III

ERGODIC GEOMETRY
and the role of
NILPOTENT GROUPS & NILMANIFOLDS
Ergodic (Measure Preserving) System

Space $X$ endowed with measure $\mu$, $\mu(X) = 1$
and transformation $T : X \to X$
preserving the measure of (measurable) sets: $\mu(T^{-1}A) = \mu(A)$

and such that one cannot split $X = X' \cup X''$
with $\mu(X') > 0$, $\mu(X'') > 0$
$T(X') \subset X'$, $T(X'') \subset X''$

whenever $\mu(A), \mu(B) > 0$, $\exists n \in \mathbb{N}$ with
$\text{Ant}^{-n}B \neq \emptyset$ ($\implies \mu(\text{Ant}^{-n}B) > 0$)

Weak Mixing System

If $\mu(A), \mu(B), \mu(C) > 0$

$\exists n \geq 1$ and two points
$x,y \in H$ with $T^n x \in B$, $T^n y \in C$

$\implies \mu(\text{Ant}^{-n}B) > 0$, $\mu(\text{Ant}^{-n}C) > 0$
**Kronecker System**

\[ X = \mathbb{Z}, \text{ a compact abelian group} \]

which contains \( \alpha \) so that

\( \{ n\alpha, 2n\alpha, 3n\alpha, \ldots \} \) is dense in \( \mathbb{Z} \)

(\( \iff \{ \ldots, -n\alpha, 0, n\alpha, \ldots \} \) is dense)

\( \mu = \) haar measure

\( T\alpha = \alpha + \alpha \)

**Classical case:** \( Z = \mathbb{R}/\mathbb{Z} \), \( \alpha \) irrational

\( \approx [0,1) \)

\( T\alpha = \alpha \mod [0,1) \)

\( \approx S^1 = \{ z \in \mathbb{C} : |z| = 1 \}, \quad T\omega = e^{2\pi i \alpha} \)

\( \text{Or, } Z = \mathbb{R}^m/\mathbb{Z}^m \), \( m \)-torus

\( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)^T, \alpha \) is \( m \)-linear independent over \( \mathbb{Q} \)

\( \text{Or, } Z = p\text{-adic integers}, \alpha = 1 \)

\( \text{Or, } Z = \mathbb{Z}/m\mathbb{Z} = \text{finite cyclic gp. } \alpha \) is \text{ej}

\( (\text{im}) = 1 \)
FACTORS

\[ X, \mu \]

\[ \pi^{-D} \]

\[ Y, \nu \]

\[ T_y \]

\[ D \]

\[ \mu(\pi^{-D}) = \nu(D) \]

\[ T \pi(y) = \pi(T_y) \]

In a Kronecker system, each character \( \chi \) is an eigenfunction: \( \pi(x) = \chi(T_y) = \chi(y + \omega) \),

\[ = \pi(\omega) \cdot \chi(y) \]

\[ \pi(\omega) : T \chi = 2 \chi \]

THM: An ergodic system is weak mixing unless it has a nontrivial eigenfunction \( \iff \) it has a nontrivial Kronecker factor

\[ Z = \{0, 1, 2, 3, 4\} \]

\[ Z = S^1 \]
CLASSICAL (MEAN) ERGODIC THEOREM (von Neumann)

If \((X, \mathcal{B}, \mu, T)\) is any measure preserving system, \(f \in L^2(X, \mathcal{B}, \mu)\), then in \(L^2\) the limit
\[
\frac{1}{N} \sum_{n=0}^{N-1} \{T^n f(T^n x) + \cdots + T^{N-1} f\}
\]
exists.

If \((X, \mathcal{B}, \mu, T)\) is ergodic, then
\[
\frac{1}{N} \left\{ T^n f(T^n x) + \cdots + T^{N-1} f \right\} \xrightarrow{L^2} \int f \, d\mu
\]
(i.e. \(\int \left[ \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) - \int f \, d\mu \right]^2 d\mu(x) \to 0\)).

WEAK MIXING (MEAN) ERGODIC THEOREM

If \((X, \mathcal{B}, \mu, T)\) is WM system and \(f_0, \ldots, f_k\) are bounded measurable functions, then in \(L^2\)
\[
\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n} x) \cdots f_k(T^{kn} x) \to \int f_1 \, d\mu \cdots \int f_k \, d\mu
\]
This cannot hold for non-WM systems!

Let \( \psi \) be non-trivial eigenfunction

\[ \psi(Tx) = \lambda \psi(x) \]

\( \psi \) is non-constant (\( \lambda \neq 1 \)) \( |\lambda| = 1 \)

Set \( f(x) = \psi(x)^2 \) \( g(x) = \psi(x)^{-1} \)

\[ f(T^nx) = \lambda^{2n} \psi(x)^2 \]
\[ g(T^nx) = \lambda^{-2n} \psi(x)^{-1} \]

\[ f(T^nx)g(T^nx) = \psi(x) \]

\[ \left( \lim_{N \to \infty} \sum f(T^nx)g(T^nx) \right) = \psi(x) \text{ is not constant!} \]

Note: RHS is function lifted from Kronecker factor of \( (X,\mathbb{B},\mu,T) \)

\[ X \xrightarrow{\pi} X \]

\[ Z \xrightarrow{f_1} X \]

\[ X \xrightarrow{f_2} Z \]

\[ Z \xrightarrow{f_3} Z \]

\[ \text{(FIRST) Non-Conventional ERGODIC THEOREM:} \]

For any ergodic system \( (X,\mathbb{B},\mu,T) \) and bold meas \( f, g \)

\[ \left\| \sum f(T^nx)g(T^nx) \right\| \to \]

\[ \int \tilde{f}(x+2) \tilde{g}(x+2) \, dx \]

in \( L^2(X) \) where \( x = \pi(x) \)
Meaning of this theorem:
The only constant on $T^n_x, T^{3n}_x$ given $x$ derives from Kronecker factor and algebraic relation between $(x, T^n x, x-2x)$

$E \in \{(x, f, f'): f-2f'+f''=0\}$

Now suppose we have $\psi$ on $X$, $1\psi=1$ and $\psi(Tx) = \varphi(x) \psi(x)$ where $\varphi(Tx) = A \varphi(x)$

$\psi(T^n x) = A^{n(2n-1)} \varphi(x)^n \psi(x)$

Set $f = \psi^3, g = \psi^{-3}, h = \psi$

$T^n f = A^{3n(2n-1)} \varphi^{3n} \psi^3$

$T^n g = A^{-6n(2n-1)} \varphi^{-6n} \psi^{-3}$

$T^{3n} h = A^{3n(2n-1)} \varphi^{3n} \psi$

$T^n f T^m g T^{3n} h = A^{(3n^2+7n^2-3n^2+6n-3n)} \varphi^{3n-6n} \psi^n$
Example of second order eigenfunction:

\[ X = \mathbb{R}^2 / \mathbb{Z}^2 \quad T(u, v) = (u + \alpha, v + \beta) \]

\[ \varphi(u, v) = e^{2\pi i u} \quad \psi(u, v) = e^{2\pi i v} \]

\[ T\varphi = e^{2\pi i \alpha} \varphi \quad T\psi = \varphi \cdot \psi \]

Kronecker factor is \( Z = \mathbb{R}^2 / \mathbb{Z} \)

\( (u, v) \rightarrow u \)

If \( f(x) = f(u, v) \) then

\[ \tilde{f}(u) = \int f(u, v) dv : \int e^{2\pi i u} dv = 0 \]

\[ \therefore \tilde{\varphi} = 0 \]

This system is also an example of Nilsystem

\[ G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathbb{Z}, y, \beta \in \mathbb{R} \right\} \]

\[ \Gamma = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : x, y, \beta \in \mathbb{Z} \right\} \quad G / \Gamma \cong \mathbb{R}^2 / \mathbb{Z}^2 \]

as top. space

\[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ 0 & 1 & \beta \end{pmatrix} \Gamma = \begin{pmatrix} 1 & x + \alpha & y + \beta \\ 0 & 1 & \beta + \delta \end{pmatrix} \Gamma = \begin{pmatrix} 1 & x & y \\ 0 & 1 & \beta \end{pmatrix} \Gamma \]
(Second) Non-conv. Ergodic Thm: (Weiss, F.)

For any ergodic system \((X, \mathcal{B}, \mu, T)\)

\[ \exists \text{ a nilsystem factor } (G/\Gamma, \alpha) \]

so that if \(f, g, h\) are bdd meas fn's

on \(X\) and \(\tilde{f}, \tilde{g}, \tilde{h}\) are proj. on \(G/\Gamma\)

then

\[
\lim \frac{1}{N} \sum f(T^n x) \circ g(T^{2n} x) \circ h(T^{3n} x) =
\lim \frac{1}{N} \sum \tilde{f}((a^n x) \circ \tilde{g}(a^{2n} x) \circ \tilde{h}(a^{3n} x))
\]

where \(x = \pi(x) \in G/\Gamma\)

Here \(G\) is nilpotent of level 2:

\[ [[G, G], G] = 1 \text{ (every } g, g_1, g_2, g_3 \text{ commutes with every } g)\]

A third analogous result was proved
by T. Ziegler with \(G\) nilpotent of level 3.

Nilsystem: \(G\) nilpotent \(g\)-flow

\(\Gamma\) subgroup with

\(\mu(G/\Gamma) < \infty\) \(\mu\) \(G\)-inv.

\(T(g \Gamma) = ag \Gamma\)

\(T^n(g \Gamma) = a^n g \Gamma\)
Algebraic Explanation of Relevance of Nilpotent Groups

What is geometric progression in a non-commutative gp?

We would like product of geom. prog. to be geometric prog.

\[(u_1, u_1 u_2, u_1 u_2^2, u_1 u_2^3) \times (u_2, u_2 u_3, u_2 u_3^2, u_2 u_3^3) =
(u_1 u_2, u_1 u_2 u_3 u_4, u_1 u_2 (u_3 u_4)^2, \ldots)\]

**Theorem of Hall-Petresco**

Let G be any group, \(G^{(1)} = [G, G]\), \(G^{(2)} = [G^{(1)}, G]\), etc. \(G^{(n+1)} = [G^{(n)}, G]\); for any \(x, y \in G\) \(\exists z \in G\) and \(w \in G^n\)

\((x, x^2, x^3, \ldots) \times (y, y^2, y^3, \ldots) = (z, z^2 w, z^3 w^2, z^4 w^3, \ldots)\)

It turns out such expressions do form a group! (Lazard, Liebman)

\(G\) is nilpotent of level 2 iff \(G^{(2)} = 1\)
**Definition:** In any group $G$, a geometric progression (g.p.) is a sequence of the form

$$ (g, g^2, g^3w_1, g^3w_1^3w_2, \ldots, g^3w_1^3\cdots w_{n-1}) $$

where $g, z$ are any elt's of $G$ and $w_i \in G^{(i)}$, $i = 1, 2, \ldots, n-1$.

*Note:* In a nilpotent group of level $l$, $l+1$ terms determine all the rest.

**Theorem:** In any group, the term by term product of geometric progressions is a g.p., and the term by term inverse of a g.p. is a g.p.

Now let $\Gamma$ be any subgroup of $G$.

**Def:** A sequence $x_1, x_2, \ldots, x_n$ in $G/\Gamma$ is a geometric progression if it can be expressed as $g_1\Gamma, g_2\Gamma, \ldots, g_n\Gamma$ with $g_1g_2\cdots g_n \in \Gamma$. 
Fundamental Thm: If $G$ is a nilpotent group of level $l$, $\Gamma \leq G$, and $x_1, \ldots, x_e, x_{e+1}, \ldots x_n$ a geometric progression in $G/\Gamma$, then the first $e+1$ terms determine the rest.

Pf: If we have two expressions for 

$$(x_1, \ldots, x_{e+1}) = (g_1 \Gamma, \ldots, g_{e+1} \Gamma) = (g'_1 \Gamma, \ldots, g'_{e+1} \Gamma)$$

with \{g_i \Gamma \neq g'_i \Gamma\} forming $g\text{p}\text{r}\text{s}$, then

$g_1^{\prime^{-1}}g_1, g_2^{\prime^{-1}}g_2, \ldots, g_{e+1}^{\prime^{-1}}g_{e+1} = \gamma_1, \gamma_2, \ldots, \gamma_e$ which is a $g\text{p}\text{r}$ in $\overline{\Gamma}$, the unique extension of this is still in $\Gamma$. Therefore for the unique continuations $g_{e+1}, g_{e+2}, \ldots, g_n$

$g_{e+1}, \ldots, g_n, \quad g_j^{\prime^{-1}}g_j \in \Gamma \Rightarrow g_j \Gamma = g_j' \Gamma$.

Therefore:

In a nilpotent system for $G$ nilpotent of level $l$: $(G/\Gamma, a)$ $Tg\Gamma = ag\Gamma$

The $e+1$ terms of $(g\Gamma, a^g\Gamma, a^{2g}\Gamma, \ldots, a^{eg}\Gamma, \ldots)$ determine all the rest.
What is a parallelepiped (prlpd) of dimension d?

$\mathbb{R}^m$ acts on $V$, on $V \times V$, on $(V \times V) \times (V \times V)$, etc.

$T_u (v_1, v_2, \ldots, v_d) = (v_1+u, v_2+u, \ldots, v_d+u)$

It is transitive on V but not on the rest.

Write $(v_1, \ldots) \cong (v'_1, \ldots)$ if $\exists u$

with $(v_1, \ldots) = T_u (v'_1, \ldots)$

A prlpd of dim 1 is a pair $(u, v_0)$

with $u \cong v_0$ (so it's any pair)

A prlpd of dim 2 is a 2-tuple $(u, v_0, v_0)$

with $\langle u, v_0 \rangle \cong \langle v_0, v_0 \rangle$ 2-prlpd

dim 3: $(u, v_0, v_0, v_1, v_0, u, v_0, v_0)$

$(u, v_0, v_0, v_0, v_0) \cong (u, v_1, u, v_0)$ 3-prlpd

ETC.
Denote by $\Pi^{[d]} \subset V^{2d}$ the subset of $d$-palais in $V^{2d}$

**Theorem** (Coordinate permutation):

- The map $(u_1 u_2) \rightarrow (u_2 u_1)$
  takes $\Pi^{[3]} \rightarrow \Pi^{[3]}$ ($= V \times V$)
- The map $(u_1 u_2 u_3 u_4) \rightarrow (u_1 u_3 u_2 u_4)$
  takes $\Pi^{[4]} \rightarrow \Pi^{[4]}$ ($\subset V \times V \times V \times V$)

The maps

$(u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8) \rightarrow (u_1 u_2 u_3 u_6 u_5 u_4 u_7 u_8)$

$t \rightarrow (u_1 u_5 u_2 u_7 u_6 u_4 u_3 u_8)$

take $\Pi^{[8]} \rightarrow \Pi^{[8]}$ ($\subset V^8$)

\[ \cdots \]

ETC.

Note: All this makes sense for any group action!
Mackey's Philosophy & Ergodic PRLPD Geometry

Ergodicity was once called metric transitivity.

Mackey: Think of ergodicity as actual transitivity. (So $X = G/L$ - a "virtual" subgroup)

Ergodic action $\leftrightarrow$ transitive
Ergodic component $\leftrightarrow$ orbit

Recall: Any meas. pres. action decomposes to ergodic "components".

These are parametrized by some measure space

$\mu = \int_{\mathcal{J}} \gamma \, d\mathcal{J}(\gamma)$

(This is well defined in "almost everywhere" sense)
HOST-XRA THEORY

Assume \((X, \mathcal{B}, \mu, T)\) is ergodic

\(X \times X\) (shorthand for \((X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \otimes \mu, T)\))

may or may not be ergodic

1. property: all pairs \((x, x_0)\)

\[ T^{(i)} x = x \]

this carries measure \(\mu^{\otimes 2} = \mu \times \mu\)

2. property: all \((x, x_0, x_1, x_2, x_3)\) with \((x, x_0) \in (x, x_1)\)

belonging to same ergodic component \(\Pi^{(i)} \subset X^4\)

\[ \mu^{\otimes 4} = \int (\mu \times \mu)^2 \times (\mu \times \mu) \, d\theta(z) \]

where

\[ \mu \times \mu = \int (\mu \times \mu) \, d\theta(z) \]

is ergodic decomp.

3. property: all \((x, x_0, \ldots, x_p)\) with \((x, \ldots, x_p) \in (x, \ldots, x_p)\)

in same ergodic component of

\((\Pi^{(i)} \times \mathcal{B}, \mu^{\otimes p})\)

\[ \mu^{\otimes p} = \int \mu^{\otimes p} \times \mu^{\otimes p} \, d\theta(y) \]
$C_y$ a particular ergodic comp. with measure $(\mu \times \mu)$,

$C_y \times C_y \subset \mathbb{T}[2]

$\mathbb{T}[2]$ of 2-prlepcl's (parallelograms) is union of $C_y \times C_y$

Measure $\mu^{[2]}$ on $\mathbb{T}[2]$ is integral of $(\mu \times \mu) \times (\mu \times \mu)$

Theorem If $(X, \beta, \mu, T)$ is WM then

$\mu^{[2]}(x) = (\mu \times \mu \times \mu \times \mu)(x)$

on $\mathbb{T}[2] = X^{2d}$

So there is no interesting geometry
(Coordinate permutation) **Theorem:**

The map \((u, u_d) \to (u, u)\) is measure preserving on \((\mathbb{T}^6, \mathcal{B}, \mu)\).

The map \((u, u_2, u_3, u_v) \to (u, u_3, u_2, u_v)\) is meas. pres. on \((\mathbb{T}^{12}, \mu)\).

The maps

\[(u, u_2, u_3, u_4, u_5, u_6, u_7, u_8) \to (u, u_2, u_3, u_4, u_5, u_6, u_7, u_8)\]

\[(u, u_3, u_2, u_4, u_5, u_6, u_7, u_8) \to (u, u_3, u_4, u_5, u_6, u_7, u_8)\]

are meas. preserving on \((\mathbb{T}^{12}, \mu)\).

**Etc.**

**Kronecker systems**

\((\mathbb{Z}, u)\) is ergodic.

\((\mathbb{Z} \times \mathbb{Z}, (u, u))\) is not ergodic; its decomposition is by cosets modulo diagonal.

So \((z, z_2) \cong (z_3, z_4) \Rightarrow z_2 - z_3 = z_4 - z_3\)
In passage to factors $p_k\rho p_k$'s $\rightarrow p_k\rho p_k$'s

**Theorem** For any ergodic system $(X, \mathbb{R}, \mu, \tau)$ the $s$-prepd's are determined by projection to its Kronecker, and $\Pi_X^{[s]}$ is lift of $\Pi_Z^{[s]}$ and $\mu_X^{[s]}$ is lift of $\mu_Z^{[s]}$.

\[
X \quad \mu = \int \mu_Z^s \, d\zeta
\]

\[
Z \quad \mu_X^{[s]} = \int_{\Pi_Z^{[s]}} \mu_Z^s \cdot \nu_Z^s \, d\nu_Z^s
\]

Something similar happens in each dimension

**Theorem** For any ergodic system $(X, \mathbb{R}, \mu, \tau)$ and for each $d$, the $d$-prepd's are determined by projection to a "characteristic factor" $W_d$, so that

\[
\Pi_X^{[d]} = \pi_d^{-1}(\Pi_{W_d}^{[d]}) \quad \text{and} \quad \mu_X^{[d]} = \int_{\Pi_{W_d}^{[d]}} \mu_{W_d}^s \cdot \nu_{W_d}^s \, d\nu_{W_d}^s
\]
Ergodic prepd Geometry produces a tower of characteristic factors

\[ \begin{array}{ccc}
X & \rightarrow & \cdots \\
W_1 & \downarrow & W_2 \\
W_2 & \downarrow & W_3 \\
W_3 & \downarrow & W_4 \\
W_4 & \downarrow & \vdots \\
W_\infty & \downarrow & \vdots \\
\end{array} \]

\[ W_2 \equiv Z = \text{Kronecker} \]

I. Each \( W_d \) is a characteristic for \((x, T^m x, T^{2m} x, \ldots, T^{dm} x)\).

That means that all constraints on these \((dm)\)-tuples are determined by their images \((3, T^3, T^{2m} x, \ldots, T^{dm} x)\) in \( W_d \).

OR...
Thm. If \( f_0, \ldots, f_d \) are odd meas fn's on \( X \) and \( \bar{f}_0, \ldots, \bar{f}_d \) are their projections on \( W_d \):

\[
\frac{1}{N} \sum f_i (T^n x) = \frac{1}{N} \sum \bar{f}_i (T^n x) \to 0
\]

in \( L^2(X) \) where \( \bar{z} = \text{proj}_{x \to W_d} (z) \)

II \( W_d \) is a nilsystem of level \( d-1 \)

Where does the nilpotent group come from?

For any ergodic system \((X, \mathcal{B}, \mu, T)\) we define the \( H-K \) group \( G(X) \) to be the group of ALL measure preserving trans. of \( X \) (like \( T^n \)) compatible with the geometry:
By coordinate permutation: if $g: \mathbb{R}^m \to \mathbb{R}^m$ preserves causal order $x < y \iff x \cdot y \leq 0$, so does $g' = x \mapsto g(x - y)$. If $g' \in G$, then for each $\gamma \in \Gamma$ given, \[ G \gamma G \cap \Gamma = G \gamma \Gamma G \] is a subgroup of $G \gamma \Gamma G$, so $g \in G \gamma G$. Therefore, $G \gamma G$.

Similarly, \[ g \in G \gamma G \iff g' \in G \gamma G \]

Taking commutators, if $g = [g_1, g_2]$ and $g_1, g_2 \in G$ preserves causal order, so does $g' = x \mapsto g(x - y)$. If $g' \in G$, then for each $\gamma \in \Gamma$ given, \[ G \gamma G \cap \Gamma = G \gamma \Gamma G \] is a subgroup of $G \gamma \Gamma G$, so $g \in G \gamma G$. Therefore, $G \gamma G$.
From this we deduce:

**Theorem** $G(W_2)$ is commutative

Idea of proof: if $g_1, g_2, g_3, g_4 \in [G,G]$ and $(x_1, x_2, x_3, x_4)$ is a $2$-pplpd

so are $(g_1 x_1, x_2, x_3, x_4)$ &

$(g_2 x_1, g_2 x_2, x_3, x_4)$ &

$(g_2 x_1, g_2 x_2, g_2 x_3, x_4)$

Defining $x \sim y$ if $y = g x, g \in [G,G]$

then $2$-pplpds on $W_2$ are determined

by $W_2/\sim$. If this relation weren't trivial could reduce $W_2$

Similarly

**Theorem** $G(W_3)$ is nilpotent of level 2.

Pf: the triple commutator $[[,[ , ]]]$ of $g_1, g_1, g_2, g_2, g_2 = 1 \times 1 \times 1 \times 1, g_2, g_2 = 1 \times 1 \times 1 \times 1, g_2, g_2 = 1 \times 1 \times 1 \times 1, g_2, g_2 = 1 \times 1 \times 1 \times 1$ is $L_{\alpha}, L_{\beta}, L_{\alpha}, L_{\beta}, L_{\alpha}, L_{\beta}$.