PROBLEMS IN PRACTICE TEST 1

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1. In the xy-plane, the curve with parametric equations $x = \cos t$ and $y = \sin t$, $0 \leq t \leq \pi$, has length

(A) 3
(B) π
(C) 3π
(D) $3/2$
(E) $\pi/2$

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2. Which of the following is an equation of the line tangent to the graph of $y = x + e^x$ at $x = 0$.

(A) $y = x$
(B) $y = x + 1$
(C) $y = x + 2$
(D) $y = 2x$
(E) $y = 2x + 1$

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3. If V and W are 2-dimensional subspaces of $\mathbb{R}^4$, what are the possible dimensions of $V \cap W$?

(A) 1 only
(B) 2 only
(C) 0 and 1 only
(D) 0, 1 and 2 only
(E) 0, 1, 2, 3 and 4

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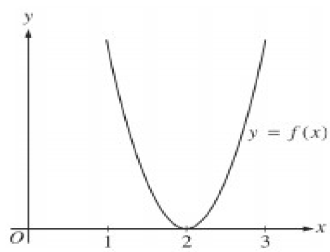
4. Let k be the number of real solutions of the equation $e^x + x - 2 = 0$ in the interval $[0, 1]$, and let n be the number of real solutions that are <u>not</u> in $[0, 1]$. Which of the following is true?

(A) $k = 0$ and $n = 1$
(B) $k = 1$ and $n = 0$
(C) $k = n = 1$
(D) $k > 1$
(E) $n > 1$



5. Suppose b is a real number and $f(x) = 3x^2 + bx + 12$ defines a function on the real line, part of which is graphed above. Then $f(5) =$

(A) 15
(B) 27
(C) 67
(D) 72
(E) 87

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6. Which of the following circles has the greatest number of points of intersection with the parabola $x^2 = y + 4$.

(A) $x^2 + y^2 = 1$
(B) $x^2 + y^2 = 2$
(C) $x^2 + y^2 = 9$
(D) $x^2 + y^2 = 16$
(E) $x^2 + y^2 = 25$

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7. $\int_{-3}^{3} |x+1|\,dx =$

(A) 0
(B) 5
(C) 10
(D) 15
(E) 20

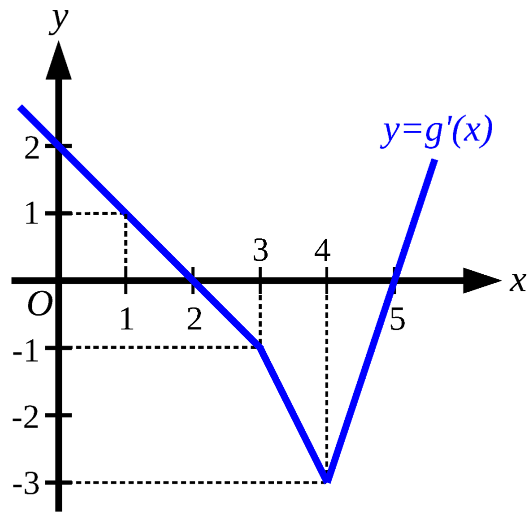
8. What is the greatest possible area of a triangular region with one vertex at the center of a circle of radius 1 and the other two vertices on the circle?

(A) $1/2$
(B) 1
(C) $\sqrt{2}$
(D) π
(E) $(1+\sqrt{2})/4$

$$\begin{aligned} J &= \int_0^1 \sqrt{1-x^4}\,dx \\ K &= \int_0^1 \sqrt{1+x^4}\,dx \\ L &= \int_0^1 \sqrt{1-x^8}\,dx \end{aligned}$$

9. Which of the following is true for the definite integral shown above?

(A) $J < L < 1 < K$
(B) $J < L < K < 1$
(C) $L < J < 1 < K$
(D) $L < J < K < 1$
(E) $L < 1 < J < K$



10. Let g be a function whose derivative g' is continuous and has the graph shown above. Which of the following values of g is the largest?

(A) $g(1)$
(B) $g(2)$
(C) $g(3)$
(D) $g(4)$
(E) $g(5)$

11. Of the following, which is the best approximation of

$$\left[\sqrt{1.5}\,\right]\left[(266)^{3/2}\right]?$$

(A) $1,000$
(B) $2,700$
(C) $3,200$
(D) $4,100$
(E) $5,300$

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12. Let A be a 2×2 matrix for which there is a constant k such that the sum of the entries in each row and each column is k. Which of the following must be an eigenvector of A?

I. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

II. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

III. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(A) I only
(B) II only
(C) III only
(D) I and II only
(E) I, II and III

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13. A total of x feet of fencing is to form three sides of a level rectangular yard. What is the maximum possible area of the yard, in terms of x?

(A) $x^2/9$
(B) $x^2/8$
(C) $x^2/4$
(D) x^2
(E) $2x^2$

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14. What is the units digit in the standard decimal expansion of the number 7^{25}?

(A) 1
(B) 3
(C) 5
(D) 7
(E) 9

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15. Let f be a continuous real-valued function defined on the closed interval $[-2, 3]$. Which of the following is NOT necessarily true?

(A) f is bounded.
(B) $\int_{-2}^{3} f(t)\,dt$ exists.
(C) For each c between $f(-2)$ and $f(3)$, there is an $x \in [-2, 3]$ such that $f(x) = c$.
(D) There is an M in $f([-2, 3])$ such that $\int_{-2}^{3} f(t)\,dt = 5M$.
(E) $\lim_{h\to 0} \frac{[f(h)] - [f(0)]}{h}$ exists.

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16. What is the volume of the solid formed by revolving, about the x-axis, the region in the first quadrant of the xy-plane bounded by: the coordinate axes and the graph of the equation $y = \frac{1}{\sqrt{1+x^2}}$?

(A) $\pi/2$
(B) π
(C) $\pi^2/4$
(D) $\pi^2/2$
(E) ∞

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17. How many real roots does the polynomial $2x^5 + 8x - 7$ have?

(A) None
(B) One
(C) Two
(D) Three
(E) Five

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18. Let V be the real vector space of all real 2×3 matrices. Let W be the real vector space of all real 4×1 column vectors. If T is a linear transformation from V onto W, what is the dimension of the subspace $\{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$ of V?

(A) 2
(B) 3
(C) 4
(D) 5
(E) 6

19. Let f and g be twice-differentiable real-valued functions defined on $\mathbb{R}$. Assume, for all $x > 0$, that $f'(x) > g'(x)$. Then which of the following inequalities must be true for all $x > 0$?

(A) $f(x) > g(x)$
(B) $f''(x) > g''(x)$
(C) $[f(x)] - [f(0)] > [g(x)] - [g(0)]$
(D) $[f'(x)] - [f'(0)] > [g'(x)] - [g'(0)]$
(E) $[f''(x)] - [f''(0)] > [g''(x)] - [g''(0)]$

20. Let f be the function defined on the real line by

$$f(x) = \begin{cases} x/2, & \text{if } x \text{ is rational;} \\ x/3, & \text{if } x \text{ is irrational.} \end{cases}$$

If D is the set of points of discontinuity of f, then D is the

(A) empty set
(B) set of rational numbers
(C) set of irrational numbers
(D) set of nonzero real numbers
(E) set of real numbers

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21. Let P_1 be the set of all primes, $\{2, 3, 5, 7, \ldots\}$, and, for each integer n, let P_n be the set of all prime multiples of n, $\{2n, 3n, 5n, 7n, \ldots\}$. What of the following intersections is nonempty?

(A) $P_1 \cap P_{23}$
(B) $P_7 \cap P_{21}$
(C) $P_{12} \cap P_{20}$
(D) $P_{20} \cap P_{24}$
(E) $P_5 \cap P_{25}$

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22. Let $C(\mathbb{R})$ be the collection of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Then $C(\mathbb{R})$ is a real vector space with vector addition defined by

$$\forall f, g \in C(\mathbb{R}),\ \forall x \in \mathbb{R}, \qquad (f+g)(x) \quad = \quad [f(x)] \ + \ [g(x)],$$

and with scalar multiplication defined by

$$\forall f \in C(\mathbb{R}),\ \forall r, x \in \mathbb{R}, \qquad (rf)(x) \quad = \quad r \cdot [f(x)].$$

Which of the following are subspaces of $\mathbb{R}$?

I. $\{f : f \text{ is twice differentiable and } f'' - 2f' + 3f = 0\}$
II. $\{g : g \text{ is twice differentiable and } g'' = 3g'\}$
III. $\{h : h \text{ is twice differentiable and } h'' = h + 1\}$

(A) I only
(B) I and II only
(C) I and III only
(D) II and III only
(E) I, II and III

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23. For what value of b is the line $y = 10x$ tangent to the curve $y = e^{bx}$ at some point in the xy-plane?

(A) $10/e$
(B) 10
(C) $10e$
(D) e^{10}
(E) e

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24. Let h be the function defined by $h(x) = \int_0^{x^2} e^{x+t}\, dt$, for all real numbers x. Then $h'(1) =$

(A) $e - 1$
(B) e^2
(C) $e^2 - e$
(D) $2e^2$
(E) $3e^2 - e$

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25. Let $\{a_n\}_{n=1}^{\infty}$ be defined recursively by $a_1 = 1$ and

$$\text{for all integers } n \geq 1, \qquad a_{n+1} = \left(\frac{n+2}{n}\right) a_n.$$

Then a_{30} is equal to

(A) (15)(31)
(B) (30)(31)
(C) 31 / 29
(D) 32 / 30
(E) [32!] / [(30!)(2!)]

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26. For all real x and y, let $f(x, y) = x^2 - 2xy + y^3$. Which of the following is true?

(A) f has all of its relative extrema on the line $x = y$.
(B) f has all of its relative extrema on the parabola $x = y^2$.
(C) f has a relative minimum at $(0, 0)$.
(D) f has an absolute minimum at $(2/3, 2/3)$.
(E) f has an absolute minimum at $(1, 1)$.

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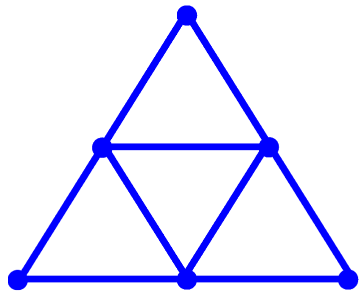
27. Consider the two planes $x + 3y - 2z = 7$ and $2x + y - 3z = 0$ in $\mathbb{R}^3$. Which of the following sets is the intersection of these two planes?

(A) $\emptyset$
(B) $\{(0, 3, 1)\}$
(C) $\{(x, y, z) \mid x = t, y = 3t, z = 7 - 2t, t \in \mathbb{R}\}$
(D) $\{(x, y, z) \mid x = 7t, y = 3 + t, z = 1 + 5t, t \in \mathbb{R}\}$
(E) $\{(x, y, z) \mid x - 2y - z = -7\}$



28. The figure above shows an undirected graph with six vertices. Enough edges are to be deleted from the graph in order to leave a spanning tree, which is a connected subgraph having the same six vertices and no cycles. How many edges must be deleted?

(A) One
(B) Two
(C) Three
(D) Four
(E) Five

29. For all positive functions f and g of the real variable x, let $\sim$ be a relation defined by

$$f \sim g \qquad \text{if and only if} \qquad \lim_{x\to\infty}\left[\frac{f(x)}{g(x)}\right] = 1.$$

Which of the following is NOT a consequence of $f \sim g$?

(A) $f^2 \sim g^2$
(B) $\sqrt{f} \sim \sqrt{g}$
(C) $e^f \sim e^g$
(D) $f + g \sim 2g$
(E) $g \sim f$

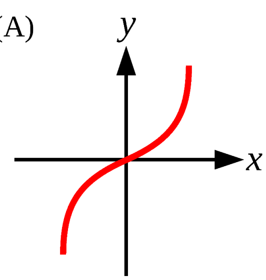
30. Let f be a function from a set X to a set Y. Consider the following statements.

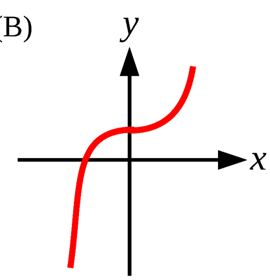
P : For each $x \in X$, there exists $y \in Y$ such that $f(x) = y$.
Q : For each $y \in Y$, there exists $x \in X$ such that $f(x) = y$.
R : There exist $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

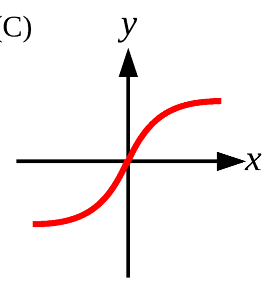
The negation of the statement "f is one-to-one and onto Y" is

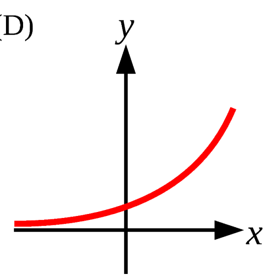
(A) P or (not R)
(B) R or (not P)
(C) R or (not Q)
(D) P and (not R)
(E) R and (not Q)

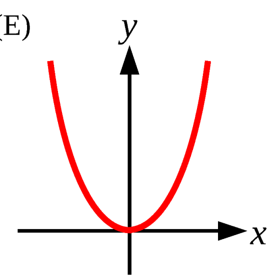
31. Which of the following most closely represents the graph of a solution to the differential equation $\dfrac{dy}{dx} = 1 + y^4$?

(A)


(B)


(C)


(D)


(E)


32. Suppose that two binary operations, denoted by $\oplus$ and $\odot$, are defined on a nonempty set S. Suppose, further, that the following conditions are satisfied:

(1) $\forall x, y \in S$, $x \oplus y \in S$ and $x \odot y \in S$.
(2) $\forall x, y, z \in S$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ and $(x \odot y) \odot z = x \odot (y \odot z)$.
(3) $\forall x, y \in S$, $x \oplus y = y \oplus x$.

For each $x \in S$ and each integer $n \geq 1$, the elements $nx, x^n \in S$ are defiend recursively by

- $1x = x^1 = x$ and
- $\forall$integer $k \geq 1$, $(k+1)x = (kx) \oplus x$ and $x^{k+1} = x^k \odot x$.

Which of the following must be true?

I. $\forall x, y \in S$, $\forall$integer $n \geq 1$, $(x \odot y)^n = x^n \odot y^n$.
II. $\forall x, y \in S$, $\forall$integer $n \geq 1$, $n(x \oplus y) = (nx) \oplus (ny)$.
III. $\forall x \in S$, $\forall$integers $m, n \geq 1$, $x^m \odot x^n = x^{m+n}$.

(A) I only
(B) II only
(C) III only
(D) II and III only
(E) I, II and III

33. The Euclidean algorithm, described below, is used to find the greatest common divisor (GCD) of two positive integers a and b.

```
INPUT(a)
INPUT(b)
WHILE b > 0
  BEGIN
    r := a mod b
    a := b
    b := r
  END
GCD:=a
OUTPUT(GCD)
```

When the algorithm is used to find the greatest common divisor of $a = 273$ and $b = 110$, which of the following is the sequence of computed values for r?

(A) 2, 26, 1, 0
(B) 2, 53, 1, 0
(C) 53, 2, 1, 0
(D) 53, 4, 1, 0
(E) 53, 5, 1, 0

34. The minimal distance between any point on the sphere

$$(x-2)^2 + (y-1)^2 + (z-3)^2 = 1$$

and any point on the sphere

$$(x+3)^2 + (y-2)^2 + (z-4)^2 = 4$$

is

(A) 0
(B) 4
(C) $\sqrt{27}$
(D) $2\left(\sqrt{2}+1\right)$
(E) $3\left(\sqrt{3}-1\right)$

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42. Let X and Y be discrete random variables on the set of positive integers. Assume, for each integer $n \geq 1$, that the probability that $X = n$ is 2^{-n}. Assume that Y has the same probability distribution as X, *i.e.*, assume, for each integer $n \geq 1$, that the probability that $Y = n$ is also 2^{-n}. Assume that X and Y are independent. What is the probability that at least one of the variables X and Y is greater than 3?

(A) 1/64
(B) 15/64
(C) 1/4
(D) 3/8
(E) 4/9

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46. Let G be the group of complex numbers $\{1, i, -1, -i\}$ under multiplication. Which of the following statements are true about the homomorphisms of G into itself?

I. $z \mapsto \overline{z}$ defines one such homomorphism, where $\overline{z}$ denotes the complex conjugate of z.
II. $z \mapsto z^2$ defines one such homomorphism
III. For every such homomorphism, there is an integer k such that the homomorphism has the form $z \mapsto z^k$.

(A) None
(B) II only
(C) I and II only
(D) II and III only
(E) I, II and III

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49. Up to isomorphism, how many additive Abelian groups G of order 16 have the property that, for all $x \in G$, $x + x + x + x = 0$?

(A) 0
(B) 1
(C) 2
(D) 3
(E) 5

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59. Let f be an analytic function of a complex variable $z = x + iy$ given by

$$f(z) = (2x + 3y) + i \cdot (g(x, y)),$$

where $g(x, y)$ is a real-valued function of the real variables x and y. If $g(2, 3) = 1$, then $g(7, 3) =$

(A) -14
(B) -9
(C) 0
(D) 11
(E) 18
