B11

## Clarifications

$X, Y$ SRVs

$\boxed{X \sim Y} \iff X, Y$ are $\boxed{\text{identically distributed}}$

$\iff$

$\forall z \in \mathbb{R},$

$\Pr[X = z] = \Pr[Y = z]$

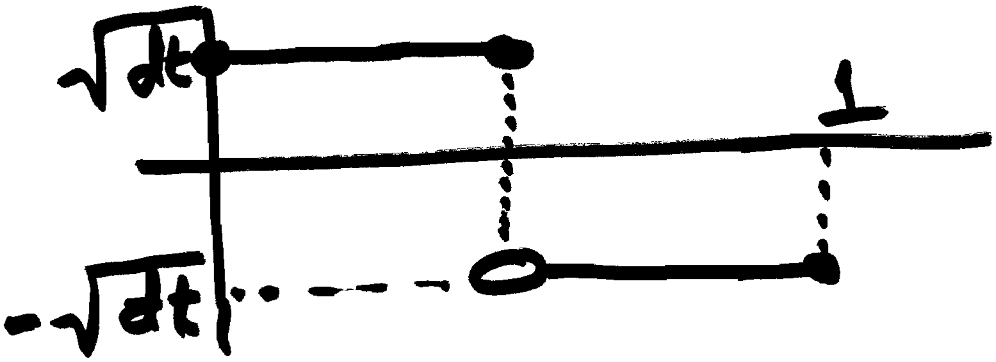
---

$X \sim Y \Rightarrow \forall f, \ \mathbb{E}[f(X)] = \mathbb{E}[f(Y)]$

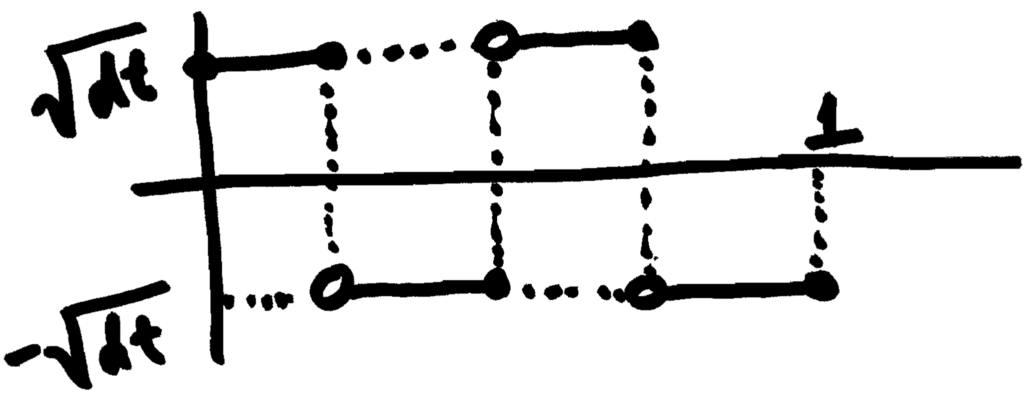
B2

$dX_t = X_{t+dt} - X_t$

E.g. $dW_0 = W_{dt} - W_0 =$




and $dW_{dt} = W_{2dt} - W_{dt} =$




$dt = 1/BP$

$I = \{0, dt, 2dt, \ldots, G?\}$

B3

$(X_t)$ is $\boxed{BM}$ if

① $X_0 = 0$

② $\forall s, t \in I^{\times}$, $s \neq t$,

$$dX_s \perp dX_t$$

$$dX_s \sim dX_t$$

③ $\forall t \in I^{\times}$,

$$E[dX_t] = 0$$

$$E[dX_t^2] = dt$$

$\boxed{\text{Approx BM}}$ same

except $E[dX_t^2] \approx dt$

$$E[X_t^2] \approx t$$

B3a

<u>CLT</u>: $(X_t)$ approx BM

$\Rightarrow$ $\forall$reasonable $t$ $\left(t \geq \frac{1}{GP}\right)$

$\forall$reasonable $f$

$$E[f(X_t)] \approx$$

$$\int_{-\infty}^{\infty} f(x)\, h^{\sqrt{t}}(x)\, dt$$

i.e. $h^{\sqrt{t}}$ is an approx PDF for $X_t$

B4

f is $\boxed{\text{0-reasonable}}$

if $f([-GP\sqrt{GP}, GP\sqrt{GP})$

$\subseteq ([-G-G-P, G-G-P])$

---

f $\boxed{\text{reasonable}}$ if

$f, f', f'', f'''$ are

0-reasonable

---

$\boxed{B5}$

**Ito-1:** $\tilde{\alpha} := \alpha(W_t)$

$X_t = \tilde{f}$, $f$ reasonable

$\Longrightarrow dX_t \approx \widetilde{f'}\, dW_t + \frac{1}{2}\widetilde{f''}\, dt$

---

**Ito:** $\tilde{\alpha} = \alpha(t, W_t)$

$X_t = \tilde{f}$, $f$ reasonable

$\Longrightarrow dX_t \approx \widetilde{\partial_1 f}\, dt + \widetilde{\partial_2 f}\, dW_t + \frac{1}{2}\widetilde{\partial_{22} f}\, dt$

B6

$$X_t = f(t, W_t)$$

$$dX_t = f(t+dt, W_t + dW_t) - f(t, W_t) =$$

$$(\partial_1 f)(t, W_t)\, dt \; +$$

$$(\partial_2 f)(t, W_t)\, dW_t \; +$$

small

$$\tfrac{1}{2}(\partial_{11} f)(t, W_t)\, dt^2 \; +$$

$$(\partial_{12} f)(t, W_t)\, dt \cdot dW_t \; +$$

$$\tfrac{1}{2}(\partial_{22} f)(t, W_t)\, \underbrace{dW_t^2}_{dt} + \cdots$$

$$d e^{e^{W_t}} = ?$$

B7

e.g. $X_t = e^{W_t} \Longrightarrow$

$$dX_t \approx e^{W_t} dW_t + \frac{1}{2} e^{W_t} dt$$

e.g. $Y_t = e^{X_t} \Longrightarrow$

$$dY_t \approx e^{X_t} dX_t + \frac{1}{2} e^{X_t} (dX_t)^2$$

$$\approx e^{X_t}\left(e^{W_t} dW_t + \frac{1}{2} e^{W_t} dt\right)$$

$$+ \frac{1}{2} e^{X_t} e^{2W_t} dt$$

repl. $X_t \to e^{W_t}$

$$X_t = f(t, W_t) \Longrightarrow$$

$$dX_t = (\text{sth})\, dW_t + (\text{sth})\, dt$$

fns of $t$ and $W_t$

---

Reverse ?

---

$$\int_a^b \alpha_t\, dW_t + \beta_t\, dt \quad :=$$

$$\sum_{\substack{a \le t < b \\ t \in I}} \alpha_t\, dW_t + \beta_t\, dt$$

---

$$X_t = X_0 + \int_0^t dX_s$$

B9

**Calculus:** $y_t = f(t)$

$$dy_t = f'(t)\,dt$$

$$y_t = y_0 + \int_0^t dy_s$$

---

**ODE:** Solve $dy_t = g_t(y_t)\,dt$

---

**SDE:** Solve

$$dX_t = \alpha_t(X_t)\,dW_t + \beta_t(X_t)\,dt$$

---

Easier if $\alpha_t, \beta_t$ all "deterministic" i.e. don't depend on $\omega \in \Omega = [0,1]$

B10

$Q^1$: Can we solve

$$dX_t = 3\,dW_t + 4\,dt$$
$$X_0 = 2$$

?

---

$Q^2$: Can we solve

$$\frac{dX_t}{X_t} = 6\,dW_t + 7\,dt$$

$$X_0 = 5$$

?

---

$A^1$: $X_t = 2 + \int_0^t 3\,dW_s + 4\,ds$

$$= 2 + 3W_t + 4t$$

B11

$$X_2 = 2 + 3W_2 + 4 \cdot 2$$
$$= 10 + 3W_2$$

---

$$\mathbb{E}[f(X_2)] = ?$$

---

$W_2$ mean $0$, var $2$

approx PDF $h^{\sqrt{2}}$

---

$$\mathbb{E}[f(X_2)] = \mathbb{E}[f(10 + 3W_2)]$$
$$\approx \int_{-\infty}^{\infty} f(10 + 3w)\, h^{\sqrt{2}}(w)\, dw$$
$$= \int_{-\infty}^{\infty} f(10 + 3z\sqrt{2})\, \underbrace{h(z)}_{e^{-z^2/2}/\sqrt{2\pi}}\, dz$$

B12

$$X_2 = 10 + 3W_2$$

$$W_2 \sim Z\sqrt{2} \quad \text{approx}$$

"std" normal

$$\mathbb{E}[f(X_2)] \approx$$

$$\int_{-\infty}^{\infty} f(10 + 3z\sqrt{2})\, e^{-z^2/2} \frac{dz}{\sqrt{2\pi}}$$

---

<u>Rmk</u> $A \sim B$

$$\Rightarrow \forall f, \quad \mathbb{E}[f(A)] = \mathbb{E}[f(B)]$$

B13

Q2: $\frac{dX_t}{X_t} = 6\,dW_t + 7\,dt$

$X_0 = 5$

$Y_t = \ln X_t$

$dY_t = \frac{dX_t}{X_t} - \frac{1}{2}\left(\frac{dX_t}{X_t}\right)^2$

$= 6\,dW_t + 7\,dt - \frac{1}{2} 36\,dt$

$Y_0 = \ln 5$

B141

$$dY_t = 6\,dW_t - 11\,dt$$

$$Y_0 = \ln 5$$

$$Y_t = \ln 5 + 6W_t - 11t$$

$$X_t = e^{Y_t} = 5\,e^{6W_t - 11t}$$

$$X_3 \sim 5e^{6z\sqrt{3} - 33} \quad \text{approx.}$$

$$\mathbb{E}[f(X_3)] =$$

$$\int_{-\infty}^{\infty} f\left(5e^{6z\sqrt{3}-33}\right) \frac{e^{-z^2/2}\,dz}{\sqrt{2\pi}}$$


$\Omega = [0,1]$

---

A **c.o.m.** or

**change of measure**

is a piecewise linear bijection $\Omega \longrightarrow \Omega$

---

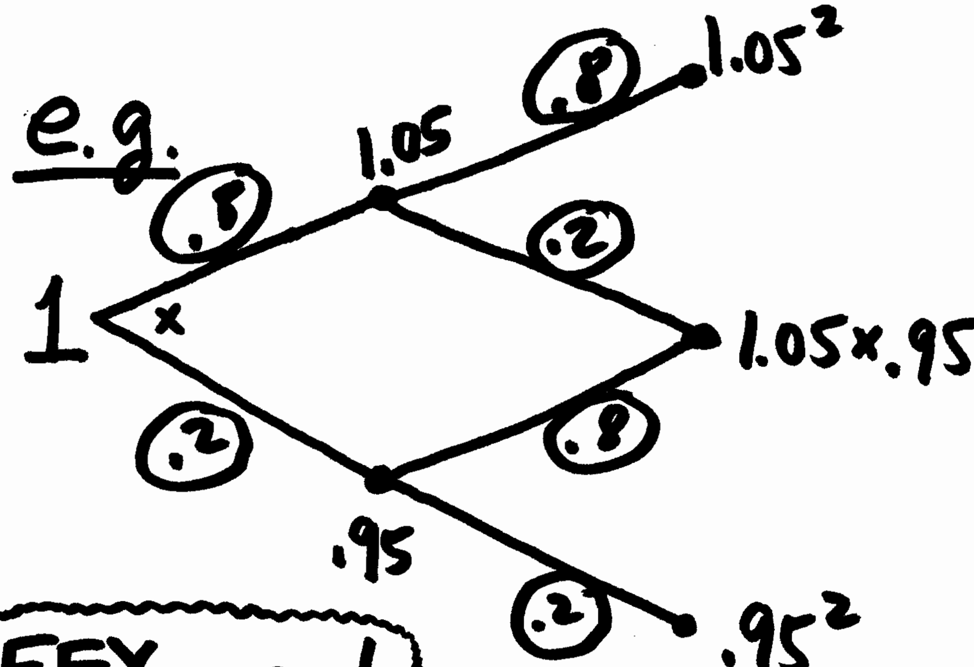
$$\boxed{X^Q} := X \circ Q$$

$\forall$ SRVs $X$

$\forall$ c.o.m. $Q$

B16

e.g.




FEX real

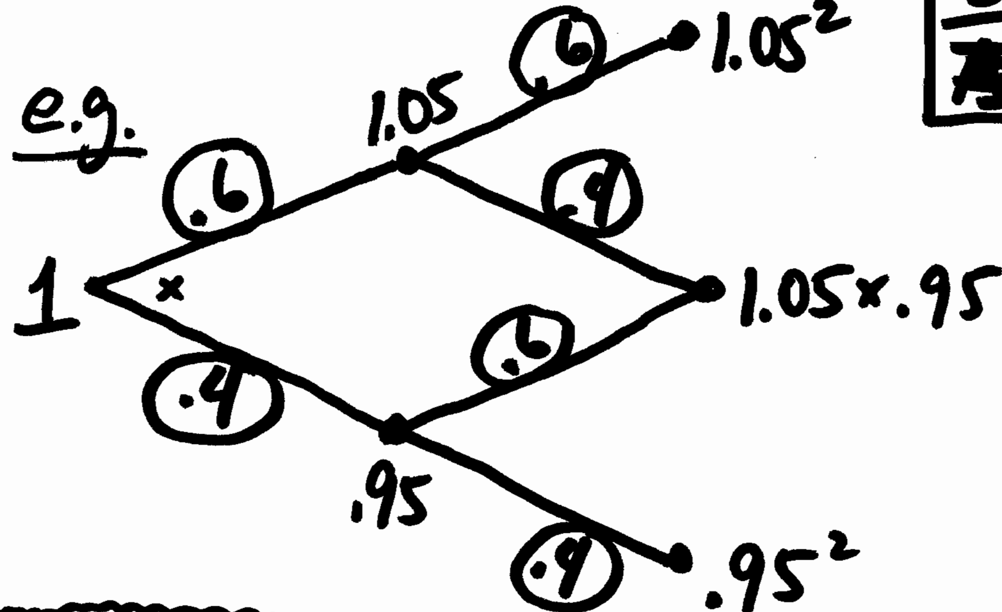
$$E_0 = 1 : \Omega \longrightarrow \mathbb{R}$$

$$E_1 = \begin{cases} 1.05 & \text{on } [0, .8] \\ .95 & \text{on } (.8, 1] \end{cases}$$

$$E_2 = \begin{cases} 1.05^2 & \text{on } [0, .64] \\ 1.05 \times .95 & \text{on } (.64, .8] \\ 1.05 \times .95 & \text{on } (.8, .96] \\ .95^2 & \text{on } (.96, 1] \end{cases}$$

B17 [redacted] [redacted]

e.g.



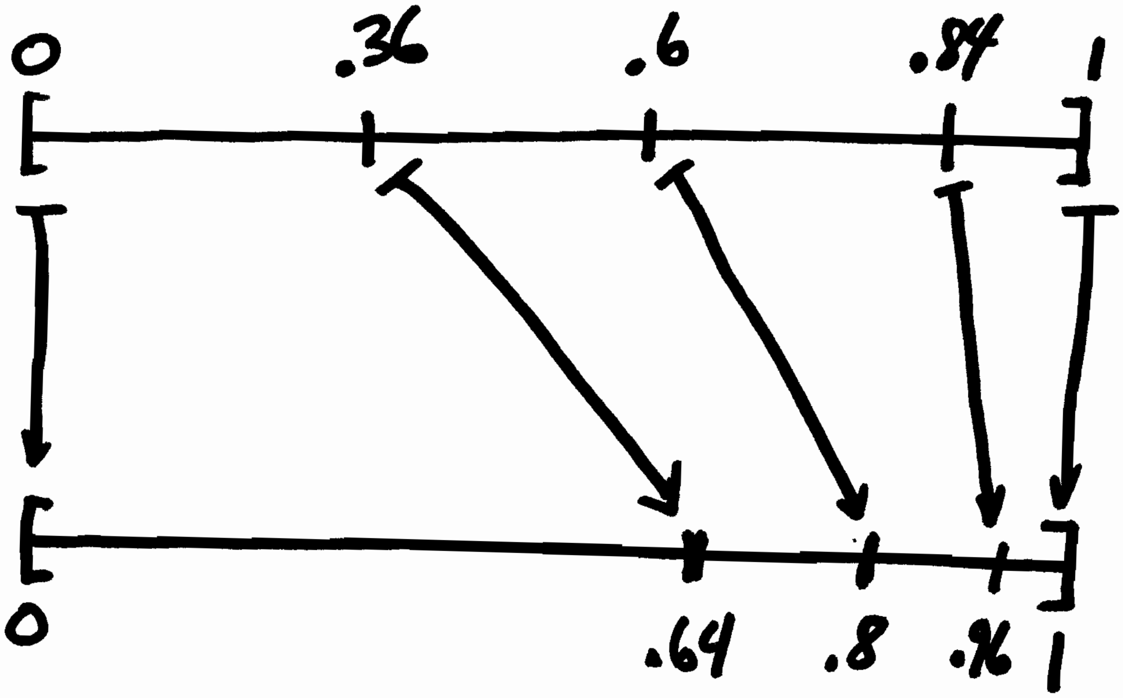

FEX risk-neutral

$$\tilde{E}_0 = 1 : \Omega \longrightarrow \mathbb{R}$$

$$\tilde{E}_1 = \begin{cases} 1.05 & \text{on } [0,.6] \\ .95 & \text{on } (.6,1] \end{cases}$$

$$\tilde{E}_2 = \begin{cases} 1.05^2 & \text{on } [0,.36] \\ 1.05 \times .95 & \text{on } (.36,.6] \\ 1.05 \times .95 & \text{on } (.6,.84] \\ .95^2 & \text{on } (.84,1] \end{cases}$$

| BIP | |
| --- | --- |
| [redacted] | [redacted] |




Q piecewise lin.
bij $\Omega \to \Omega$

$$\tilde{E}_t = E_t^Q \quad \forall t \in \{0, 1, 2\}$$

B19

<u>Girsanov</u>-1: $|\alpha| < GP$

$\tilde{W}_0 = 0$

$d\tilde{W}_t = dW_t + \alpha\, dt, \ \forall t \in I^*$

$\Longrightarrow \exists$ c.o.m. $Q$

$\ni: (\tilde{W}_t^Q)$ is $\approx$ a B. M.

---

"can compensate for changing drift by changing measure"

---

<u>Pf</u>: ...

B20 [redacted]

$\tilde{W}_t$ given by

$$\begin{tikzcd}[column sep=large] & \sqrt{dt} + \alpha\, dt =: U \\ 0 \arrow[ur, ".5"] \arrow[dr, ".5"'] \arrow[r, phantom, "+"] & \cdots \text{GP} \times \text{BP} \\ & -\sqrt{dt} + \alpha\, dt =: V \end{tikzcd}$$

---

$$V < 0 < U$$

---

Choose $p \ni: (1-p)V + pU = 0$

---

Choose c.o.m. $Q \ni:$

$\tilde{W}_t^Q$ is given by

$$\begin{tikzcd}[column sep=large] & U \\ 0 \arrow[ur, "p"] \arrow[dr, "1-p"'] \arrow[r, phantom, "+"] & \cdots \text{GP} \times \text{BP} \\ & V \end{tikzcd}$$

$U = \sqrt{dt} + \alpha\, dt$, $V = -\sqrt{dt} + \alpha\, dt$

B21

[redacted] [redacted]

---

$(d\tilde{W}_t^Q)$ is iid w/ mean 0,

$$\text{var} = pU^2 + (1-p)V^2$$

$$= dt + \alpha^2 dt^2 + (p-(1-p))(2\alpha\, dt\sqrt{dt})$$

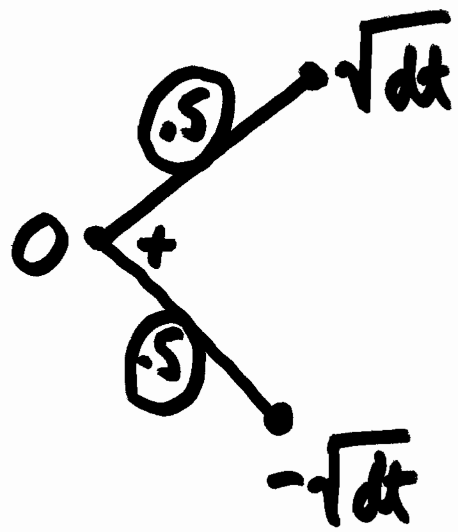
$$\approx dt$$

---

$\tilde{W}_t^Q$ is $\simeq$ B.M. QED

$dt = 1/BP$

$\boxed{\alpha}$




... BP

NO WAY! $\approx h(\omega)\,d\omega \;\; \forall \omega$

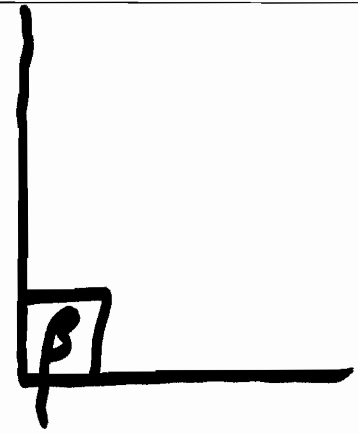
dt yrs

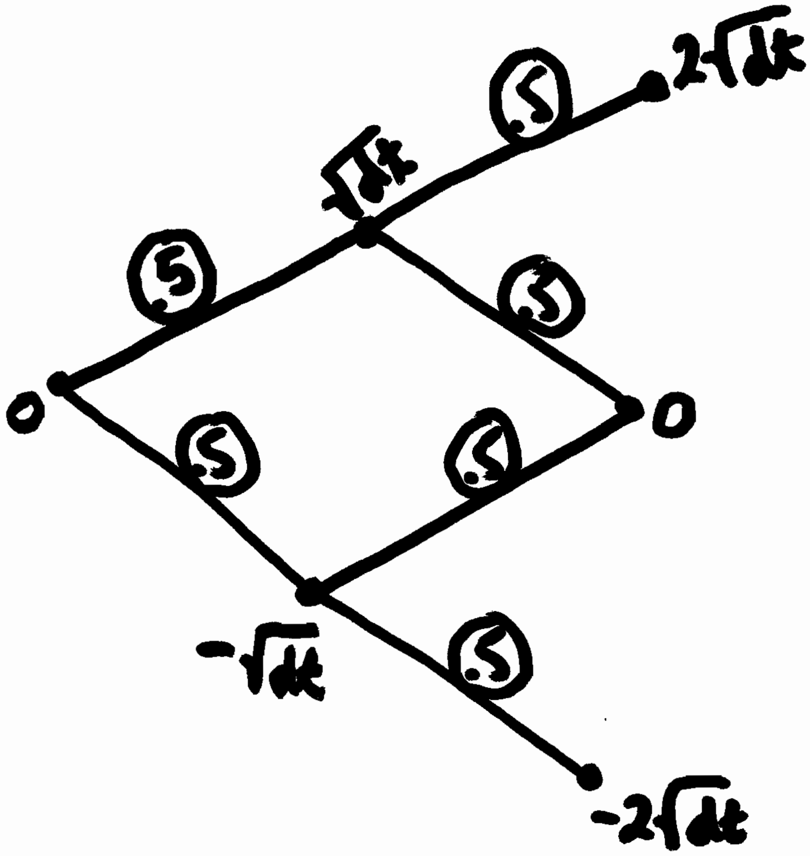
1 "split second" = dt yrs

Time horizon = ~~1 yr~~ GP yrs

Move to SPA

God $\longrightarrow$ Tyche



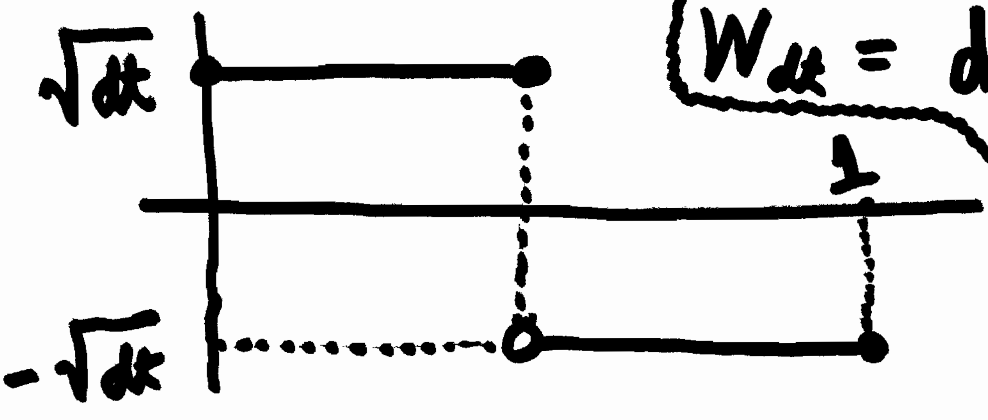

... BP × GP

$W_0, \quad W_{dt}, \quad W_{2dt}, \ldots, \quad W_{GP}$

$\overset{\parallel}{0} : \Omega \to \mathbb{R}$

$\boxed{\Omega = [0,1]}$

$W_{dt} = dW_0 = CF_0 \sqrt{dt}$



$\boxed{\gamma}$

$\Pr[-1 \le W_1 \le 1] =$

NO WAY! $\approx$

$\int_{-1}^{1} h(w)\,dw \approx .68$

---

$\mathbb{E}[f(W_1)] = \sum$ NO WAY!

$\approx \int_{-\infty}^{\infty} f(w)\, h(w)\, dw$

---

$h^{\sqrt{t}}(w)$ is an approx. PDF for $W_t$

---

$W_t = \sqrt{t}\, Z$, $\quad h(z) \approx$ PDF for $Z$

$\boxed{\delta}$

$(X_t)$ $\boxed{\text{B.M.}}$ if

(init) $\{ X_0 = 0$

(iid) $\begin{cases} t \neq u \Rightarrow & dX_t \sim dX_u \\ & dX_t \perp dX_u \end{cases}$

(mean 0) $\begin{cases} \forall t, & \mathbb{E}[dX_t] = 0 \end{cases}$

(var. dt) $\quad \mathbb{E}[(dX_t)^2] = dt$

---

approx B.M. same, but

$\mathbb{E}[(dX_t)^2] \approx dt$

$|\text{error}| < dt/GP$

B'1

$\boxed{\text{simple set}}$ = finite union of intervals

$\boxed{\text{simple partition}}$ = partition of $\Omega = [0,1]$ by simple sets

$X : \Omega \to \mathbb{R}$ SRV is $\boxed{\mathcal{F}\text{-msbl}}$ means: $\forall S \in \mathcal{F}$, $X|S$ is const.

$$\boxed{(\mathbb{E}[X|\mathcal{F}])(\omega)} := \frac{1}{|S|}\int_S X, \text{ if } \omega \in S \in \mathcal{F}$$

Tyche answers: $\omega \in ? \in \mathcal{F}$

We guess: $X(\omega)$

coarse fine

$\mathcal{F} \leq \mathcal{G} \overset{\text{def}}{\iff}$ $\boxed{B'_2}$

$\left(\forall F \in \mathcal{F}, \exists \mathcal{G}_0 \subseteq \mathcal{G} \ni: F = \bigcup \mathcal{G}_0\right) \implies$ [redacted]

$\mathbb{E}\left[\mathbb{E}[X \mid \mathcal{F}] \mid \mathcal{G}\right] =$

$\forall X,$ $\mathbb{E}[X \mid \mathcal{G}]$ "tower law"

---

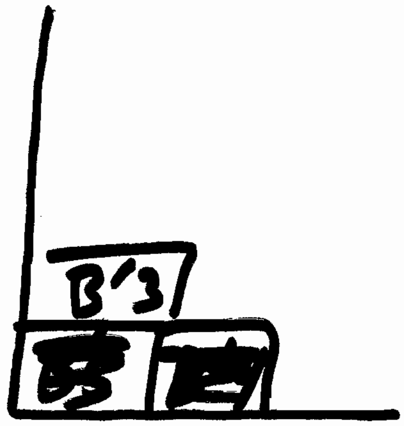
$\forall X,$ $\overline{X} := \mathbb{E}[X \mid \mathcal{F}]$

$\phi, \psi$ $\mathcal{F}$-msbl SRVs

$\implies \overline{\phi A + \psi B} = \phi \overline{A} + \psi \overline{B},$ $\forall A, B$

"$\mathcal{F}$-msbl linearity"

$$\overset{\text{coarse}}{\mathcal{F}} \leq \overset{\text{fine}}{\mathcal{G}}$$




$$\forall X, \quad \tilde{X} := \mathbb{E}[X \mid \mathcal{F}]$$

$$\bar{X} := \mathbb{E}[X \mid \mathcal{G}]$$

$\phi, \psi$ $\mathcal{F}$-msbl SRVs

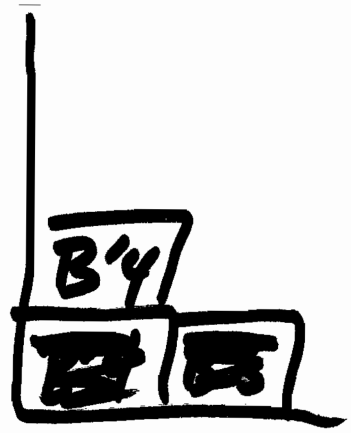
$$\Downarrow$$

$$\overline{\phi A + \psi B} =$$

$$\overline{\widetilde{\phi A + \psi B}} =$$

$$\overline{\phi \tilde{A} + \psi \tilde{B}}, \quad \forall A, B$$

"tower law trick"




$\boxed{\mathcal{F}(X)} := \{X^{-1}(x) \ni: x \in \mathbb{R}\}$

coarsest $\ni:$ $X$ msbl

---

$X(\omega) = ?$ same as

$\omega \in ? \in \mathcal{F}(X)$

---

$\boxed{\mathcal{F}(X_1,\dots,X_n)} := \{S_1 \cap \cdots \cap S_n \ni:$

$S_1 \in \mathcal{F}(X_1), \dots, S_n \in \mathcal{F}(X_n)\}$

coarsest $\ni:$ $X_1,\dots,X_n$ msbl

---

$(X_1(\omega),\dots,X_n(\omega)) = ?$ same as

$\omega \in ? \in \mathcal{F}(X_1,\dots,X_n)$

$B'4a$

$Y$ is $\mathcal{F}$-msbl

$\Rightarrow$ if Tyche answers

$\omega \in\ ? \in \mathcal{F}$

then we can compute

$Y(\omega)$

---

$Y$ is $\mathcal{F}(X_1,\ldots,X_n)$-msbl

$\Rightarrow$ if Tyche answers

$(X_1(\omega),\ldots,X_n(\omega)) = ?$

then we can compute

$Y(\omega)$

---

$I = \{0, dt, 2dt, \ldots, GP\}$

E | B'S

$(\phi_t)$ $\boxed{\text{adapted}}$ means

$\forall t \in I$, $\phi_t$ is $\mathcal{F}(W_0, \dots, W_t)$-msbl

---

Tyche reveals $W_0(\omega), \dots, W_t(\omega)$

and we compute $\phi_t(\omega)$,

but not a moment before

---

$$\varepsilon = 2^{-t/dt} \Longrightarrow \boxed{\mathcal{F}_t} :=$$

$$\mathcal{F}(W_0, \dots, W_t) =$$

$$\{[0, \varepsilon], (\varepsilon, 2\varepsilon], \dots, (1-\varepsilon, 1]\}$$

---

$\mathcal{F}_0 = \{[0, 1]\}$ ← coarsest

$\mathcal{F}_0$-msbl = const.

$$\mathbb{E} = \mathbb{E}[\cdot \mid \mathcal{F}_0]$$

B6

$Y$ possibly not
$\mathcal{F}(X_1, \dots, X_n)$ - msbl

---

Tyche tells
$X_1(\omega), \dots, X_n(\omega)$,

we <u>guess</u> $Y(\omega)$

---

$$\boxed{\mathbb{E}[Y \mid X_1, \dots, X_n]} := \mathbb{E}[Y \mid \mathcal{F}(X_1, \dots, X_n)]$$

B'7

**Fact:** $\mathbb{E} = \mathbb{E}[\cdot \mid \overset{\text{coarsest}}{\mathcal{F}_0}]$

$\forall t, \quad \mathbb{E}_t := \mathbb{E}[\cdot \mid \underbrace{W_0, \dots, W_t}_{\text{fine}}]$

$(\sigma_t), (\mu_t)$ adapted

**Fix $t$.**

$$\mathbb{E}[\sigma_t \underbrace{dW_t}_{\mathbb{E}_t \to 0} + \mu_t \underbrace{dt}_{\mathbb{E}_t \to dt}] = ?$$

$$\therefore \mathbb{E}[\sigma_t \, dW_t + \mu_t \, dt] = \mathbb{E}[\mu_t \, dt] = (\mathbb{E}[\mu_t]) \, dt$$

# Black-Scholes Model

[illegible] BS

$$S_0 = 1 = B_0$$

$$\frac{dB_t}{B_t} = r\,dt \quad ; \quad B_t = e^{rt}$$

$$\frac{dS_t}{S_t} = ?$$

$$\omega \in \Omega \to CF_0, CF_{dt}, CF_{2dt}, \ldots$$

$$\to W_0, W_{dt}, W_{2dt}, \ldots$$

$$\frac{dS_t}{S_t} = \sigma\,dW_t + \alpha\,dt$$

[redacted] B9

$$d \ln S_t \approx$$

$$\sigma \, dW_t + \left(\alpha - \frac{\sigma^2}{2}\right) dt$$

---

$$\ln S_t \approx \sigma W_t + \left(\alpha - \frac{\sigma^2}{2}\right) t$$

---

$$S_t \approx e^{\sigma W_t + \mu t}$$

$$\mu := \alpha - \frac{\sigma^2}{2}$$

---

Market analysis:

Find $\sigma$

(std. dev. of $\ln S_1$)

[redacted] | B10

<u>Goal</u>: Price a deriv. paying $(S_T - K)^+$ at time $T$

---

ABCDEF:

$\exists$ adapted $\phi_t, \psi_t, V_t$

$\ni$: $$V_t = \phi_t S_t + \psi_t B_t$$

$$dV_t = \phi_t \, dS_t + \psi_t \, dB_t$$

$$V_T = (S_T - K)^+$$

---

<u>Goal</u>: $V_0 \leftarrow \mathcal{F}_0$-msbl

i.e. const

B11

"rebalance portfolio"

$$V_t = \phi_t S_t + \psi_t B_t$$

$$\phi_t S_{t+dt} + \psi_t B_{t+dt} = \phi_{t+dt} S_{t+dt} + \psi_{t+dt} B_{t+dt} = V_{t+dt}$$

$t$ $\qquad\qquad$ $t+dt$

Portfolio IGL "Investment Gain / Loss"

$$dV_t = V_{t+dt} - V_t$$

$$= \phi_t (S_{t+dt} - S_t) + \psi_t (B_{t+dt} - B_t)$$

$$= \underbrace{\phi_t \, dS_t}_{\text{Stock IGL}} + \underbrace{\psi_t \, dB_t}_{\text{Bond IGL}}$$

B/12

[redacted]

$$\frac{dB_t}{B_t} = r\,dt$$

$$\frac{dS_t}{S_t} = \sigma\,dW_t + \alpha\,dt$$

$$= \sigma\,d\tilde{W}_t + r\,dt$$

---

$$d\tilde{W}_t = dW_t + \frac{\alpha - r}{\sigma}\,dt$$

$$\tilde{W}_0 = 0$$

---

$\exists$ com $Q \ni$: $\tilde{W}_t^Q$ approx BM

$\frac{dB_t}{B_t} = r\,dt$, $\frac{dS_t}{S_t} = \sigma\,d\tilde{W}_t + r\,dt$ $\boxed{B'13}$

---

$\overline{V}_t := \mathbb{E}[V_t^Q] \qquad \forall t \in I$

$\overline{V}_T = \mathbb{E}[(S_T^Q - K)^+]$

**Claim**: $\frac{d\overline{V}_t}{dt} = r\,\overline{V}_t$ (IOU)

$\overline{V}_t \approx \overline{V}_0 e^{rt} = V_0 e^{rt}$

$V_0 = \overline{V}_T e^{-rT}$

**Goal**: $\underbrace{\mathbb{E}[(S_T^Q - K)^+]}_{\overline{V}_T =}$

B'14

[redacted]

<u>Goal</u>: $\mathbb{E}\left[(S_T^Q - K)^+\right]$

$$\frac{dS_t}{S_t} = \sigma \, d\tilde{W}_t + r\,dt$$

$$d \ln S_t \approx \sigma \, d\tilde{W}_t + \left(r - \frac{\sigma^2}{2}\right) dt$$

$$\ln S_t \approx \sigma \tilde{W}_t + \left(r - \frac{\sigma^2}{2}\right) t$$

$$\ln S_T^Q \approx \sigma \tilde{W}_T^Q + \left(r - \frac{\sigma^2}{2}\right) T$$

$$\sim \sigma Z \sqrt{T} + \left(r - \frac{\sigma^2}{2}\right) T$$

(approx.)

normal
std dev 1
mean 0

B/S
[redacted]

$$\mathbb{E}\left[(S_T^Q - K)^+\right] \approx$$

$$\mathbb{E}\left[\left(e^{\sigma Z\sqrt{T} + (r-\sigma^2/2)T} - K\right)^+\right] =$$

$$\int_{-\infty}^{\infty}\left(e^{\sigma z\sqrt{T}+(r-\sigma^2/2)T}-K\right)^+ \cdot \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz$$

$= \cdots$ (Black-Scholes)
(w/ $S_0=1$)

---

e.g., $\sigma\sqrt{T}=3$, $(r-\sigma^2/2)T=-7$, $K=9$

B'15a

# Calc. exercise

$$h(z) = e^{-z^2/2} / \sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} \left(e^{3z-7} - 9\right)^{+} h(z)\, dz =$$

$$\int_{a}^{\infty} \left(e^{3z-7} - 9\right) h(z)\, dz =$$

$$\underbrace{\int_{a}^{\infty} e^{3z-7} \cdot h(z)\, dz}_{\int_{a-3}^{\infty} e^{3(z+3)} h(z+3)\, dz = \cdots} - 9\Phi(-a)$$

$$\overline{V}_t = \mathbb{E}[V_t^Q]$$

<u>Claim</u>: $d\overline{V}_t = r\overline{V}_t\,dt$ [redacted]

$$\mathbb{E}_t^Q := \mathbb{E}[\cdot \mid W_0^Q, \dots, W_t^Q]$$

$$\stackrel{\text{Iou}}{=} \mathbb{E}[\cdot \mid \tilde{W}_0^Q, \dots, \tilde{W}_t^Q]$$

---

$$dV_t^Q = \phi_t^Q\, dS_t^Q + \psi_t^Q\, dB_t^Q$$

$$dS_t^Q \xrightarrow[\text{Iou}]{\mathbb{E}_t^Q} rS_t^Q\,dt \qquad dB_t^Q \xrightarrow{\mathbb{E}_t^Q} rB_t^Q\,dt$$

---

$$\mathbb{E}_t[dV_t^Q] = r\big(\underbrace{\phi_t^Q S_t^Q + \psi_t^Q B_t^Q}_{V_t^Q}\big)\,dt$$

B'17

$$\mathbb{E}_t[dV_t^Q] = r V_t^Q \, dt$$

$$\mathbb{E}[dV_t^Q] = r \, \mathbb{E}[V_t^Q] \, dt$$

(tower law trick)

$$d\bar{V}_t = r \bar{V}_t \, dt \quad \text{QED}$$

$$\frac{dS_t}{S_t} = \sigma \, d\tilde{W}_t + r \, dt$$

$$dS_t^Q = S_t^Q(\sigma \, d\tilde{W}_t^Q + r \, dt)$$

B18

<u>Want</u>: $\mathbb{E}_t^Q[dS_t^Q] = r S_t^Q dt$

$$dS_t^Q = S_t^Q(\sigma \underbrace{d\tilde{W}_t^Q}_{\xrightarrow{\mathbb{E}_t^Q} 0} + r \underbrace{dt}_{\xrightarrow{\mathbb{E}_t^Q} dt})$$

QED

$$\mathbb{E}_t^Q = \mathbb{E}[\cdot \mid W_0^Q, \dots, W_t^Q]$$

$$\overset{\text{IOU?}}{=} \mathbb{E}[\cdot \mid \tilde{W}_0^Q, \dots, \tilde{W}_t^Q]$$

B/19

<u>Want</u>: $\mathcal{F}(W_0, \dots, W_t) = \mathcal{F}(\tilde{W}_0, \dots, \tilde{W}_t), \quad \forall t$

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$\gamma := \frac{\alpha - \beta}{\sigma}$

$\tilde{W}_0 = 0, \qquad d\tilde{W}_t = dW_t + \gamma\, dt$

---

$\forall t, \ \tilde{W}_t = W_t + \gamma t$

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Fix $t$. $\quad N = \#\{0, \dots, t\}$

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$\Gamma : \mathbb{R}^N \hookrightarrow\!\!\!\!\twoheadrightarrow \mathbb{R}^N$

$(\tilde{W}_0, \dots, \tilde{W}_t) = \Gamma(W_0, \dots, W_t)$

B'20

**Fact:** $X_1, \dots, X_N$ SRVs

$Y_1, \dots, Y_N$ SRVs

$$\Gamma : \mathbb{R}^N \hookrightarrow\!\!\!\rightarrow \mathbb{R}^N$$

$$(Y_1, \dots, Y_N) = \Gamma(X_1, \dots, X_N)$$

$$\Longrightarrow \mathcal{F}(X_1, \dots, X_N) = \mathcal{F}(Y_1, \dots, Y_N)$$