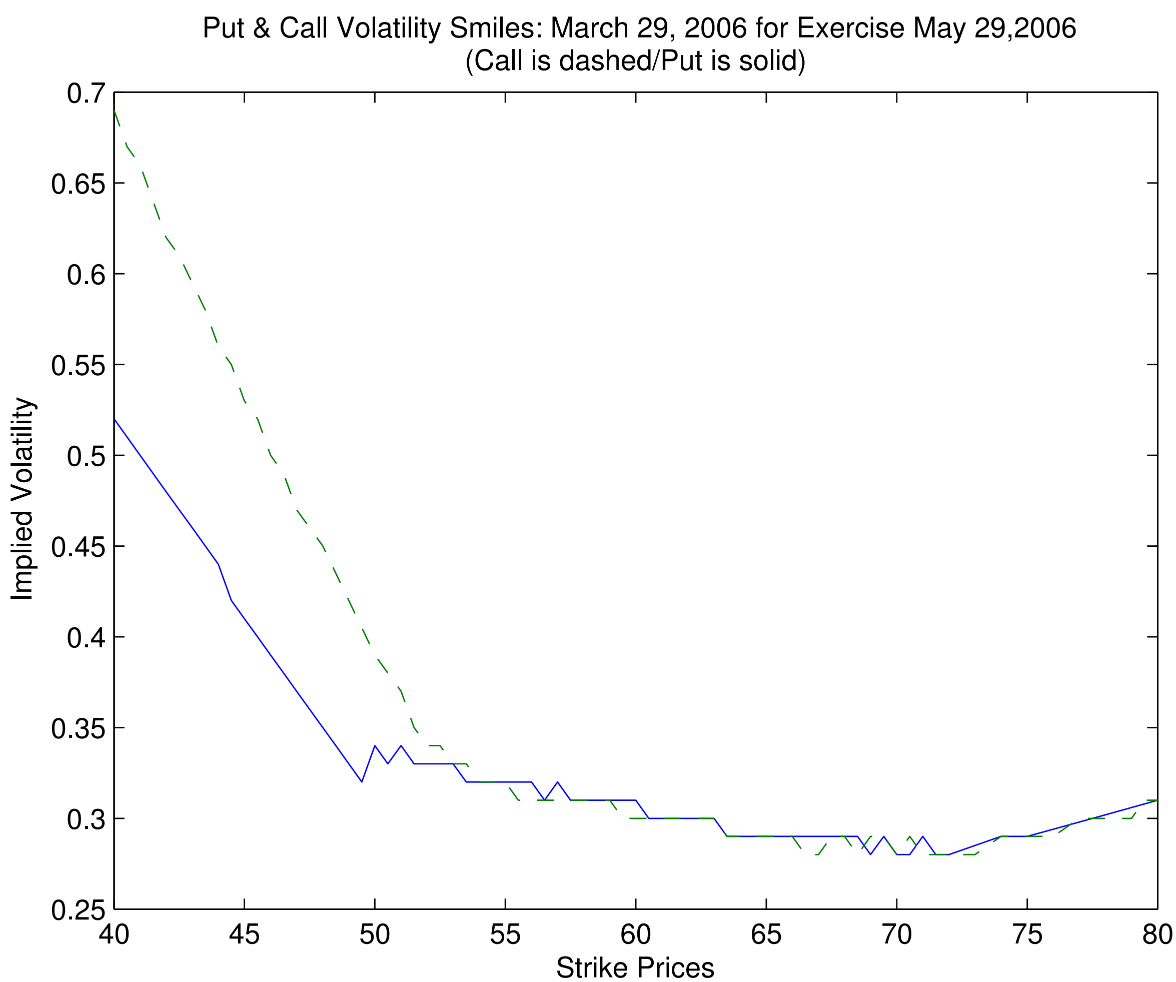
Volatility smiles also occur with commodities. Below are examples of smiles for both calls and puts for crude oil.



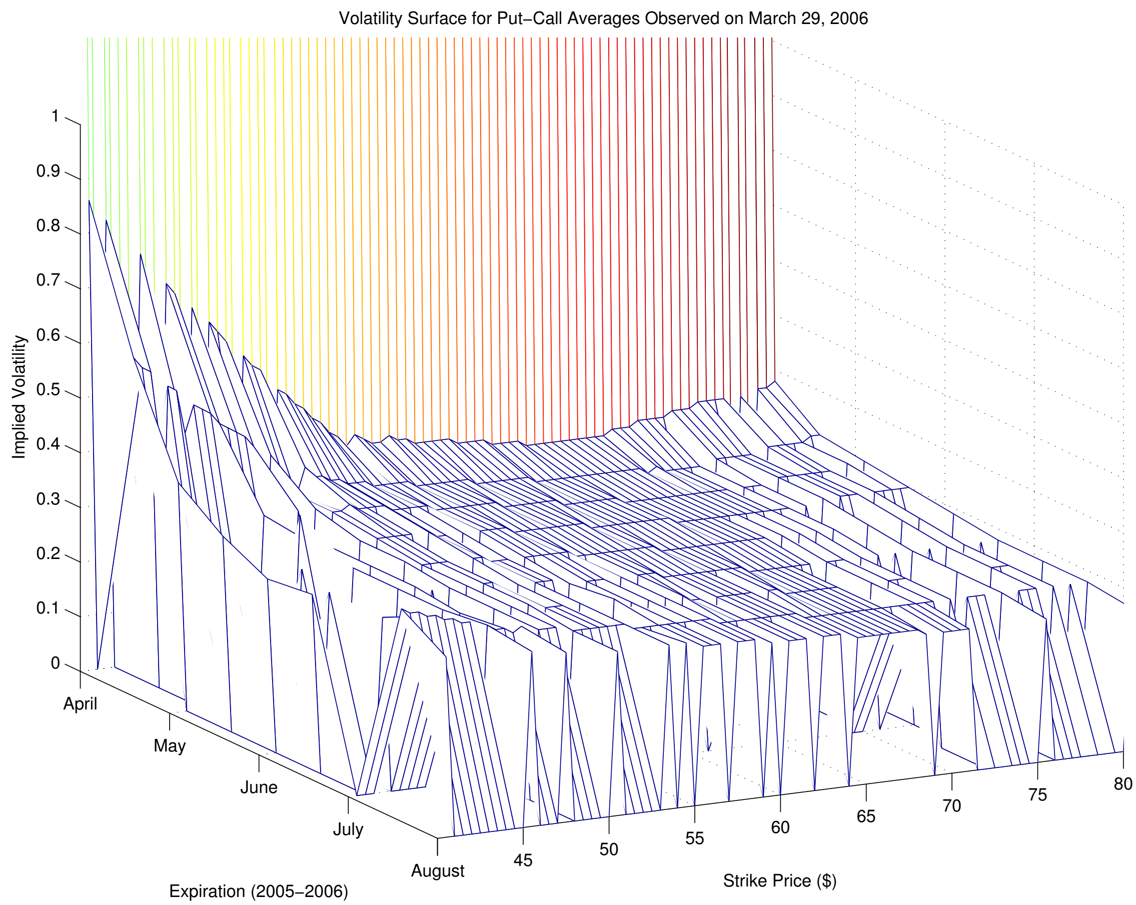



So $\sigma$ not only varies with the strike price, but also depends on whether we are pricing a call or a put.

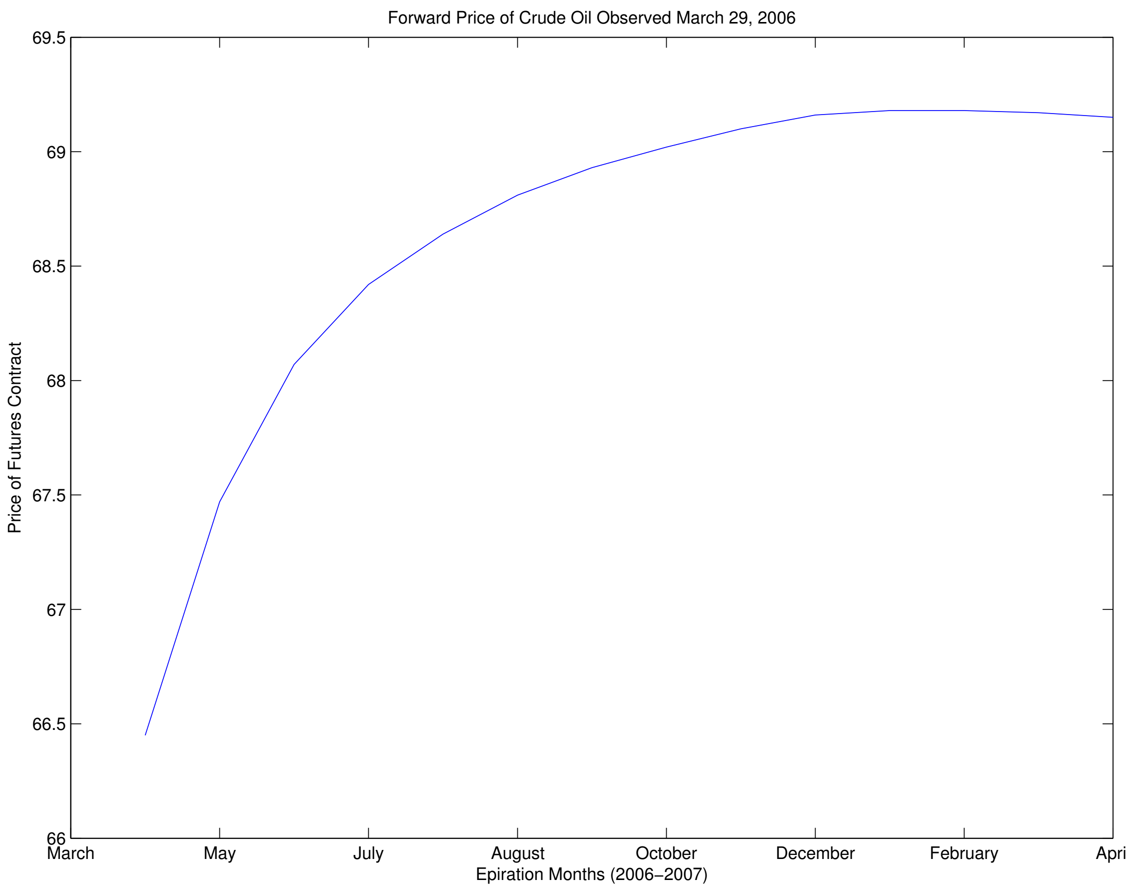
Below are the volatility smiles of the call and put above in one plot.



As a final kicker, implied volatility varies with the expiration of the option. We may therefore plot a volatility $surface$.



Forwards are not generally as easy as the gold example made it seem either. Below is the forward curve for crude oil.



Remember risk-neutral measures? We may obtain the *implied* risk-neutral distribution from a volatility smile.

If $g$ is the risk-neutral pdf of $S_T$, we have for fixed interest rate, $r$, that

$$c = e^{-r(T-t)} \int_{S_T=K}^{\infty} (S_T - K) g(S_T) dS_T$$

Taking partials with respect to $K$ twice, we get

$$\frac{\partial^2 c}{\partial K^2} = e^{-r(T-t)} g(K)$$

So that our risk-neutral probability may be written

$$g(K) = e^{r(T-t)} \frac{\partial^2 c}{\partial K^2}$$

We may approximate the risk-neutral probability density function of $S_T$ by

$$g(K) \approx e^{r(T-t)} \frac{c_1 + c_3 - 2c_2}{\delta^2}$$

where $c_1$, $c_2$, $c_3$ are the observed call prices for strikes $K - \delta$, $K$, and $K + \delta$ respectively.

This is theoretically nice, but usually market data is not sufficiently rich to obtain good approximations.

In the end, BS is used to show that BS is, well...lacking. We could enrich the model. Some prime suggestions are

- Assume volatility is *level-dependent*. That is, let $dS = \mu(S_t, t)dt + \sigma(S_t, t)dW$.

- Assume volatility is stochastic.

- Assume volatility is $uncertain$, and optimize accordingly.

- Model $dr = (\theta - \alpha r_t)dt + \sigma d\tilde{W}$.