A Derivation of the Black-Scholes-Merton PDE

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1 Introduction

To derive the Black-Scholes-Merton (BSM) PDE, we require a model for a security $S = S_t$ and a bond (which we consider a riskless asset) $B = B_t$. We will assume

$$\frac{dS}{S_t} = \mu dt + \sigma_t dW. \tag{1}$$

Here W is a Brownian motion, and σ_t is a deterministic function of time. When σ_t is constant, (1) is the original Black-Scholes model of the movement of a security, S. In this formulation, μ is the mean return of S, and σ is the variance. We note in passing that σ is no longer seen as the historical volatility of an underlying in real market applications. The bond will satisfy $B_0 = 1$, and

$$dB = r_t B_t dt, \tag{2}$$

where r_t is the risk-free interest rate available at time t. For convenience, we also define the discount process belonging to B by $\beta_t = 1/B_t$. β allows us to discount future values of money to present day values.

The BSM PDE is a partial differential equation which any contingent claim, f with underlying S following (1) must satisfy. The derivation uses only Ito's formula and the idea of self financing. While this is a standard result, the two references that were primarily used (based on how close they were on the bookshelf) were [1] and [2].

2 Derivation

For a given contingent claim, $f = f(S_t, t)$, with expiration T, we denote the value at time t of the claim by $F_t = f(S_t, t)$. We want to construct a portfolio, Π, consisting of securities and bonds with the property that $\Pi_t = F_t$ at all times $t \leq T$. That is, we want to simulate the claim with other known quantities. In this vein, we will define an $\mathbb{R}^2$ valued process $\theta_t = (\theta_t^0, \theta_t^1)$, where we will hold θ_t^0 units of the riskless bond and θ_t^1 units of the underlying. Before determining θ, we note that if Π does indeed replicate the value of the claim, we *require* that the portfolio is self financing via arbitrage arguments. Recall that a portfolio, $\pi = \gamma_t B_t + \alpha_t S_t$, is self financing exactly when $d\pi = \gamma_t dB + \alpha_t dS$. It is the requirement of self financing, coupled with insight from Ito's formula that yields the BSM PDE.

By Ito, we have that

$$dF = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} \mu S_t + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S_t^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dW. \tag{3}$$

Since the portfolio we are constructing will have $\Pi_t = F_t$, we will have $d\Pi = dF$. The self financing requirement yields

$$\begin{aligned} d\Pi &= \theta_t^0 dB + \theta_t^1 dS \\ &= (\theta_t^0 r_t B_t + \theta_t^1 \mu S) dt + \theta_t^1 \sigma S dW. \end{aligned} \tag{4}$$

Comparing the stochastic differential equations in (3) and (4), we must have

$$\theta_t^1 = \frac{\partial}{\partial S} f(S_t, t). \tag{5}$$

For simplicity of exposition (and to use relatively standard notation), we will define Δ_t as $\frac{\partial}{\partial S} f(S_t, t)$.

We have now that $\Pi = \theta_t^0 B_t + \Delta_t S_t$. We find θ_t^0 by solving

$$F_t = \theta_t^0 B_t + \Delta_t S_t,$$

which yields immediately that

$$\theta_t^0 = \beta_t (F_t - \Delta_t S_t). \tag{6}$$

This completes the construction of Π.

We are now in a position to compare the deterministic parts of (3) and (4). This yields that

$$\begin{aligned} \theta_t^0 r_t B_t + \theta_t^1 \mu S_t &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} \mu S_t + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S_t^2 \\ (F_t - \Delta_t S_t) r_t + \Delta_t \mu S_t &= \frac{\partial f}{\partial t} + \Delta_t \mu S_t + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S_t^2. \end{aligned}$$

Reverting back to all original notations and putting all terms on one side gives the BSM PDE:

$$\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - r_t f + r_t \frac{\partial f}{\partial S} S = 0. \tag{7}$$

We have proved that in the Black-Scholes world where securities are modelled by (1), every contingent claim based on such a security must satisfy (7).

The value of a particular claim is therefore determined by its boundary conditions, $f(S_T, T)$. For example, a European call with strike price K has final payoff $\max(S_T - K, 0)$. We could use this condition together with (7) to determine the Black-Scholes price of the call.

3 Delta Hedged

From the portfolio we have just constructed, we have already noted that

$$dF = \Delta_t dS + \theta_t^0 dB,$$

and hence

$$dF - \Delta_t dS = (F_t - \Delta_t S_t) r_t dt. \tag{8}$$

A portfolio, π_Δ consisting of one option and $-\Delta_t$ units of the underlying is said to be delta hedged. This portfolio is not self financing, however, so we cannot say (mathematically) that $d(F_t - \Delta_t S_t) = dF - \Delta_t dS$. However if Δ_t is sufficiently well behaved, we may assume it is constant over a small (enough) time interval. In this case, equation (8) demonstrates that π_Δ grows at the risk free rate.

The portfolio π_Δ is often times used in deriving the BSM PDE. In fact, Black and Scholes used this portfolio in their original derivation. While the derivation using this portfolio is slightly more straightforward, it is not mathematically sufficient.

References

[1] Buff, R. (2002): Uncertain Volatility Models-Theory and Application. Springer-Verlag, Berlin.

[2] Hull, J. (2003): Options, Futures and Other Derivatives-Fifth Edition. Prentice Hall, New Jersey.