Patterns in the Stars, Recurrence in Dynamical Systems & The Combinatorial Background for Non-Conventional Ergodic Theorems

\[ T: x \rightarrow x \]
Conventional Ergodic Averages:

\[ (*) \quad \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \frac{1}{N} \left( f(x) + f(Tx) + f(T^2 x) + \ldots + f(T^{N-1} x) \right) \]

where

\[ x \in X, \text{ a space endowed with "measure" } \mu \]

with \( \mu(X) < \infty \)

and \( T : X \to X \) is measure preserving

Ergodic Theorem: The limit of (*), as \( N \to \infty \), exists (i) almost everywhere \( \int \)

(ii) in \( L^p \), \( p \geq 1 \)

for \( f \in L^p \)

Non-Conventional Ergodic Averages:

\[ (**) \quad \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) g(T^{2n} x) h(T^{3n} x) \]

\[ \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) g(T^{2n} x) h(T^{3n} x) k(T^{5n} x) \]

Theorem (B. Host & B. Kra): The limits of averages as in (**) exist in \( L^2 \) when all functions are bounded and measurable.
The interest in expressions as in (???) comes from RAMSEY THEORY.

For any 2-coloring of the edges of the complete graph with 6 vertices, there will be at least one monochromatic triangle.

The RAMSEY phenomenon:

Inside certain "rich" structures, certain patterns will be inevitably found.
**Constellation Theorem** (Katznelson & F.)

Let \( S \subset \mathbb{R}^m \) be a set of positive (upper) density; let \( F \subset \mathbb{R}^m \) be any finite set. Then \( \exists \) \( m \in \mathbb{N} \), \( \hat{u} \in \mathbb{R}^m \) so that \( mF + \hat{u} \subset S \)

\( (S \subset \mathbb{R}^m \) has positive upper density if \( \exists \) ball \( B_r \) with arbitrarily large radius \( r \) and a \( d > 0 \) with \( \frac{\text{vol}(S \cap B_r)}{\text{vol}(B_r)} > d \) for each \( B_r \)

(Only) Proof of this uses ergodic theory

**Example**

\( F = \ldots \ldots \)

\( S = \) stars (as tiny splotches in visual plane or in space)
What is analogous statement for $\mathbb{Z}$?

If $S \subseteq \mathbb{Z}$ has positive upper density (e.g., if $\frac{1}{N} \#(S \cap \{1, 2, \ldots, NS\}) \to S > 0$)
and $F$ is finite, $\exists m, n$ with

$$mF + n \subseteq S$$

$$F = \{a, a+1, a+2, \ldots, a+l-1\}; \text{ take } F = \{a, \ldots, a+l-1\}$$

$mF + n$ is arithmetic progression with $l$ terms

**THEOREM (Szemerédi)** A subset of $\mathbb{Z}$ of positive upper density contains arbitrarily long arithmetic progressions.

**THEOREM (van der Waerden)** In any partition

$$\mathbb{Z} = C_1 \cup C_2 \cup \ldots \cup C_N$$

one of the $C_j$ contains arbitrarily long arithmetic progressions.

(Conjectured by Erdős-Turán in 30's)
We think of \( \mathbb{Z} \) as measure space, with

**Density** as measure:

\[
\text{DEN} (\text{even}) = \frac{1}{2}, \quad \text{DEN} (7\mathbb{Z} + 3) = \frac{1}{7}
\]

\[
\text{DEN} (\text{primes}) = 0, \quad \text{DEN} (A \cup B) = \text{DEN} (A) + \text{DEN} (B) - \text{DEN} (A \cap B)
\]

(But \( \text{DEN} \) is not \( \sigma \)-additive:
\[
\mathbb{Z} = \bigcup \{ n \}, \quad \text{DEN} \{ n \} = 0
\]

Define \( T: \mathbb{Z} \rightarrow \mathbb{Z} \) by \( T_n = x + 1 \)

\( T \) preserves density!

\[
\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
T: & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& x & T_x & T^2_x & T^3_x & \cdots & \cdots & \cdots \\
\end{array}
\]

Szemerédi's Theorem \( \iff \)

\textbf{(MRT) Multiple Recurrence Theorem in Ergodic Theory}

\[
\mu (A) > 0 \implies
\]

for any \( k \in \mathbb{N} \)

\[
\mu (\bigcap_{n=0}^{m-1} A 
\]
What about \( k=1 \)? \( \mu(\mathcal{A} T^{-m} \mathcal{A}) > 0 \)

Must every set return to itself?

This is Poincaré's Recurrence Theorem.

**THEOREM:** If \((X, \mathcal{B}, \mu)\) is a measure space with \( \mu(X) < \infty \), \( T: X \to X \) a measure preserving transformation, \( A \in \mathcal{B} \) with \( \mu(A) > 0 \), then for some \( m > 0 \)
\[
\mu(\mathcal{A} T^{-m} \mathcal{A}) > 0.
\]

MRT is special case of **Commuting Multi. Rec. Th.**

**THEOREM:** If \((X, \mathcal{B}, \mu)\) is a measure space with \( \mu(X) < \infty \), \( T_1: X \to X, T_2: X \to X, \ldots \)
\( T_k: X \to X \) \( k \) commuting measure preserving transformations, \( A \in \mathcal{B} \) with \( \mu(A) > 0 \), then for some \( m > 0 \)
\[
\mu(\mathcal{A} T_1^{-m} \mathcal{A} \cap T_2^{-m} \mathcal{A} \cap \ldots \cap T_k^{-m} \mathcal{A}) > 0.
\]
Comm. Mult. Rec. Thm $\Rightarrow$ Constellation Thm

$T_i$ is translation by corresponding vector:

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Proof of Poincaré Recurrence via Ergodic Theorem

We'll prove that if $\mu(A) > 0$ then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^{-n}A) > 0$$

Principle: Behavior on the average is easier to control
Lemma: If $T$ is measure preserving, then for $f$ integrable

$$\int f(Tx) \, d\mu(x) = \int f(x) \, d\mu(x).$$

Pf: True for $f = 1_A$ where $f(Tx) = 1_{T^{-A}}(x)$ and

$\text{LHS} = \mu(T^{-A}) \; \& \; \text{RHS} = \mu(A) \; \& \; \text{these are equal.}$

Linear combinations of these are dense in $L^2$. \[ \checkmark \]

By Ergodic thm, if $f = 1_A = \sum_{n}^{N-1} \{ 0 \times \delta \}$

$$f(x) = \lim_{n \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \; \text{exists (in } L^2)$$

Note that $f(x) = f(Tx)$ and $\int f \, d\mu = \mu(A)$

In particular, $f$ is not a.e. 0. \[ \mu(A) \]

Claim $\int f(x) f(x) \, d\mu(x) > 0$.

Suppose = 0. Replace $x$ by $T^n x$:

$$\int f(x) f(T^n x) \, d\mu(x) = 0$$

Average over $n$. This $\Rightarrow$

$$\int f(x) \cdot f(T^n x) \, d\mu(x) = 0$$

i.e. $f = 0$ a.e. But $\int f \, d\mu = \mu(A) > 0$. So

$$\int f(x) f(x) \, d\mu(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \int f(T^n x) f(x) \, d\mu(x)$$

$$\int 1_A(T^n x) 1_A(x) \, d\mu(x) = \mu(A \cap T^{-n}A)$$
Similarly one can prove:

\[
\lim \frac{1}{N} \sum_{n=0}^{N-1} \mu(\Lambda^n, A \cap T_{2}^{n} A \cap \cdots \cap T_{k}^{n} A) > 0.
\]

This is

\[
\lim \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{n!} \left(\int_A (T_{2}^{n} x) \Lambda A(t_{2}^{n} x) \cdots \Lambda A(t_{k}^{n} x) - \Lambda A(t_{k}^{n} x)\right) d\mu(x).
\]

Does limit exist?

?? ?? ?? ??

By Host-Kra limit exists for

\[ T_{1} = T_{2}^{2} = \cdots = T_{k}^{k} \]

and we can identify limit!!

In all cases, multiple recurrence is established by analyzing

(non-conventional) ergodic averages.

In addition to commuting mult. recurrence

(Katnelson,Y) \[
\lim \frac{1}{N} \sum \mu(\Lambda^n, A \cap \cdots \cap T_{k}^{n} A) > 0
\]
we have for $\mu(A) > 0$:

\[
(\text{Bergelson-Leibman}) \lim \frac{1}{N} \sum \mu(AnT^{-p_1(n)}A \cap T^{-p_2(n)}A \cap \ldots \cap T^{-p_k(n)}A) > 0
\]

where $p_1(n), \ldots, p_k(n)$ are polynomials

taking integer values on $\mathbb{Z}$ and with $p_i(0) = 0$

For example:

\[
\lim \frac{1}{N} \sum \mu(AnT^{-n}A \cap T^{-n^2}A) > 0
\]

⇒ Polynomial Multiple Recurrence:

e.g.: $\exists n: \mu(AnT^{-n}A \cap T^{-n^2}A) > 0$

General Principle: A recurrence pattern in dynamics is also a recurrence pattern for sets of positive (upper) density in $\mathbb{Z}$

⇒ If $S \subseteq \mathbb{Z}$ has positive (upper) density then it contains arbitrary "polynomial progressions"
\[
a, a+n, a+n^2; a+n^3
\]
&
\[
b, b+m, b+2m, b+3m^2, b+4m^2
\]

etc.

(But not necessarily \( c, c+\mod{m} \))

\[x - y = m^2 \quad \text{(Sarkozy)}\]

Underlying idea: In presence of sufficient randomness/mixing (ergodicity, 
\[
\frac{1}{N} \sum f_i (T^n x) f_2 (T^{2n} x) ... f_k (T^{kn} x)
\]
\[\xrightarrow{L^2} \int f_i(y) dp(x) f_2(y) dp(x) ... f_k(y) dp(x)\]

In case of (pure) periodicity:

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} f_i (T^n x) f_2 (T^{2n} x) ... f_k (T^{kn} x) = \\
\frac{1}{S} \sum_{j=0}^{S-1} f_i (x+j) f_2 (x+2j) ... f_k (x+kj)
\]

\[x = \mathbb{Z}/S\mathbb{Z}\]

Idea: Integrate out the random effects
We assume herefore that \( \mu(X) = 1 \) or \((X, \mathcal{B}, \mu)\) is probability space

**Independence:** \( \mu(A \cap B) = \mu(A) \mu(B) \)

**Strong Mixing:** \( \mu(A \cap T^{-n}B) \to \mu(A) \mu(B) \)

E.g. **Baker's Transformation (kneading)**

\[
\begin{align*}
T : A & \to B \\
B & \to \frac{1}{n} \text{ "Uniform spread"}
\end{align*}
\]

\[
\mu(A \cap T^{-n}B) = \mu(T^n A \cap B)
\]
Weak MIXING: (chaotic behavior)

If $A, B, C \in \mathcal{B}$ and

$$\mu(A) > 0, \mu(B) > 0, \mu(C) > 0$$

then $\exists n$ so that

$$\mu(\text{AnT}^{-n}B) > 0 \text{ and } \mu(\text{AnT}^{-n}C) > 0$$

Note that (strong) mixing $\Rightarrow$ weak mixing

**Theorem:** If $(X, \mathcal{B}, \mu, T)$ is weak mixing,

$$f_1, f_2, \ldots, f_k \text{ bounded meas. functions,}$$

then

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n} x) \cdots f_k(T^{kn} x) \xrightarrow{L^2}$$

$$\int f_1 d\mu \int f_2 d\mu \cdots \int f_k d\mu$$

In particular: If $f_1 = f_2 = \cdots = f_k = 1_A$, limit is $\mu(A)^k$

and

$$\frac{1}{N} \sum \mu(\text{AnT}^{-n}A \cdots \text{nT}^{-kn}A) = \frac{1}{N} \sum \int 1_A (\text{AnT}^{-n}A) \cdots 1_A (T^{kn}A) d\mu$$

$$\rightarrow \mu(A)^{kn} > 0$$
\[ T: X_i \rightarrow X_{i+1} \]

\[ X = X_0 \cup X_1 \cup X_2 \cup X_3 \cup X_4 \]

Suppose each step is (weak) mixing

Note: If \( T \) is weak mixing, so is \( T^5 \)

Write \( S = T^5 \)

Want \( \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) g(T^{2n} x) \theta(T^{3n} x) \)

\[ N = 5M \quad n = 5m + r \]

\[ \frac{1}{N} \left\{ \frac{1}{M} \sum f(T^m x) g(T^{2m} x) \theta(T^{3m} x) + \right. \]

\[ \left. \frac{1}{M} \sum f(T^m x) g(T^{2m} x) \theta(T^{3m} x) + \right. \]

\[ \cdots + \frac{1}{M} \sum f(T^m x) g(T^{2m} x) \theta(T^{3m} x) \right\} - \]

\[ \frac{1}{5} \left\{ \alpha_1 f Y_1 + \alpha_2 f_1 \beta_1 + \cdots + \alpha_5 f_1 \beta_5 \right\} \]

where \( \alpha_j = \frac{1}{X_j} \int f d\mu \quad \beta_j = \frac{1}{X_j} \int g d\mu \quad X_j = \frac{1}{X_j} \int \theta d\mu \)

and \( x \in X_i \)
This will be model for general (ergodic) case

\[ \mathbb{Z}/\mathbb{Z} \] 

compact abelian group \[ \rightarrow \] rotation in gp

\[ (X, \mu) \]

\[ (G/\Gamma, m) \]

G will be nilpotent gp, \( \Gamma \) subgroup
m = haar measure