1-1. Let $C$ be the part of the graph of $y = \ln(\cos x)$ between $x = 0$ and $x = \pi/4$. Find the length of $C$.

**Solution:** We parametrize the curve $C$ by $x = t$ and $y = \ln(\cos t)$, for $0 \leq t \leq \pi/4$. The velocity is given by $\dot{x} = 1$ and $\dot{y} = \tan t$. For $0 < t < \pi/4$, the speed is given by $[1^2 + (\tan t)^2]^{1/2} = [\sec^2 t]^{1/2} = \sec t$. The length of $C$ is then

$$\int_0^{\pi/4} \sec t \, dt = \left[ \ln((\tan t) + (\sec t)) \right]_{t=0}^{t=\pi/4} = \left[ \ln \left(1 + \sqrt{2}\right) \right] - \ln(0 + 1) = \ln \left(1 + \sqrt{2}\right). \quad \square$$

1-2. In $xyz$-space, let $C$ be the curve with parametric equations $x = 2t$, $y = t^2$ and $z = t^3/3$, $0 \leq t \leq 1$. Find the length of $C$.

**Solution:** The velocity is given by $\dot{x} = 2$, $\dot{y} = 2t$ and $\dot{z} = t^2$. The speed is given by $[2^2 + (2t)^2 + (t^2)^2]^{1/2} = [4 + 4t^2 + t^4]^{1/2} = 2 + t^2$. The length of $C$ is then

$$\int_0^1 (2 + t^2) \, dt = \left[ 2t + \frac{t^3}{3} \right]_{t=0}^{t=1} = 2 + \frac{1}{3} = \frac{7}{3}. \quad \square$$

2-1. Give an equation of the line tangent to the graph of $y = 5x + \sin x$ at $x = \pi$.

**Solution:** The slope is

$$[(d/dx)(5x + \sin x)]_{x=\pi} = [5 + \cos x]_{x=\pi} = 5 + (-1) = 4.$$

The $y$-coordinate of the point of tangency is

$$[5x + \sin x]_{x=\pi} = 5\pi + 0 = 5\pi.$$

so the point of tangency is $(\pi, 5\pi)$. An equation of the line is therefore $y - 5\pi = 4(x - \pi)$, or, equivalently, $y = 4x + \pi$. \quad \square
3-1. If $V$ is a 3-dimensional subspace of $\mathbb{R}^7$ and $W$ is a 5-dimensional subspace of $\mathbb{R}^7$, what are the possible dimensions of $V \cap W$?

**Solution:** The minimum possible dimension of $V + W$ is

$$\max\{\dim V, \dim W\} = \max\{3, 5\} = 5.$$  

The maximum possible dimension of $V + W$ is

$$\min\{(\dim V) + (\dim W), \dim \mathbb{R}^7\} = \min\{3 + 5, 7\} = 7.$$  

So the set of possible dimensions of $V + W$ is $\{5, 6, 7\}$. As

$$(\dim V) + (\dim W) = (\dim (V + W)) + (\dim (V \cap W)),$$

we see that $8 = 3 + 5 = (\dim (V + W)) + (\dim (V \cap W))$, so

$$\dim (V \cap W) = 8 - (\dim (V + W)).$$  

Then the set of possible dimensions of $V \cap W$ is

$$\{8 - 5, 8 - 6, 8 - 7\} = \{3, 2, 1\}.$$  

□

4-1. Let $k$ be the number of real solutions of the equation $7 - x^5 - x = 0$ in the interval $[0, 1]$, and let $n$ be the number of real solutions that are not in $[0, 1]$. Which of the following is true?

(A) $k = 0$ and $n = 1$
(B) $k = 1$ and $n = 0$
(C) $k = n = 1$
(D) $k > 1$
(E) $n > 1$

**Solution:** Since $[7 - x^5 - x]_{x\to-\infty} > 0$ and $[7 - x^5 - x]_{x\to\infty} < 0$, it follows that the equation $7 - x^5 - x = 0$ has at least one solution in $\mathbb{R}$, so $k + n \geq 1$. For all $x \in \mathbb{R}$, $[d/dx][7 - x^5 - x] = -5x^4 - 1 < 0$, so $7 - x^5 - x$ is decreasing in $x$. Thus the equation $7 - x^5 - x = 0$ has at most solution in $\mathbb{R}$, so $k + n \leq 1$. Then $k + n = 1$.

Since $[7 - x^5 - x]_{x\to0} = 7 > 0$, since $[7 - x^5 - x]_{x\to1} = 7 - 1 - 1 > 0$ and since $7 - x^5 - x$ is decreasing in $x$, it follows that the equation $7 - x^5 - x = 0$ has no solutions in $[0, 1]$. Then $k = 0$.

Then $n = (k + n) - k = 1 - 0 = 1$. Answer: (A)  

□
5-1. Suppose $b$ is a real number and $f(x) = 4x^2 + bx + 9$ defines a function on the real line, part of which is graphed above. Compute $f(5)$.

**Solution:** We have $f'(x) = 8x + b$, so $f'(2) = 16 + b$. From the graph, $f'(2) = 0$. Thus $16 + b = 0$, so $b = -16$. Then $f(x) = 4x^2 - 16x + 9$, so $f(5) = 4 \cdot 5^2 - 16 \cdot 5 + 9 = 100 - 80 + 9 = 29$. □

6-1. For what values of $b$ does the curve $4x^2 + (y - b)^2 = 1$ have exactly one intersection point with $y = 2x$?

**Solution:** Let $b \in \mathbb{R}$. For all $x, y \in \mathbb{R}$,

\[
\begin{aligned}
&[ (4x^2 + (y - b)^2 = 1) \text{ and } (y = 2x) ] \iff \\
&[ (y^2 + (y - b)^2 = 1) \text{ and } (y = 2x) ] \iff \\
&[ (y^2 + y^2 - 2yb + b^2 = 1) \text{ and } (x = y/2) ] \iff \\
&[ (2y^2 - 2by + (b^2 - 1) = 0) \text{ and } (x = y/2) ] \iff \\
\end{aligned}
\]

The expression $2y^2 - 2by + (b^2 - 1)$ is a quadratic in $y$ with coefficients $2, -2b, b^2 - 1$.

The discriminant of this quadratic is

\[
(-2b)^2 - 4 \cdot 2 \cdot (b^2 - 1) = 4b^2 - 8(b^2 - 1) = -4b^2 + 8.
\]

Thus

\[
\begin{aligned}
[ \text{there is one intersection point} ] &\iff \\
[ -4b^2 + 8 = 0 ] &\iff [ b^2 = 2 ] \iff \\
[ (b = \sqrt{2}) \text{ or } (b = -\sqrt{2}) ] &. \quad \Box
\end{aligned}
\]
7-1. Compute \( \int_{-3}^{3} e^{x+1} \, dx \).

\textbf{Solution:} The integral is equal to
\[
\left[ \int_{-3}^{-1} e^{x+1} \, dx \right] + \left[ \int_{-1}^{3} e^{x+1} \, dx \right] = \left[ -e^{-x-1} \right]_{x=-3}^{x=-1} + \left[ e^{x+1} \right]_{x=-1}^{x=3} = [-e^{0} - (-e^{2})] + [e^{4} - e^{0}] = -1 + e^{2} + e^{4} - 1 = e^{4} + e^{2} - 2. \]

8-1. Let \( R \) be a rectangle whose vertices are \((x, y), (-x, y), (-x, 0)\) and \((x, 0)\). Assume that \(0 < x < 3\), that \(0 < y < 3\) and that \(x^{4} + y^{4} = 1\). What is the maximum possible area inside such a rectangle \( R \)?

\textbf{Solution:} The rectangle has width \(2x\) and has height \(y\). Therefore, since \(y = (1 - x^{4})^{1/4}\), the area is \(2xy = 2x(1 - x^{4})^{1/4}\). We have
\[
f'(x) = \frac{2(1 - x^{4})^{1/4}}{(1 - x^{4})^{3/4}} + \frac{2x(1/4)(1 - x^{4})^{-3/4}(-4x^{3})}{(1 - x^{4})^{3/4}} = \frac{2(1 - x^{4}) + [-2x^{4}]}{(1 - x^{4})^{3/4}} = \frac{2(1 - 2x^{4})}{(1 - x^{4})^{3/4}}.
\]
Then \(f'(x) > 0\) on \(0 < x < 2^{-1/4}\). Also, \(f'(x) < 0\) on \(2^{-1/4} < x < 1\). Thus \(f(x)\) is increasing on \(0 < x < 2^{-1/4}\). Also, \(f(x)\) is decreasing on \(2^{-1/4} < x < 1\). Then \(f(x)\) attains a global maximum at \(x = 2^{-1/4}\). So, since \(f(x) = 2x(1 - x^{4})^{1/4}\), the maximum value is
\[
f(2^{-1/4}) = 2 \cdot 2^{-1/4} \cdot (1 - 2^{-1})^{1/4} = 2 \cdot 2^{-1/4} \cdot (1/2)^{1/4} = 2^{1-(1/4)-(1/4)} = 2^{1/2} = \sqrt{2}. \]
9-1. Define

\begin{align*}
J & := \int_{1}^{2} \sqrt{256 - x^4} \, dx \\
K & := \int_{1}^{2} \sqrt{256 + x^4} \, dx \\
L & := \int_{1}^{2} \sqrt{256 - x^8} \, dx
\end{align*}

Order 16, $J$, $K$, $L$ from smallest to largest.

*Solution:* On $1 < x < 2$, we have:

\[-256 < -x^8 < -x^4 \prec 0 < x^4,\]

which implies \[0 < 256 - x^8 < 256 - x^4 < 256 < 256 + x^4,\]

which implies \[\sqrt{256 - x^8} < \sqrt{256 - x^4} < 16 < \sqrt{256 + x^4}.

Thus $L < J < 16 < K$. \hfill \square

10-1. Let $g$ be a function whose derivative $g'$ is continuous and has the graph shown above. On $0 < x < 5$, what are the maximal open intervals of concavity for $g(x)$?

*Solution:* Since $g'(x)$ is decreasing on $0 < x < 4$, it follows that $g(x)$ is concave down on $0 < x < 4$. Since $g'(x)$ is increasing on $4 < x < 5$, it follows that $g'(x)$ is concave up on $4 < x < 5.$
It follows, on $0 < x < 5$, that the only maximal open interval of concave down for $g(x)$ is $0 < x < 4$. It also follows, on $0 < x < 5$, that the only maximal open interval of concave up for $g(x)$ is $4 < x < 5$. □

11-1. Approximate $\left[ 3.59 \right] \left( (10)^{5/2} \right)$.

Solution: We have:

\[
\begin{align*}
\left[ \sqrt{3.59} \right] \left( (10)^{5/2} \right) &= \left[ \sqrt{3.59} \right] \left[ \sqrt{(10)^5} \right] \\
&= \left[ \sqrt{3.59} \right] \left[ \sqrt{10} \right] \left[ \sqrt{(10)^4} \right] \\
&= \left[ \sqrt{(3.59)(10)} \right] [10^2] \\
&= \left[ \sqrt{35.9} \right] [100] \\
&\approx < \left[ \sqrt{36} \right] [100] \\
&= [6] [100] \\
&= 600 \quad □
\end{align*}
\]

NOTE: Two two decimals, the exact answer is 599.17.

12-1. Let $A$ be a $5 \times 5$ matrix such that the entries in each row add up to 10. Let $B := 6A^3 + 4A^2 + 7A$. True or False: The entries any row of $B$ will add up to 6470.

Solution: True. Proof: Let $v$ be the row vector $\left[ 1 \ 1 \ 1 \ 1 \ 1 \right]$. Let $w := v^t$ be the transpose of $v$, so that $w$ is a $5 \times 1$ column vector, with all entries equal to 1. Since sum of the entries in each row of $A$ is 10, we get $Aw = 10w$. Then $A^2w = 100w$ and $A^3w = 1000w$. Then

\[
Bw = 6A^3w + 4A^2w + 7Aw = 6000w + 400w + 70w = 6470w.
\]

Then the entries in any row of $B$ add up to 6470. □
13-1. We have available 75 square feet of material, and wish to use it to form the sides and bottom of an open-topped rectangular box. What is the maximum volume of the box?

Solution: Let \( x \) and \( y \) be the dimensions of the base of an open-topped rectangular box constructed from 75 square feet of material. Let \( h \) denote the height of the box. The total of the surface areas of the bottom and the four sides is \( xy + xh + yh + xh + yh \), or \( xy + (2x + 2y)h \). Then \( xy + (2x + 2y)h = 75 \). Then \( h = (75 - xy)/(2x + 2y) \). Also, because \( x, y, h > 0 \), it follows that \( xy < 75 \).

Let 
\[
D := \{ (x, y) \in \mathbb{R}^2 \mid x, y > 0, \ xy < 75 \}
\]

For all \( (x, y) \in D \), let \( V(x, y) \) denote the volume of the box. Then \( V(x, y) = xyh = xy(75 - xy)/(2x + 2y) \). We wish to maximize \( V \) on \( D \).

For all \( s \in (0, \infty) \), let \( D_s := (0, \sqrt{75/s}) \) and define \( V_s : D_s \to \mathbb{R} \) by \( V_s(x) = V(x, sx) \). Then, because \( D = \bigcup_{s>0} \{ (x, sx) \mid x \in D_s \} \), it follows that \( V(D) = \bigcup_{s>0} \{ V(x, sx) \mid x \in D_s \} = \bigcup_{s>0} [V_s(D_s)] \).

We will show, for all \( s > 0 \), that, on \( D_s \), the function \( V_s \) attains its global maximum at \( 5/\sqrt{s} \). We will also show that the function \( s \mapsto V_s(5/\sqrt{s}) : (0, \infty) \to \mathbb{R} \) attains its global maximum at \( s = 1 \). It will then follow that the global maximum value of \( V \) is \( V_1(5/\sqrt{1}) \).

We first show that the function \( s \mapsto V_s(5/\sqrt{s}) : (0, \infty) \to \mathbb{R} \) attains its global maximum at 1, as follows: For all \( s > 0 \), we have
\[
V_s(5/\sqrt{s}) = V(5/\sqrt{s}, s \cdot (5/\sqrt{s})) = V(5/\sqrt{s}, 5 \cdot \sqrt{s})
\]
\[
= \frac{(5/\sqrt{s})(5 \cdot \sqrt{s})(75 - (5/\sqrt{s})(5 \cdot \sqrt{s}))}{2 \cdot (5/\sqrt{s}) + 2 \cdot (5 \cdot \sqrt{s})}
\]
\[
= \frac{25 \cdot (75 - 25)}{10 \cdot (5/\sqrt{s}) + 10 \cdot \sqrt{s}}
\]
\[
= \frac{25 \cdot 50}{10 \cdot [(1/\sqrt{s}) + \sqrt{s}]} \cdot \frac{\sqrt{s}}{\sqrt{s}} = \frac{125}{1 + s} \cdot \sqrt{s}.
\]

Differentiating with respect to \( s \), we find, on \( s > 0 \), that
\[
\frac{d}{ds}[V(5/\sqrt{s})] = 125 \frac{(1 + s)((1/2)s^{-1/2}) - \sqrt{s}}{(1 + s)^2} \cdot \frac{2\sqrt{s}}{2\sqrt{s}}
\]
\[
= \frac{125(1 + s) - 2s}{2 \cdot (1 + s)^2 \sqrt{s}} = \frac{125}{2} \frac{1 - s}{(1 + s)^2 \sqrt{s}}.
\]
Thus $V_s(5/\sqrt{s})$ is increasing on $0 < s \leq 1$ and decreasing on $1 \leq s$. It follows that the function $s \mapsto V_s(5/\sqrt{s}) : (0, \infty) \to \mathbb{R}$ attains its global maximum at 1. It remains to show, for all $s > 0$, that, on $D_s$, the function $V_s$ attains its global maximum at $5/\sqrt{s}$. Given $s > 0$. We wish show, on $D_s$, that $V_s$ attains its global maximum at $5/\sqrt{s}$.

By definition, for all $x \in D_s$, we have

$$V_s(x) = V(x, sx) = \frac{x \cdot (sx) \cdot (75 - x \cdot (sx))}{2x + 2 \cdot (sx)},$$

so

$$V_s(x) = \frac{sx^2(75 - sx^2)}{(2 + 2s)x} = \left[\frac{s}{2 + 2s}\right] \left[\frac{x^2(75 - sx^2)}{x}\right],$$

so $V_s(x) = \left[\frac{s}{2 + 2s}\right][75x - sx^3]$. Differentiating with respect to $x$, we find, on $0 < x < \sqrt{75/s}$, that

$$\frac{d}{dx}[V_s(x)] = \left[\frac{s}{2 + 2s}\right][75 - 3sx^2] = \left[\frac{3s}{2 + 2s}\right][25 - sx^2].$$

Thus $V_s(x)$ is increasing on the interval $0 < x < 5/\sqrt{s}$ and $V_s(x)$ is decreasing on the interval $5/\sqrt{s} < x < \sqrt{75/s}$. Thus $V_s$ attains its global maximum at $5/\sqrt{s}$.

Thus the global maximum value of $V$ is $V_1(5/\sqrt{1})$, which is equal to

$$V_1(5) = V(5, 5) = \frac{5 \cdot 5 \cdot (75 - 5 \cdot 5)}{2 \cdot 5 + 2 \cdot 5} = \frac{25 \cdot 50}{20} = \frac{125}{2}. \quad \Box$$

NOTE: To simplify the problem we could specify, in advance, that the rectangular box is to have a square base. That would result in a problem in one-variable calculus, instead of multi-variable calculus.

14-1. What is the hundreds digit in the standard decimal expansion of the number $7^{26}$?

**Solution:** In this solution, congruences ($\equiv$) are all mod 100. We have

$$7^2 = 7 \cdot 7 = 49, \quad \text{so}$$

$$7^3 = 7 \cdot 7^2 \equiv 7 \cdot 49 = 343 \equiv 43, \quad \text{so}$$

$$7^4 = 7 \cdot 7^3 \equiv 7 \cdot 43 = 301 \equiv 1.$$

Then $7^{26} = 7^2 \cdot (7^4)^6 \equiv 49 \cdot 1^6 = 49$. Thus the hundreds digit of $7^{25}$ is the same as that of 49, namely 4. \hfill \Box
15-1. True or False: Let \( f \) be a continuous real-valued function defined on the open interval \((-2, 3)\). Then \( f \) is bounded.

**Solution:** False. Counterexample: Let \( f : (-2, 3) \to \mathbb{R} \) be defined by \( f(x) = 1/(x + 2) \). Then \( \lim_{x \to -2^+} (f(x)) = \infty \), so \( f \) is not bounded above, so \( f \) is not bounded. \( \square \)

15-2. True or False: Let \( f \) be a continuous real-valued function defined on the closed interval \([-2, 3]\). There exists \( c \in (-2, 3) \) such that \( f \) is differentiable at \( c \) and such that \( 5 \cdot [f'(c)] = [f(3)] - [f(-2)]. \)

**Solution:** False. Counterexample: Let \( f : [-2, 3] \to \mathbb{R} \) be defined by \( f(x) = |x| \). Then \( [f(3)] - [f(-2)] = |3| - |-2| = 1. \)

On the other hand,
- for all \( c \in (-2, 0) \), \( 5 \cdot [f'(c)] = 5 \cdot [-1] = -5, \)
- \( f \) is not differentiable at 0, and
- for all \( c \in (0, 3) \), \( 5 \cdot [f'(c)] = 5 \cdot 1 = 5. \)

Thus there is no \( c \in (-2, 3) \) such that \( f \) is differentiable at \( c \) and such that \( 5 \cdot [f'(c)] = 1. \) \( \square \)

15-3. True or False: Let \( f \) be a continuous real-valued function defined on the closed interval \([-2, 3]\). Assume that \( f \) is differentiable at 0 and that \( f'(0) = 0 \). Then \( f \) has a local extremum at 0.

**Solution:** False. Counterexample: Let \( f : [-2, 3] \to \mathbb{R} \) be defined by \( f(x) = x^3 \). Then \( f \) is differentiable at 0 and \( f'(0) = 3 \cdot 0^2 = 0 \), but \( f \) is increasing on \([-2, 3]\), so \( f \) has no local extremum at 0. \( \square \)

15-4. True or False: Let \( f \) be a continuous real-valued function defined on the closed interval \([-2, 3]\). Assume that all of the following are true:

- \( f \) is twice-differentiable at 0,
- \( f'(0) = 0 \), and
- \( f''(0) \neq 0. \)

Then \( f \) has a local extremum at 0.

**Solution:** True. Proof: By the Second Derivative Test, if \( f''(0) > 0 \), then \( f \) has a local minimum at 0. Also, by the Second Derivative Test, if \( f''(0) < 0 \), then \( f \) has a local maximum at 0. In either case, \( f \) has a local extremum at 0. \( \square \)
16-1. What is the volume of the solid formed by revolving, about the $x$-axis, the region in the first quadrant of the $xy$-plane bounded by: the coordinate axes and the graph of the equation $y = \sqrt{\frac{x}{1 + x^4}}$?

Solution: By the disk method, with $u = x^2$, $du = 2x\,dx$, the volume is

$$
\int_{0}^{\infty} \pi \left(\sqrt{\frac{x}{1 + x^4}}\right)^2 \, dx = \pi \int_{0}^{\infty} \frac{x\,dx}{1 + x^4} = \pi \int_{0}^{\infty} \frac{du/2}{1 + u^2} = \frac{\pi}{2} \left[\arctan u\right]_{u \to 0}^{u \to \infty} = \frac{\pi^2}{4}.
$$

16-2. What is the volume of the solid formed by revolving, about the $y$-axis, the region in the first quadrant of the $xy$-plane bounded by: the coordinate axes and the graph of the equation $y = \frac{x^2}{(1 + x^4)^{3/2}}$?

Solution: By the shell method, with $u = 1 + x^4$, $du = 3x^2\,dx$, the volume is

$$
\int_{0}^{\infty} 2\pi x \left[\frac{x^2}{(1 + x^4)^{3/2}}\right] \, dx = 2\pi \int_{0}^{\infty} \frac{x^3\,dx}{(1 + x^4)^{3/2}} = 2\pi \int_{1}^{\infty} \frac{du/3}{u^{3/2}} = \frac{2\pi}{3} \int_{1}^{\infty} u^{-3/2} \, du = \frac{2\pi}{3} \left[\frac{u^{-1/2}}{-1/2}\right]_{u=1}^{u=\infty} = \frac{4\pi}{3}.
$$
17-1. How many real roots does the polynomial $x^5 - 5x + 3$ have?

**Solution:** Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^5 - 5x + 3$. We want to know how many real roots $f$ has.

For all $x \in \mathbb{R}$, we have $f'(x) = 5x^4 - 5 = 5(x - 1)(x + 1)(x^2 + 1)$. Thus $f'$ is positive on $(1, \infty)$, is negative on $(-1, 1)$ and is positive on $(-\infty, -1)$. Thus, by the Increasing Test and the Decreasing Test, $f$ is increasing on $(1, \infty)$, is decreasing on $(-1, 1)$ and is increasing on $(-\infty, -1)$. Thus $f$ has at most one root on $(1, \infty)$, has at most one root on $(-1, 1)$ and has at most one root on $(-\infty, -1)$.

We have $\lim_{x \to -\infty} [f(x)] = \lim_{x \to -\infty} x^5 = -\infty$ and $f(-1) = -1 + 5 + 3 > 0$ and $f(1) = 1 - 5 + 3 < 0$ and $\lim_{x \to \infty} [f(x)] = \lim_{x \to \infty} x^5 = \infty$.

Because $\lim_{x \to -\infty} [f(x)] = -\infty$ and $f(-1) > 0$, it follows, from the Intermediate Value Theorem, that $f$ has at least one root on $(-\infty, -1)$. So, since $f$ has at most one root on $(-\infty, -1)$, we conclude that $f$ has exactly one root on $(-\infty, -1)$.

Because $f(-1) > 0$ and $f(1) < 0$, it follows, from the Intermediate Value Theorem, that $f$ has at least one root on $(-1, 1)$. So, since $f$ has at most one root on $(-1, 1)$, we conclude that $f$ has exactly one root on $(-1, 1)$.

Because $f(1) = 1 - 5 + 3 < 0$ and $\lim_{x \to \infty} [f(x)] = \infty$, it follows, from the Intermediate Value Theorem, that $f$ has at least one root on $(1, \infty)$. So, since $f$ has at most one root on $(1, \infty)$, we conclude that $f$ has exactly one root on $(1, \infty)$.

Because $f$ has exactly one root on $(-\infty, -1)$ and on $(-1, 1)$ and on $(1, \infty)$, and because $f(-1) \neq 0$ and $f(1) \neq 0$, we conclude that $f$ has exactly three real roots. \qed
18-1. Let $V$ be the real vector space of all real homogeneous polynomials in $x$ and $y$ of degree 7 (together with the zero polynomial). Let $W$ be the real vector space of all real polynomials in $x$ of degree $\leq 3$ (together with the zero polynomial). If $T$ is a linear transformation from $V$ onto $W$, what is the dimension of the subspace $\{v \in V \mid T(v) = 0\}$ of $V$?

\textit{Solution:} By definition of kernel, we have
\[
\ker[T] = \{v \in V \mid T(v) = 0\}.
\]
We therefore wish to calculate $\dim(\ker[T])$. A basis for $V$ is
\[
\{x^7, x^6y, x^5y^2, x^4y^3, x^3y^4, x^2y^5, xy^6, y^7\},
\]
so $\dim(V) = 8$. A basis for $W$ is $\{1, x, x^2, x^3\}$, so $\dim(W) = 4$. Then
\begin{itemize}
  \item $\dim(\text{dom}[T]) = \dim(V) = 8$ and
  \item $\dim(\text{im}[T]) = \dim(W) = 4$.
\end{itemize}
So, since
\[
\dim(\ker[T]) + \dim(\im[T]) = \dim(\text{dom}[T]),
\]
we conclude that $\dim(\ker[T]) + 4 = 8$, and so $\dim(\ker[T]) = 4$. \qed
18-2. Let $V$ be the real vector space of all real polynomials in $x$ and $y$ of degree $\leq 7$ (together with the zero polynomial). Let $W$ be the real vector space of all real polynomials in $x$ of degree $\leq 3$ (together with the zero polynomial). If $T$ is a linear transformation from $V$ onto $W$, what is the dimension of the subspace $\{ v \in V \mid T(v) = 0 \}$ of $V$?

**Solution:** By definition of kernel, we have

$$ \ker[T] = \{ v \in V \mid T(v) = 0 \}. $$

We therefore wish to calculate $\dim (\ker[T])$. A basis for $V$ is

$$ \{ x^7, x^6y, x^5y^2, x^4y^3, x^3y^4, x^2y^5, xy^6, y^7, $$

$$ x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6, $$

$$ x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5, $$

$$ x^4, x^3y, x^2y^2, xy^3, y^4, $$

$$ x^3, x^2y, xy^2, y^3, $$

$$ x^2, xy, y^2, $$

$$ x, y, $$

$$ 1 \} $$

so $\dim (V) = 8 + 7 + \cdots + 1 = 8 \cdot 9 / 2 = 36$. A basis for $W$ is $\{ 1, x, x^2, x^3 \}$, so $\dim (W) = 4$. Then

- $\dim (\text{dom} [T]) = \dim V = 36$ and
- $\dim (\text{im} [T]) = \dim W = 4.$

So, since

$$ \dim (\ker[T]) + \dim (\text{im}[T]) = \dim (\text{dom}[T]), $$

we conclude that $\dim (\ker[T]) + 4 = 36$, and so $\dim (\ker[T]) = 32$. $\square$
19-1. True or False: Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function such that, for all \( x \in \mathbb{R} \), we have \(-x^2 \leq f(x) \leq x^2\). Then, for all \( x \in \mathbb{R} \), we have \(-2x \leq f'(x) \leq 2x\).

Solution: False. Counterexample: Define \( f : \mathbb{R} \to \mathbb{R} \) by

\[
  f(x) = \begin{cases} 
  x^2 \cdot \left(\sin \left(\frac{1}{x}\right)\right), & \text{if } x \neq 0; \\
  0, & \text{if } x = 0.
  \end{cases}
\]

Then, for all \( x \in \mathbb{R} \), we have \(-x^2 \leq f(x) \leq x^2\). Also, \( f \) is differentiable, and, for all \( x \in \mathbb{R} \setminus \{0\} \), we have

\[
  f'(x) = 2x \cdot \left(\sin \left(\frac{1}{x}\right)\right) + x^2 \cdot \left(\cos \left(\frac{1}{x}\right)\right) \cdot \left(-\frac{1}{x^2}\right)
  = 2x \cdot \left(\sin \left(\frac{1}{x}\right)\right) - \left(\cos \left(\frac{1}{x}\right)\right).
\]

In particular, \( f'(1/\pi) = (2/\pi) \cdot 0 - (1) = 1 > 2/\pi \). So it is NOT true, for all \( x \in \mathbb{R} \), that \( f'(x) \leq 2x \). \( \square \)

19-2. True or False: Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function such that, for all \( x \in \mathbb{R} \), we have \(-x^2 \leq f(x) \leq x^2\). Then \( f'(0) = 0 \).

Solution: True. Proof: We have \(-0^2 \leq f(0) \leq 0^2\), so \( f(0) = 0 \). Then, for all \( h \in \mathbb{R} \setminus \{0\} \), we have

\[
  \frac{|f(0 + h)| - |f(0)|}{h} = \frac{f(h)}{h}.
\]

We therefore wish to show that \( \lim_{h \to 0} \left[ \frac{f(h)}{h} \right] = 0 \). It therefore suffices to show both that \( \lim_{h \to 0^+} \left[ \frac{f(h)}{h} \right] = 0 \) and that \( \lim_{h \to 0^-} \left[ \frac{f(h)}{h} \right] = 0 \).

For all \( h > 0 \), we have

\[
  -h = -\frac{h^2}{h} \leq \frac{f(h)}{h} \leq \frac{h^2}{h} = h.
\]

It follows, from the Squeeze Theorem, that \( \lim_{h \to 0^+} \left[ \frac{f(h)}{h} \right] = 0 \). It remains to show that \( \lim_{h \to 0^-} \left[ \frac{f(h)}{h} \right] = 0 \).

For all \( h < 0 \), we have

\[
  -h = -\frac{h^2}{h} \geq \frac{f(h)}{h} \geq \frac{h^2}{h} = h.
\]
It follows, from the Squeeze Theorem, that \( \lim_{h \to 0} \frac{f(h)}{h} = 0. \) □

19-3. True or False: Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function such that \( f'(1) = 5 \) and \( f'(3) = 9. \) Then \( \exists c \in (1, 3) \) such that \( f'(c) = 7. \)

Solution: For every \( h \in \mathbb{R} \setminus \{0\}, \) define \( g_h : \mathbb{R} \to \mathbb{R} \) by
\[
g_h(x) = \frac{[f(x + h)] - [f(x)]}{h}.
\]
We have
\[
\lim_{h \to 0} [g_h(1)] = f'(1) = 5 \quad \text{and} \quad \lim_{h \to 0} [g_{-h}(3)] = f'(3) = 9.
\]
Choose \( b > 0 \) so small that \( g_b(1) < 6 \) and \( g_{-b}(3) > 8. \) Then
\[
g_b(3 - b) = \frac{[f((3 - b) + b)] - [f(3 - b)]}{b} = \frac{[f(3)] - [f(3 - b)]}{b} = \frac{[f(3) - f(3)]}{b} = \frac{0}{b} = g_{-b}(3) > 8.
\]
Since \( g_b(1) < 6, \) since \( g_b(3 - b) > 8 \) and since \( g_b : \mathbb{R} \to \mathbb{R} \) is continuous, by the Intermediate Value Theorem, we choose \( a \in (1, 3 - b) \) such that \( g_b(a) = 7. \) Then
\[
\frac{[f(a + b)] - [f(a)]}{b} = g_b(a) = 7.
\]
Then, by the Mean Value Theorem, we choose \( c \in (a, a + b) \) such that \( f'(c) = 7. \) It remains to show that \( c \in (1, 3). \)

Since \( a \in (1, 3 - b), \) it follows that \( 1 \leq a \) and that \( a + b \leq 3. \) Then \( c \in (a, a + b) \subseteq (1, 3), \) as desired. □

NOTE: It is a general fact that if a real-valued function is differentiable on \( \mathbb{R}, \) then the function satisfies the Intermediate Value Theorem. Problem 19-3 simply asks whether this is true in a particular case, using the interval \([1, 3].\) The solution given above is based on a proof of that general fact.
True or False: Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function. Then there exists \( c \in \mathbb{R} \) such that \( f' \) is continuous at \( c \).

NOTE: Let’s talk about measure theory . . .

In measure theory, one thinks of a property as holding “almost everywhere” (or, sometimes, “generically”) if it holds except on a set of measure zero. An old joke has it that doing measure theory is torture because you have to say “almost everywhere” almost everywhere.

A set is **null** if it has measure zero. A set is **conull** if its complement is null. Then a generic property is one that holds on a conull set.

Now we move from measure theory to topology . . .

For any \( A, B \subseteq \mathbb{R} \), “\( B \) is **dense in** \( A \)” means that \( A \) is a subset of the closure in \( \mathbb{R} \) of \( B \). A subset of \( \mathbb{R} \) is **somewhere dense**, or **somewhere dense in** \( \mathbb{R} \), if it is dense in a nonempty open subset of \( \mathbb{R} \). So, for example, \( \mathbb{Q} \cap (0, 1) \) is not dense in \( \mathbb{R} \), but it is dense in \( (0, 1) \), and, consequently, is somewhere dense in \( \mathbb{R} \). A subset of \( \mathbb{R} \) is

- **nowhere dense** if it is not somewhere dense,
- **meager** if it is a countable union of nowhere dense sets and
- **comeager** if its complement in \( \mathbb{R} \) is meager.

Warning: In older books, instead of “meager” you’ll see “of first category”, and, instead of “comeager”, you’ll see “residual”.

Next, let’s compare topology and measure theory . . .

We use Lebesgue measure on \( \mathbb{R} \), and so a subset of \( \mathbb{R} \) is null iff it has Lebesgue measure zero. While

- meager does not imply null, and
- null does not imply meager,

it’s work to actually find a meager set that isn’t null, or a null set that isn’t meager. More importantly, the intuition for meager sets is about the same as for null sets, and, in some sense, “meager” is a topological analogue of “null”. So, for example, topologists typically say that a property is “generic” if it happens on a comeager set.

Finally, let’s talk about Problem 19-4 . . .

If a real-valued function is differentiable on \( \mathbb{R} \), then its derivative is a pointwise limit of its difference quotients, each of which is continuous. While a limit of continuous functions need not be continuous on all of \( \mathbb{R} \), it is a general fact that it must be “generically continuous”, i.e., continuous on a comeager subset of \( \mathbb{R} \). By the Baire Category Theorem,
any comeager subset of $\mathbb{R}$ is nonempty, and it follows that the answer to Problem 19-4 is “true”. The solution given below simply follows a detailed proof of that general fact, where $g := f'$ is the pointwise limit of continuous difference quotients $g_1, g_2, \ldots$ of $f$.

**Solution:** True. Proof: Let $g := f'$. For all $h > 0$, let $\gamma_h : \mathbb{R} \to \mathbb{R}$ be defined by $\gamma_h(x) = \frac{[f(x + h)] - [f(x)]}{h}$. Then, for all $x \in \mathbb{R}$, we have $\gamma_h(x) \to g(x)$, as $h \to 0$. Since $f$ is differentiable, $f$ is continuous. Then, for all $h > 0$, the function $\gamma_h$ is continuous. For all integers $j \geq 1$, let $g_j := \gamma_{1/j}$; then $g_j$ is continuous. Also, for all $x \in \mathbb{R}$, we have $g_j(x) \to g(x)$, as $j \to \infty$.

Let $D$ be the set of $x \in \mathbb{R}$ such that $g$ is discontinuous at $x$. We wish to show that $D \neq \mathbb{R}$. By the Baire Category Theorem, it suffices to show that $D$ is meager. Assume that $D$ is nonmeager. We aim for a contradiction.

For all $\varepsilon > 0$, let $D_\varepsilon$ denote the set of all $x \in \mathbb{R}$ such that

- for all $\delta > 0$, $g((x - \delta, x + \delta)) \not\subseteq ([g(x)] - \varepsilon, [g(x)] + \varepsilon)$.

Then $D = D_1 \cup D_{1/2} \cup D_{1/3} \cup \cdots$. Fix an integer $m \geq 1$ such that $D_{1/m}$ is nonmeager. Let $\varepsilon := 1/m$. Then $D_\varepsilon$ is nonmeager.

We define $\eta := \varepsilon/5$ and $S := \{\ell \eta | \ell \in \mathbb{Z}\}$. For all $s \in S$, we define $I_s := (s - \eta, s + \eta]$. Then $\bigcup_{s \in S} I_s = \mathbb{R}$, so $\bigcup_{s \in S} [g^{-1}(I_s)] = g^{-1}(\mathbb{R})$. Then

$$\bigcup_{s \in S} [(g^{-1}(I_s)) \cap D_\varepsilon] = [g^{-1}(\mathbb{R})] \cap D_\varepsilon = \mathbb{R} \cap D_\varepsilon = D_\varepsilon.$$  

As $S$ is countable and $D_\varepsilon$ is nonmeager, fix $s \in S$ such that $(g^{-1}(I_s)) \cap D_\varepsilon$ is nonmeager. Let $A := (g^{-1}(I_s)) \cap D_\varepsilon$. Then $A$ is nonmeager.

For all integers $k \geq 1$, let $L_k$ be the set of all $x \in \mathbb{R}$ such that

- for all integers $j \geq k$, $|[g_j(x)] - [g(x)]| < \eta$.

Recall, for all $x \in \mathbb{R}$, that $g_j(x) \to g(x)$, as $j \to \infty$. It follows that $\bigcup_{k=1}^\infty L_k = \mathbb{R}$. Then $\bigcup_{k=1}^\infty [L_k \cap A] = \mathbb{R} \cap A = A$. Choose an integer $k \geq 1$ such that $L_k \cap A$ is nonmeager.

For any function $\phi : \mathbb{R} \to \mathbb{R}$, for any $\Omega \subseteq \mathbb{R}$, for any $\rho \geq 0$, let’s agree that “$\phi$ is $\rho$-constant on $\Omega$” means:

- for all $\mu, \nu \in \Omega$, $|\phi(\mu) - \phi(\nu)| \leq \rho$.

Because $I_s = [s - \eta, s + \eta]$, we see that $g$ is $(2\eta)$-constant on $g^{-1}(I_s)$. Since $A = (g^{-1}(I_s)) \cap D_\varepsilon$, we get $A \subseteq D_\varepsilon$. Then $g$ is $(2\eta)$-constant.
on $A$. Then, by construction of $L_k$, we see, for all integers $j \geq k$, that $g_j$ is $(4\eta)$-constant on $L_k \cap A$.

Every nowhere dense set is meager, while $L_k \cap A$ is nonmeager. Thus $L_k \cap A$ somewhere dense. Let $C$ denote the closure in $\mathbb{R}$ of $L_k \cap A$. Let $U$ be a open subset of $\mathbb{R}$ s.t. $\emptyset \neq U \subseteq C$. For all integers $j \geq k$, $g_j$ is continuous and $(4\eta)$-constant on $L_k \cap A$, and so $g_j$ is $(4\eta)$-constant on $C$, and so $g_j$ is $(4\eta)$-constant on $U$. So, as $g_j \to g$ pointwise, as $j \to \infty$, we conclude that $g$ is $(4\eta)$-constant on $U$. On the other hand, since $4\eta < \varepsilon$, by definition of $D_\varepsilon$, we see that $g$ is not $(4\eta)$-constant on any open neighborhood of any point of $D_\varepsilon$. Then $U \cap D_\varepsilon = \emptyset$. So, since $L_k \cap A \subseteq A = (g^{-1}(I_s)) \cap D_\varepsilon \subseteq D_\varepsilon$, we conclude that $U \cap (L_k \cap A) = \emptyset$.

That is, $L_k \cap A \subseteq \mathbb{R} \setminus U$. So, since $\mathbb{R} \setminus U$ is closed in $\mathbb{R}$, it follows that $C \subseteq \mathbb{R} \setminus U$, and so $U \cap C = \emptyset$. Because $U \subseteq C$, we see that $U = U \cap C$. Then $\emptyset \neq U = U \cap C = \emptyset$, contradiction. □

20-1. Let $f$ be the function defined on the real line by

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational;} \\ 2x, & \text{if } x \text{ is irrational.} \end{cases}$$

Compute the set of points of discontinuity of $f$.

Solution: Because the rationals and irrationals are both dense in $\mathbb{R}$, we see, for all $c > 2$, that $\liminf_{x \to c} [f(x)] = 2c$ and that $\limsup_{x \to c} [f(x)] = c^2$, and, therefore, that $f$ is not continuous at $c$. Because the rationals and irrationals are both dense in $\mathbb{R}$, we see, for all $c < 0$, that $\liminf_{x \to c} [f(x)] = 2c$ and that $\limsup_{x \to c} [f(x)] = c^2$, and, therefore, that $f$ is not continuous at $c$. Because the rationals and irrationals are both dense in $\mathbb{R}$, we see, for all $c \in (0, 2)$, that $\liminf_{x \to c} [f(x)] = c^2$ and that $\limsup_{x \to c} [f(x)] = 2c$, and, therefore, that $f$ is not continuous at $c$.

We have $\liminf_{x \to 0} [f(x)] \geq 0$ and $\limsup_{x \to 0} [f(x)] \leq 0$ and $f(0) = 0$, and so $f$ is continuous at $0$. Finally, $\liminf_{x \to 2} [f(x)] \geq 4$ and $\limsup_{x \to 2} [f(x)] \leq 4$ and $f(2) = 4$, and so $f$ is continuous at $2$.

Thus the set of points of discontinuity of $f$ is $\mathbb{R} \setminus \{0, 2\}$. □
21-1. Let \( p := 7919 \), which is a prime number. Let \( Q := \{ p, 2p, 3p, \ldots \} \) be the set of multiples of \( p \). Let \( K := \{ 0, 1, \ldots, p \} \) denote the set of integers from 0 to \( p \). For all \( k \in K \), let \( C^p_k \) denote the binomial coefficient \( \binom{p}{k} \). Let \( S := \{ k \in K \mid C^p_k \in Q \} \). So, for example, because \( C^p_1 = p \in Q \) and \( C^p_2 = \lfloor(p-1)/2\rfloor p = 3959p \in Q \), we get \( 2 \in S \). Compute the maximum element of \( S \).

Solution: Claim: \( \forall k \in \{1, \ldots, p-1\} \), we have \( C^p_k \in Q \). Proof of claim: Given \( k \in \{1, \ldots, p\} \), let \( c := C^p_k \). We wish to show that \( c \in Q \).

Let \( \ell := p - k \). Then \( c = \lfloor (k!) (\ell!) \rfloor / \lfloor (k!) (\ell!) \rfloor \). Since \( p \) is a prime, it follows, for all integers \( m, n \geq 1 \), that:

\[
(*) \quad [mn \in Q] \Rightarrow [(m \in Q) \lor (n \in Q)].
\]

Taking the contrapositive, for any positive integers \( m \) and \( n \), we have:

\[
(**) \quad [(m \notin Q) \land (n \notin Q)] \Rightarrow [mn \notin Q].
\]

Since \( 1, \ldots, k < p \), it follows that \( 1, \ldots, k \notin Q \). Repeatedly applying \((**)\), we see that \( (1)(2) \cdots (k) \notin Q \), i.e., that \( k! \notin Q \). Since \( 1, \ldots, \ell < p \), it follows that \( 1, \ldots, \ell \notin Q \). Repeatedly applying \((**)\), we see that \( (1)(2) \cdots (\ell) \notin Q \), i.e., that \( \ell! \notin Q \).

We have \( c(k!)(\ell!) = p! \in Q \). Then either \( c \in Q \) or \( (k!)(\ell!) \in Q \). Since both \( k! \notin Q \) and \( \ell! \notin Q \), it follows, from \((**)\), that \( (k!)(\ell!) \notin Q \). Then \( c \in Q \), as desired. End of proof of claim.

By the claim, \( C^p_0, C^p_1, \ldots, C^p_{p-1} \in Q \). Then \( p-1 \in S \). On the other hand, \( C^p_p = 1 \notin Q \), so \( p \notin S \). So, since \( S \subseteq K = \{0,1,\ldots,p\} \), it follows that \( \max S = p - 1 = 7918 \). \( \square \)

22-1. Let \( C(\mathbb{R}) \) be the collection of all continuous functions from \( \mathbb{R} \) to \( \mathbb{R} \). Then \( C(\mathbb{R}) \) is a real vector space with vector addition defined by

\[
\forall f, g \in C(\mathbb{R}), \forall x \in \mathbb{R}, \quad (f + g)(x) = [f(x)] + [g(x)],
\]

and with scalar multiplication defined by

\[
\forall f \in C(\mathbb{R}), \forall r, x \in \mathbb{R}, \quad (rf)(x) = r \cdot [f(x)].
\]

Let \( S \) denote the set of \( f \in C(\mathbb{R}) \) such that all of the following hold:

- \( f \) is twice differentiable,
- for all \( x \in \mathbb{R} \), \( f(x + 2\pi) = f(x) \).
- \( f'' = -f \).

True or False: \( S \) is a subspace of \( C(\mathbb{R}) \).
Solution: True. Proof: Let $V$ denote the set of all $\phi \in C(\mathbb{R})$ such that $\phi$ is twice differentiable. Then $V$ is a subspace of $C(\mathbb{R})$.

For all $f \in C(\mathbb{R})$, let $f^+ \in C(\mathbb{R})$ be defined by $f^+(x) = f(x + 2\pi)$. Define $\Gamma : C(\mathbb{R}) \to C(\mathbb{R})$ by $\Gamma(f) = f - f^+$. Then $\Gamma$ is a linear map. Then $\ker[\Gamma]$ is a subspace of $C(\mathbb{R})$. Define $\Delta : V \to C(\mathbb{R})$ by $\Delta(f) = f'' + f$. Then $\Delta$ is a linear map. Then $\ker[\Delta]$ is a subspace of $V$, and, therefore, is a subspace of $C(\mathbb{R})$. Then, because $S = (\ker[\Gamma]) \cap (\ker[\Delta])$, we see that $S$ is a subspace of $C(\mathbb{R})$. □

23-1. True or False: There exists a real number $b$ such that the line $y = 10x$ tangent to the curve $y = bx^2 + 10x + 1$ at some point in the $xy$-plane.

Solution: False. Proof: Let $b \in \mathbb{R}$, and assume that the line $y = 10x$ is tangent to the curve $y = bx^2 + 10x + 1$. We aim for a contradiction.

Let $x \in \mathbb{R}$ be the first coordinate of the point of tangency. Then

$$10x = bx^2 + 10x + 1 \quad \text{and} \quad 10 = 2bx + 10.$$

By the second equation, we see that $bx = 0$. By the first equation, we see that $x \neq 0$. Since $bx = 0$ and $x \neq 0$, we get $b = 0$. Then the first equation says $10x = 10x + 1$, and so $0 = 1$, contradiction □

24-1. Let $h$ be the function defined by $h(x) = \int_0^{x^2 + x} e^{(x+t)^2} \, dt$, for all real numbers $x$. Compute $h'(1)$.

Solution: Claim: For all $x \in \mathbb{R}$, we have $h(x) = \int_x^{x^2+x} e^{s^2} \, ds$. Proof of claim: Given $x \in \mathbb{R}$. We wish to prove that $h(x) = \int_x^{x^2+x} e^{s^2} \, ds$.

Make the change of variables $s = x + t$ and $ds = dt$, in the definition of $h(x)$. This yields $h(x) = \int_x^{x^2+x} e^{s^2} \, ds$. End of proof of claim.

Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = \int_0^x e^{s^2} \, ds$. By the claim, for all $x \in \mathbb{R}$, we have $f(x) = [F(x^2 + x)] - [F(x)]$. Differentiating, for all $x \in \mathbb{R}$, we have $f'(x) = [F'(x^2 + x)][2x + 1] - [F'(x)]$. Evaluating this at $x :\to 1$ yields $f'(1) = [F'(2)][3] - [F'(1)]$. By the Fundamental Theorem of Calculus, for all $x \in \mathbb{R}$, $F'(x) = e^{x^2}$. Then $F'(2) = e^4$ and $F'(1) = e$. Then $f'(1) = [F'(2)][3] - [F'(1)] = 3e^4 - e$. □
25-1. Let \( \{a_n\}_{n=1}^{\infty} \) be defined recursively by \( a_1 = 7 \) and for all integers \( n \geq 1 \),
\[
a_{n+1} = \left( \frac{n}{n+3} \right) a_n.
\]
Compute \( a_{25} \).

**Solution:** We have
\[
a_{25} = \left( \frac{24}{27} \right) a_{24} = \left( \frac{24}{27} \right) \left( \frac{23}{26} \right) a_{23} = \ldots = \left( \frac{24}{27} \right) \left( \frac{23}{26} \right) \left( \frac{22}{25} \right) \left( \frac{4}{1} \right) a_1 = \frac{24!}{(27!)/(3!)} = \frac{(3)(2)(1)}{(27)(26)(25)} = \frac{1}{(9)(13)(25)} = \frac{1}{2925}. \]

26-1. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(x, y) = 2x^2 - 4xy + y^4 \). Find all the absolute extreme values of \( f \), and where they occur.

**Solution:** For all \( x, y \in \mathbb{R} \),
\[
f(x, y) = 2(x - y)^2 + y^4 - 2y^2 = 2(x - y)^2 + (y^2 - 1)^2 - 1,
\]
so \(-1\) is an absolute minimum value for \( f \). Also, for all \( x, y \in \mathbb{R} \), we have: \( f(x, y) = -1 \) iff \( (x - y = 0) \) and \( (y^2 - 1 = 0) \). Thus \( f \) attains its absolute minimum value at, and only at, \((1, 1)\) and \((-1, -1)\).
Since \( \lim_{x \to \infty} [f(x, 0)] = \infty \), \( f \) has no absolute maximum value. □

27-1. Find the dimension of the solution space, in \( \mathbb{R}^4 \), of
\[
\begin{align*}
3w + 4x - 2y - 3z &= 1 \\
2w + x - y &= 2 \\
-w + 7x - y - 9z &= -7.
\end{align*}
\]
Solution: Multiply the first equation by $-3$ and add to the last:

\[
\begin{align*}
3w + 4x - 2y - 3z &= 1 \\
2w + x - y &= 2 \\
-10w - 5x + 5y &= -10.
\end{align*}
\]

The third equation is $-5$ times the second equation; we eliminate it:

\[
\begin{align*}
3w + 4x - 2y - 3z &= 1 \\
2w + x - y &= 2.
\end{align*}
\]

For any $w, x \in \mathbb{R}$, there is a unique solution of this system, and all solutions are obtained this way.

So the dimension of the solution space is 2. \hfill \square

27-2. Find the dimension of the solution space, in $\mathbb{R}^4$, of

\[
\begin{align*}
3w + 4x - 2y - 3z &= 1 \\
2w + 2x - y &= 2 \\
- w + 7x - y - 9z &= -7.
\end{align*}
\]

Solution: Multiply the first equation by $-3$ and add to the last:

\[
\begin{align*}
3w + 4x - 2y - 3z &= 1 \\
2w + 2x - y &= 2 \\
-10w - 5x + 5y &= -10.
\end{align*}
\]

Multiply the second equation by 5 and add to the last:

\[
\begin{align*}
3w + 4x - 2y - 3z &= 1 \\
2w + 2x - y &= 2 \\
5x &= 0.
\end{align*}
\]

Divide the last equation by 5:

\[
\begin{align*}
3w + 4x - 2y - 3z &= 1 \\
2w + 2x - y &= 2 \\
x &= 0.
\end{align*}
\]

Multiply the last equation by $-4$ and add to the first:

\[
\begin{align*}
3w - 2y - 3z &= 1 \\
2w + 2x - y &= 2 \\
x &= 0.
\end{align*}
\]
Multiply the last equation by $-2$ and add to the second:

\[
\begin{align*}
3w & - 2y - 3z = 1 \\
2w & - y = 2 \\
x & = 0.
\end{align*}
\]

For any $w \in \mathbb{R}$, there is a unique solution of this system, and all solutions are obtained this way.

So the dimension of the solution space is 1. \hfill \Box

27-3. Find the solution space, in $\mathbb{R}^4$, of

\[
\begin{align*}
3w + 4x - 2y - 3z &= 1 \\
2w + x - y &= 2 \\
-w + 7x - y - 9z &= 5.
\end{align*}
\]

**Solution:** Multiply the first equation by $-3$ and add to the last:

\[
\begin{align*}
3w + 4x - 2y - 3z &= 1 \\
2w + x - y &= 2 \\
-10w - 5x + 5y &= 2.
\end{align*}
\]

Multiply the second equation by 5 and add to the last:

\[
\begin{align*}
3w + 4x - 2y - 3z &= 1 \\
2w + x - y &= 2 \\
0 &= 12.
\end{align*}
\]

Because of the inconsistency of the last equation ($0 = 12$), we see that the solution space is the empty set, $\emptyset$. \hfill \Box

28-1. Let $T$ be a graph with 378 vertices. Assume $T$ is a tree, which is a connected graph with no cycles. How many edges does $T$ have?

**Solution:** Let $V$ be the set of vertices of $T$ and let $E$ be the set of edges of $T$. We know that $\#V = 378$, and we wish to compute $\#E$.

An induction proof shows that any tree has one more vertex than it has edges. Then $\#V = 1 + (\#E)$.

Then $\#E = (\#V) - 1 = 378 - 1 = 377$. \hfill \Box
29-1. For all positive functions \( f \) and \( g \) of the real variable \( x \), let \( \sim \) be a relation defined by

\[ f \sim g \quad \text{if and only if} \quad \lim_{x \to \infty} \left[ \frac{f(x)}{g(x)} \right] = 1. \]

True or False: Let \( f, g, \phi, \psi \) be positive functions of \( x \). Assume that \( f \sim g \) and that \( \phi \sim \psi \). Then \( f + \phi \sim g + \psi \).

Solution: True. Proof: Given \( \varepsilon > 0 \). We wish to show that there exists \( M \in \mathbb{R} \) such that, for all \( x > M \), we have

\[ \left| \frac{f(x) + \phi(x)}{g(x) + \psi(x)} - 1 \right| < \varepsilon. \]

Choose \( L > 0 \) such that, for all \( x > A \), we have \( \left| \frac{f(x)}{g(x)} - 1 \right| < \varepsilon \).

Choose \( \Lambda > 0 \) such that, for all \( x > \Lambda \), we have \( \left| \frac{\phi(x)}{\psi(x)} - 1 \right| < \varepsilon \).

Let \( M := \max\{L, \Lambda\} \). Given \( x > M \). We wish to prove that

\[ \left| \frac{f(x) + \phi(x)}{g(x) + \psi(x)} - 1 \right| < \varepsilon. \]

Let \( s := f(x) \), \( t := g(x) \), \( \sigma := \phi(x) \), \( \tau := \psi(x) \). We wish to prove that

\[ \left| \frac{s + \sigma}{t + \tau} - 1 \right| < \varepsilon. \]

Since \( x > M \geq \Lambda \), it follows that \( \left| \frac{\sigma}{\tau} - 1 \right| < \varepsilon \), and so

\[ 1 - \varepsilon < \frac{\sigma}{\tau} < 1 + \varepsilon. \]

Multiplying by the positive number \( \tau \), we get \((1 - \varepsilon)\tau < \sigma < (1 + \varepsilon)\tau\).

Since \( x > M \geq L \), it follows that \( \left| \frac{s}{t} - 1 \right| < \varepsilon \), and so

\[ 1 - \varepsilon < \frac{s}{t} < 1 + \varepsilon. \]

Multiplying by the positive number \( t \), we get \((1 - \varepsilon)t < s < (1 + \varepsilon)t\).

Adding this to \((1 - \varepsilon)\tau < \sigma < (1 + \varepsilon)\tau \) yields

\[ (1 - \varepsilon)(t + \tau) < s + \sigma < (1 + \varepsilon)(t + \tau). \]

Dividing by the positive number \( t + \tau \), we get

\[ 1 - \varepsilon < \frac{s + \sigma}{t + \tau} < 1 + \varepsilon, \]
30-1. Let \( S \) and \( T \) be sets and assume that there exists a function \( f : S \to T \) such that \( f \) is onto \( T \). True or False: There must exist a function \( g : T \to S \) such that \( g \) is one-to-one.

**Solution:** True. Proof: By the Axiom of Choice, for all \( y \in T \), choose \( x_y \in S \) such that \( f(x_y) = y \). Define \( g : T \to S \) by \( g(y) = x_y \). We wish to show that \( g \) is one-to-one. Given \( t, \tau \in T \). Assume \( g(t) = g(\tau) \). We wish to prove that \( t = \tau \).

We have \( x_t = g(t) = g(\tau) = x_\tau \), and so \( f(x_t) = f(x_\tau) \). So, since \( f(x_t) = t \) and \( f(x_\tau) = \tau \), we get \( t = f(x_t) = f(x_\tau) = \tau \), as desired. \( \square \)

30-2. Let \( S \) and \( T \) be sets. Assume that there does NOT exist a function \( f : S \to T \) such that \( f \) is one-to-one. True or False: There must exist a function \( g : T \to S \) such that \( g \) is one-to-one.

**Solution:** True. Proof: Let \( I \) denote the set of functions \( f : S_0 \to T \) such that \( S_0 \subseteq S \) and such that \( f \) is one-to-one. We define a partial ordering \( \leq \) on \( I \) by \( [ f \leq g ] \iff [ \exists A \subseteq \text{dom } [g] \text{ s.t. } g|A = f ] \). Then every \( \leq \)-chain has an upper bound, so, by Zorn, let \( f \in I \) be a maximal element with respect to \( \leq \).

Let \( S_0 \subseteq S \) be the domain of \( f \) and let \( T_0 \) be the image of \( f \). By assumption, \( S_0 \neq S \). Then \( T_0 = T \); otherwise, we could extend \( f \), contradicting maximality of \( f \). Then \( f \) is a bijection from \( S_0 \) onto \( T \). Let \( g \) be the inverse of \( f \). Then \( g \) is an bijection from \( T \) onto \( S_0 \), so \( g : T \to S \) is one-to-one. \( \square \)

31-1. True or False: There exists a solution \( y : \mathbb{R} \to \mathbb{R} \) to the differential equation \( y' = x^4 + 2x^2y^2 + y^4 \) with the property that, for every \( x \in \mathbb{R} \), we have \(-1000 < y(x) < 1000\).

**Solution:** False. Proof: Following the notation given in the problem, \( y \) and \( y(x) \) are used interchangeably. Also, \( y' \) and \( y'(x) \) are used interchangeably. Let \( y : \mathbb{R} \to \mathbb{R} \) be a solution to the ODE \( y' = x^4 + 2x^2y^2 + y^4 \). We will show that there exists \( x \in \mathbb{R} \) such that \( y(x) \geq 1000 \).

For all \( x \geq 1 \), we have \( y'(x) = x^4 + 2x^2y^2 + y^4 \geq x^4 \geq 1 \). So, by the Mean Value Theorem, for all \( x \geq 1 \), we have \([y(x)] - [y(1)] \geq x - 1\).

and so \( \left| \frac{s + \sigma}{t + \tau} - 1 \right| < \varepsilon \), as desired. \( \square \)
Let $x := \max\{1, 1001 - [y(1)]\}$. Then $[y(x)] - [y(1)] \geq x - 1$. Then $y(x) \geq [y(1)] + x - 1 \geq [y(1)] + 1001 - [y(1)] - 1 = 1000$. □

32-1. True or False: Let $G$ be a group. Assume, for all $a,b \in G$, for all integers $n \geq 1$, that $(ab)^n = a^n b^n$. Then $G$ is Abelian.

Solution: True. Proof: Given $a,b \in G$. We wish to show that $ab = ba$.

For all integers $n \geq 1$, $(ab)^n = a^n b^n$. In particular, $(ab)^2 = a^2 b^2$. Then $abab = aabb$, so $a^{-1}[abab]b^{-1} = a^{-1}[aabb]b^{-1}$, so $ba = ab$. □

33-1. True or False: Let $p$ and $q$ be prime numbers, and let $n$ be an integer. Assume that $p \neq q$. Then there exist integers $k$ and $\ell$ such that $\frac{n}{p^2 q} = \frac{k}{p^2} + \frac{\ell}{q}$.

Solution: True. Proof: By the Euclidean algorithm, choose $a,b \in \mathbb{Z}$ such that $ap^2 + bq = 1$. Let $k := nb$ and let $\ell := na$. We wish to prove that $\frac{n}{p^2 q} = \frac{k}{p^2} + \frac{\ell}{q}$. Equivalently, we wish to prove that $n = kq + \ell p^2$.

We have $n = n \cdot 1 = n(ap^2 + bq) = (na)p^2 + (nb)q = \ell p^2 + kq$. □

33-2. True or False: Let $p$ and $q$ be prime numbers, and let $n$ be an integer. Assume that $p \neq q$. Then there exist integers $r,s,t,u$ such that $0 \leq s < p$ and $0 \leq t < p$ and $0 \leq u < q$ and $\frac{n}{p^2 q} = r + \frac{s}{p} + \frac{t}{p^2} + \frac{u}{q}$.

Solution: True. Proof: By 33-1, choose integers $k$ and $\ell$ such that

$$\frac{n}{p^2 q} = \frac{k}{p^2} + \frac{\ell}{q}.$$}

By the Division Algorithm, choose integers $a$ and $t \in [0,p)$ such that $k = ap + t$, then choose integers $b$ and $s \in [0,p)$ such that $a = bp + s$, then choose integers $c$ and $u \in [0,q)$ such that $\ell = cq + u$. We then define $r := b + c$. We wish to prove that $\frac{n}{p^2 q} = r + \frac{s}{p} + \frac{t}{p^2} + \frac{u}{q}$.
We have $\frac{n}{p^2q} = \frac{k}{p^2} + \frac{\ell}{q} = \frac{ap + t}{p^2} + \frac{cq + u}{q}$, and so

$$\frac{n}{p^2q} = \frac{a}{p} + \frac{t}{p^2} + c + \frac{u}{q} = \frac{bp + s}{p} + \frac{t}{p^2} + c + \frac{u}{q} = b + \frac{s}{p} + \frac{t}{p^2} + \frac{u}{q} = r + \frac{s}{p} + \frac{t}{p^2} + \frac{u}{q}.$$ □

33-3. True or False: Let $\mathbb{R}[x]$ denote the ring of polynomials, with real coefficients, in the indeterminate $x$. Let $p, q \in \mathbb{R}[x]$ be irreducible polynomials, and let $f \in \mathbb{R}[x]$. Assume that $p \neq q$. Then there exist $r, s, t, u \in \mathbb{R}[x]$ such that $\deg[s] < \deg[p]$ and $\deg[t] < \deg[p]$ and $\deg[u] < \deg[q]$ and $f = \frac{r}{p^2q} = \frac{s}{p} + \frac{t}{p^2} + \frac{u}{q}$.

Solution: True. The proof is the same as for 33-2, except: We are using “$f$” instead of “$n$”, and we must follow the Euclidean Algorithm and the Division Algorithm in $\mathbb{R}[x]$, rather than in $\mathbb{Z}$. □

NOTE: This is an example of a partial fractions decomposition, often taught as a technique of integration in first year Calculus.

34-1. Define $N : \mathbb{R}^2 \to [0, \infty)$ by $N(x, y) = [x^4 + y^4]^{1/4}$. (This is sometimes called the $L^4$-norm on $\mathbb{R}^2$.) Let $C := (1, 2) \in \mathbb{R}^2$ and let $D := (3, 5) \in \mathbb{R}^2$. Let

$$S := \{ A \in \mathbb{R}^2 \mid N(A - C) = 1 \}$$
$$T := \{ B \in \mathbb{R}^2 \mid N(B - D) = 2 \}$$

(These are two $L^4$-spheres in $\mathbb{R}^2$.) Minimize $N(A - B)$ subject to the constraints $A \in S$ and $B \in T$. (That is, compute how close the one $L^4$-sphere gets to the other.)

Solution: In this problem $\text{dist}$ denotes $L^4$-distance, so, for all $P, Q \in \mathbb{R}^2$, we define $\text{dist}(P, Q) := N(P - Q)$. The $L^4$-triangle inequality asserts: For all $P, Q, R \in \mathbb{R}^2$, $\text{dist}(P, R) \leq [\text{dist}(P, Q)] + [\text{dist}(Q, R)]$. Also, note: For all $P, Q, R \in \mathbb{R}^2$, if $Q$ is on the line segment from $P$ to $R$, then $\text{dist}(P, R) = [\text{dist}(P, Q)] + [\text{dist}(Q, R)]$. (These facts hold for any
norm on $\mathbb{R}^2$, and, in particular, for the $L^4$-norm.) We define the length of a line segment in $\mathbb{R}^2$ to be the $L^4$-distance between its endpoints. We define the length of a polygonal path in $\mathbb{R}^2$ to be the sum of the lengths of its line segments.

Let $L$ be the line segment from $C$ to $D$. Let $A$ be the point of intersection of $S$ and $L$. Let $B$ be the point of intersection of $T$ and $L$. By the $L^4$-triangle inequality (and mathematical induction), there is no polygonal path from $C$ to $D$ whose total $L^4$-length is $< \text{dist}(C,D)$.

If $\exists A_1 \in S, B_1 \in T$ s.t. $\text{dist}(A_1,B_1) < \text{dist}(A,B)$, then, concatenating

- the line segment from $C$ to $A_1$ (which has length 1),
- the line segment from $A_1$ to $B_1$,
- the line segment from $B_1$ to $D$ (which has length 2),

we would arrive at a polygonal path from $C$ to $D$ whose total length is $1 + \text{dist}(A,B) + 2 = \text{dist}(C,D)$, which, as we just pointed out, is impossible. Thus the minimal distance between any point on $S$ and any point on $T$ is $\text{dist}(A,B)$, and this is the number we seek. Because $A$ and $B$ are points on $L$, we have

$$\text{dist}(A,B) = [\text{length}(L)] - [\text{dist}(C,A)] - [\text{dist}(D,B)].$$

We have $C = (1, 2)$ and $D = (3, 5)$, so the $L^4$-length of $L$ is

$$\text{dist}(C,D) = N(C-D) = [(1-3)^4 + (2-5)^4]^{1/4}$$

$$= [(-2)^4 + (-3)^4]^{1/4} = [16 + 81]^{1/4} = \sqrt[4]{97}.$$

The distance from $C$ to $A$ is the radius of $S$, which is 1. The distance from $D$ to $B$ is the radius of $T$, which is 2.

Thus $\text{dist}(A,B) = \sqrt[4]{97} - 1 - 2 = \sqrt[4]{97} - 3$. \hfill \Box$

Alternate Solution: The gradient of $(p-s)^4 + (q-t)^4$ w.r.t. $p,q,s,t$ is

$$(4(p-s)^3, 4(q-t)^3, -4(p-s)^3, -4(q-t)^3).$$

The gradients of $(p-1)^4 + (q-2)^4$ and $(s-3)^4 + (t-5)^4$, w.r.t. $p,q,s,t$ are

$$(4(p-1)^3, 4(q-2)^3, 0, 0) \quad \text{and}$$

$$(0, 0, 4(s-3)^3, 4(t-5)^3)$$

Choose $p,q,s,t$ so as to minimize

$$(p-s)^4 + (q-t)^4$$
subject to
\[(p - 1)^4 + (q - 2)^4 = 1 \quad \text{and} \quad (s - 3)^4 + (t - 5)^4 = 16\]

We wish to compute \[\sqrt[4]{(p - s)^4 + (q - t)^4}.\]

By Lagrange Multipliers, choose \(\lambda_1, \mu_1 \in \mathbb{R}\) such that
\[
(4(p - s)^3, 4(q - t)^3, -4(p - s)^3, -4(q - t)^3).
\]
is equal to the \(\lambda_1, \mu_1\) linear combination of
\[
(4(p - 1)^3, 4(q - 2)^3, 0, 0) \quad \text{and} \quad (0, 0, 4(s - 3)^3, 4(t - 5)^3).
\]
Let \(\lambda := \sqrt[4]{\lambda_1}\) and let \(\mu := \sqrt[4]{\mu_1}\). Then
\[
(p - s, q - t) = \lambda \cdot (p - 1, q - 2) \quad \text{and} \quad (s - p, t - q) = \mu \cdot (s - 3, t - 5).
\]

Throughout this problem dist is \(L^4\)-distance, so, for all \(V, W \in \mathbb{R}^2\), we define \(\text{dist}(V, W) := N(V - W)\). Let \(A := (p, q)\) and let \(B := (s, t)\). Let \(C := (1, 2)\) and let \(D := (3, 5)\). Then \(A - B = \lambda(A - C)\) and \(B - A = \mu(B - D)\). Let \(L\) be the line in \(\mathbb{R}^2\) through \(C\) and \(D\). Because
\[
(p - 1)^4 + (q - 2)^4 = 1 \quad \text{and} \quad (s - 3)^4 + (t - 5)^4 = 16,
\]
it follows that \(\text{dist}(A, C) = 1\) and \(\text{dist}(B, D) = 2\). Then
\[
\text{dist}(C, D) = \sqrt[4]{(1 - 3)^4 + (2 - 5)^4} = \sqrt[4]{97} > 3 = [\text{dist}(A, C)] + [\text{dist}(B, D)],
\]
Thus
\[
\text{dist}(C, D) > \text{dist}(C, A) + \text{dist}(B, D).
\]
On the other hand, by the \(L^4\)-triangle inequality, we have
\[
\text{dist}(C, D) \leq \text{dist}(C, A) + \text{dist}(A, D).
\]
Therefore \(A \neq B\). Let \(L\) be the line through \(A\) and \(B\).

Because \(A - B = \lambda(A - C)\), it follows that \(B = \lambda C + (1 - \lambda)A\), and so \(B\) is on the line in \(\mathbb{R}^3\) through \(C\) and \(A\). This line, which passes through \(A\) and \(B\), must be \(L\). Then \(C \in L\).
Because $B - A = \mu(B - D)$, it follows that $A = \mu D + (1 - \mu)B$, and so $A$ is on the line in $\mathbb{R}^3$ through $D$ and $B$. This line, which passes through $A$ and $B$, must be $L$. Then $D \in L$.

Since $C, D \in L$, we conclude that $L$ is the line through $C$ and $D$. So, since $A \in L$, choose $v \in (0, 1)$ such that $A = (1 - v)C + vD$. Similarly, since $B \in L$, choose $w \in (0, 1)$ such that $B = wC + (1 - w)D$.

Then $A - C = v(D - C)$ and $B - D = w(C - D)$. That is,
\[
(p - 1, q - 2) = v(-2, -3) \quad \text{and} \quad (s - 3, t - 5) = w(2, 3).
\]

Recall that
\[
(p - 1)^4 + (q - 2)^4 = 1 \quad \text{and} \quad (s - 3)^4 + (t - 5)^4 = 16.
\]

Then
\[
v^4((-2)^4 + (-3)^4) = 1 \quad \text{and} \quad w^4(2^4 + 3^4) = 16.
\]

Then $v = \pm \sqrt[4]{1/97}$ and $w = \pm \sqrt[4]{16/97}$. Let $z := 1/\sqrt[4]{97}$. Then $v = \pm z$ and $w = \pm 2z$. Since $A = (1 - v)C + vD$, we see that
\[
(p, q) = A \in \{(1 - z)C + zD, (1 + z)C - zD\}.
\]

Since $B = wC + (1 - w)D$, we see that
\[
(s, t) = B \in \{2zC + (1 - 2z)D, -2zC + (1 + 2z)D\}.
\]

Thus there are two possibilities for $(p, q)$ and there are two possibilities for $(s, t)$. This gives four possibilities for $(p, q, s, t)$, each of which satisfies the constraints.

We evaluate the objective $(p - s)^4 + (q - t)^4 = [N(A - B)]^4$ assuming $(p, q) = A = (1 - z)C + zD$ and $(s, t) = B = 2zC + (1 - 2z)D$, and obtain
\[
[ N( (1 - z)C + zD ) - [2zC + (1 - 2z)D] ) ]^4,
\]
which equals
\[
[ N( (1 - 3z)C + (3z - 1)D ) ) ]^4 = [ N( (1 - 3z)(C - D) ) ]^4,
\]
which equals
\[
(1 - 3z)^4[N(C - D)]^4 = (1 - 3z)^4[N((-2, -3))]^4 = 97(1 - 3z)^4.
\]
Evaluating the objective assuming the other three possibilities yields
\[
97(1 + z)^4 \quad \text{and} \quad 97(1 - z)^4 \quad \text{and} \quad 97(1 + 3z)^4.
\]
Recall: \( z = 1/\sqrt{97} \). Among these four possibilities, the smallest value of the objective occurs at the first, so \((p, q) = A = (1 - z)C + zD\) and \((s, t) = B = 2zC + (1 - 2z)D\). Then \([N(A - B)]^4 = 97(1 - 3z)^4\).

Recall that we wish to compute

\[
\sqrt[4]{(p - s)^4 + (q - t)^4},
\]

which equals \(N(A - B)\). Thus our final answer is

\[
N(A - B) = \sqrt[4]{N(A - B)}^4 = \sqrt[4]{97(1 - 3z)^4} = \sqrt[4]{97(1 - 3z)},
\]

which equals

\[
\frac{1}{z}(1 - 3z) = \frac{1}{z} - 3 = \sqrt[4]{97} - 3. \quad \square
\]

42-1. Let \( p : \mathbb{R} \to \mathbb{R} \) be defined by \( p(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \). Let \( X \) and \( Y \) be independent random variables. Assume that \( X \) and \( Y \) are both standard normal, \( i.e., \) that both \( X \) and \( Y \) have probability density function \( p \). Compute the probability that \( X < 9Y \).

Solution: Let \( Z := (X, Y) \), which is an \( \mathbb{R}^2 \)-valued random variable. Let \( v := (-1, 9) \in \mathbb{R}^2 \). Then \( v \cdot Z = (-1, 9) \cdot (X, Y) = -X + 9Y \). We therefore wish to calculate \( \Pr[v \cdot Z > 0] \)

Let \( v_0 := (\sqrt{82}, 0) \). Then \( v \) and \( v_0 \) have the same length. Denote by \( R : \mathbb{R}^2 \to \mathbb{R}^2 \) the rotation such that \( R(v) = v_0 \). Let \( Z_0 := R(Z) \). Then \( v \cdot Z = v_0 \cdot Z_0 \). We therefore wish to calculate \( \Pr[v_0 \cdot Z_0 > 0] \).

Define \( P : \mathbb{R}^2 \to \mathbb{R} \) by \( P(x, y) = [p(x)][p(y)] \). Then \( P \) is the probability density function of \( Z \), and so \( P \circ R^{-1} \) is the probability density function of \( Z_0 \). For all \( x, y \in \mathbb{R} \), we have \( P(x, y) = \frac{e^{-(x^2+y^2)/2}}{2\pi} \).

Because \( (x, y) \mapsto x^2 + y^2 : \mathbb{R}^2 \to \mathbb{R} \) is rotationally invariant, it follows that \( P \circ R = P \). Thus \( Z \) and \( Z_0 \) have the same distribution. We therefore wish to calculate \( \Pr[v_0 \cdot Z > 0] \).

We have \( v_0 \cdot Z = (\sqrt{82}, 0) \cdot (X, Y) = \sqrt{82}X \). Then

\[
[ v_0 \cdot Z > 0 ] \iff [ X > 0 ].
\]

We therefore wish to calculate \( \Pr[X > 0] \).
Because $p$ is the PDF of $X$, it follows both that $\int_{-\infty}^{\infty} p = 1$ and that

$$\Pr[X > 0] = \int_{0}^{\infty} p.$$ 

Since $p$ is even, we have

$$\int_{0}^{\infty} p = \int_{-\infty}^{0} p.$$ 

Then

$$2 \left[ \int_{0}^{\infty} p \right] = \left[ \int_{-\infty}^{0} p \right] + \left[ \int_{0}^{-\infty} p \right] = \int_{-\infty}^{\infty} p = 1,$$

and so $\int_{0}^{\infty} p = \frac{1}{2}$. Then $\Pr[X > 0] = \int_{0}^{\infty} p = \frac{1}{2}$. \hfill \Box

46-1. TRUE OR FALSE: For any cyclic group $G$, for any homomorphism $f : G \to G$, there exists an integer $n$ such that, for all $x \in G$, we have $f(x) = x^n$.

Solution: True. Proof: Given a cyclic group $G$ and a homomorphism $f : G \to G$. We wish to show that there exists an integer $n$ such that, for all $x \in G$, we have $f(x) = x^n$.

Since $G$ is cyclic, choose a generator $a$ of $G$. Then, for all $g \in G$, there exists an integer $k$ such that $g = a^k$. Choose an integer $n$ such that $f(a) = a^n$. Given $x \in G$. We wish to show that $f(x) = x^n$.

Choose an integer $m$ such that $x = a^m$. Then

$$f(x) = f(a^m) = [f(a)]^m = [a^n]^m = [a^m]^n = x^n,$$

as desired. \hfill \Box

46-2. TRUE OR FALSE: For any Abelian group $G$, for any homomorphism $f : G \to G$, there exists an integer $n$ such that, for all $x \in G$, we have $f(x) = x^n$.

Solution: False. Counterexample: Let $C$ denote the multiplicative group $\{ -1, 1 \}$. Let $G := C \oplus C$. Then $G$ is Abelian. Let $e := (1,1)$, $a := (1,-1)$, $b := (-1,1)$, $c := (-1,-1)$. Then $G = \{ e, a, b, c \}$. Also, $ab = c$, $bc = a$ and $ca = b$. Let $f : G \to G$ be the function defined by:

$$f(e) = e, \quad f(a) = b, \quad f(b) = c \quad f(c) = a.$$ 

Let $n$ be an integer, and assume, for all $x \in G$, that $f(x) = x^n$. We aim for a contradiction.
We have \( f(a) = a^n \). Then
\[
\begin{align*}
  b &= f(a) = a^n \\
  &\in \{a^k | k \text{ is an even integer}\} \cup \{a^k | k \text{ is an odd integer}\} \\
  &= \{e\} \cup \{a\} = \{e, a\}.
\end{align*}
\]
However, \( b \notin \{e, a\} \), contradiction. \( \square \)

49-1. Up to isomorphism, how many additive Abelian groups are there of order 12?

Solution: For any integer \( n \geq 1 \), let \( C_n := \mathbb{Z}/(n\mathbb{Z}) \) be the finite additive cyclic group of order \( n \). By the Structure Theorem for Finite Abelian Groups, any additive Abelian group is isomorphic to a direct sum of additive cyclic groups of prime power order. The prime powers that divide 12 are 1, 2, 3 and 4. Consequently, up to isomorphism, the only additive Abelian groups of order 12 are
\[
C_4 \oplus C_3, \quad C_2 \oplus C_2 \oplus C_3.
\]

Thus the answer is: two. \( \square \)

49-2. Up to isomorphism, how many additive Abelian groups \( G \) of order 12 have the property that, for all \( x \in G \), \( x + x + x + x + x + x = 0 \)?

Solution: For all \( x \in G \), the condition \( x + x + x + x + x + x = 0 \) is equivalent to \( \text{the order of } x \text{ is a divisor of } 6 \), and this, in turn, is equivalent to \( \text{the order of } x \text{ is } 1 \text{ or } 2 \text{ or } 3 \text{ or } 6 \). By 49-1, we need only check \( C_4 \oplus C_3 \) and \( C_2 \oplus C_2 \oplus C_3 \). In \( C_4 \oplus C_3 \), the element \((1,1)\) has order 12, so \( G \) cannot be isomorphic to \( C_4 \oplus C_3 \). In \( C_2 \oplus C_2 \oplus C_3 \), every element has order 1 or 2 or 3 or 6. Thus \( G \) can be isomorphic to \( C_2 \oplus C_2 \oplus C_3 \). Thus the answer is: one. \( \square \)

49-3. Up to isomorphism, how many additive Abelian groups are there of order 24?

Solution: For any integer \( n \geq 1 \), let \( C_n := \mathbb{Z}/(n\mathbb{Z}) \) be the finite additive cyclic group of order \( n \). By the Structure Theorem for Finite Abelian Groups, any additive Abelian group is isomorphic to a direct sum of additive cyclic groups of prime power order. The prime powers that divide 24 are 1, 2, 3, 4 and 8. Consequently, up to isomorphism, the
only additive Abelian groups of order 12 are 
\[ C_8 \oplus C_3, \quad C_4 \oplus C_2 \oplus C_3, \quad C_2 \oplus C_2 \oplus C_2 \oplus C_3. \]
Thus the answer is: three. \[\square\]

49-4. Up to isomorphism, how many additive Abelian groups \( G \) of order 24 have the property that, for all \( x \in G \), \( x + x + x + x + x = 0 \)?

\textit{Solution:} We claim that no such group \( G \) exists. Let \( G \) be a group of order 24 such that, for all \( x \in G \), \( x + x + x + x + x = 0 \). We aim for a contradiction.

Choose \( x \in G \setminus \{0\} \). Let \( n \) denote the order of \( x \). Because
\[ x + x + x + x + x = 0, \]
it follows that \( n \) is a divisor of 5, so \( n \in \{1, 5\} \). Because \( \# G = 24 \), it follows that \( n \) is a divisor of 24, so \( n \in \{1, 2, 3, 4, 6, 8, 12, 24\} \). Then \( n \in \{1, 5\} \cap \{1, 2, 3, 4, 6, 8, 12, 24\} = \{1\} \), so \( n = 1 \). That is, the order of \( x \) is 1, and it follows that \( x = 0 \). However, \( x \in G \setminus \{0\} \), so \( x \neq 0 \), contradiction, completing the proof of the claim.

Since no such group \( G \) exists, the answer is: zero. \[\square\]

49-5. Up to isomorphism, how many additive Abelian groups \( G \) of order 24 have the property that, for all \( x \in G \), \( x + x + x + x + x = 0 \)?

\textit{Solution:} For all \( x \in G \), the condition \( x + x + x + x = 0 \) is equivalent to [ the order of \( x \) is a divisor of 4 ], and this, in turn, is equivalent to [ the order of \( x \) is 1 or 2 or 4 ]. By 49-3, we need only check \( C_8 \oplus C_3 \) and \( C_4 \oplus C_2 \oplus C_3 \) and \( C_2 \oplus C_2 \oplus C_2 \oplus C_3 \). In \( C_8 \oplus C_3 \), the element \( (1, 1) \) has order 24, so \( G \) cannot be isomorphic to \( C_8 \oplus C_3 \). In \( C_4 \oplus C_2 \oplus C_3 \), the element \( (1, 1, 1) \) has order 12, so \( G \) cannot be isomorphic to \( C_4 \oplus C_2 \oplus C_3 \). In \( C_2 \oplus C_2 \oplus C_2 \oplus C_3 \), the element \( (1, 1, 1, 1) \) has order 6, so \( G \) cannot be isomorphic to \( C_2 \oplus C_2 \oplus C_2 \oplus C_3 \). Thus the answer is: zero. \[\square\]
59-1. Let \( f \) be an analytic function of a complex variable \( z = x + iy \) given by
\[
f(z) = (3x + 5y) + i \cdot (g(x, y)),
\]
where \( g(x, y) \) is a real-valued function of the real variables \( x \) and \( y \). If \( g(0, 0) = 1 \), then \( g(7, 3) = \)

**Solution:** We will compute \( g(7, 3) - g(7, 0) \) and \( g(7, 0) - g(0, 0) \) separately, and then add the results to get \( g(7, 3) - g(0, 0) \). We will then add \( g(0, 0) \), which is given in the problem as 1, and obtain \( g(7, 3) \).

Define \( Z : \mathbb{R}^2 \to \mathbb{C} \) by \( Z(x, y) = x + iy \). Define \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( h(x, y) = 3x + 5y \). Then \( f \circ Z = h + ig \).

According to the Cauchy-Riemann equations, a counterclockwise \( 90^\circ \) rotation of \((\partial_1 h, \partial_1 g)\) gives \((\partial_2 h, \partial_2 g)\). That is,
\[
\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \partial_1 h \\ \partial_1 g \end{bmatrix} = \begin{bmatrix} \partial_2 h \\ \partial_2 g \end{bmatrix}.
\]
That is, \(-\partial_1 g = \partial_2 h \) and \( \partial_1 h = \partial_2 g \).

For all \( x, y \in \mathbb{R} \), \( h(x, y) = 3x + 5y \). Computing partial derivatives, for all \( x, y \in \mathbb{R} \), we get \((\partial_1 h)(x, y) = 3\) and \((\partial_2 h)(x, y) = 5\), and so
\[
-(\partial_1 g)(x, y) = 5 \quad \text{and} \quad (\partial_2 g)(x, y) = 3.
\]
Multiplying the first equation by \(-1\), and substituting \( y :\to 0 \), we see, for all \( x \in \mathbb{R} \), that \((\partial_1 g)(x, 0) = -5\). Integrating this equation from \( x = 0 \) to \( x = 7 \), we see that \([g(7, 0)] - [g(0, 0)]\) = \(\int_0^7 (-5) \, dx \). Then \([g(7, 0)] - [g(0, 0)] = (-5)(7) = -35\). Recall that, for all \( x, y \in \mathbb{R} \), \((\partial_2 g)(x, y) = 3\). Substituting \( x :\to 7 \), we see, for all \( y \in \mathbb{R} \), that \((\partial_2 g)(7, y) = 3\). Integrating this equation from \( y = 0 \) to \( y = 3 \) yields \([g(7, 3)] - [g(7, 0)]\) = \(\int_0^3 3 \, dx = (3)(3) = 9\). Then
\[
[g(7, 3)] - [g(0, 0)] = ([g(7, 3)] - [g(7, 0)]) + ([g(7, 0)] - [g(0, 0)])
= 9 + (-35) = -26.
\]
Then \( g(7, 3) = [g(0, 0)] + (-26) = 1 + (-26) = -25 \).