SOLUTIONS OF VARIATIONS, PRACTICE TEST 1

1-1. Let C be the part of the graph of $y = \ln(\cos x)$ between $x = 0$ and $x = \pi/4$. Find the length of C.

Solution: We parametrize the curve C by $x = t$ and $y = \ln(\cos t)$, for $0 \leq t \leq \pi/4$. The velocity is given by $\dot{x} = 1$ and $\dot{y} = \tan t$. For $0 < t < \pi/4$, the speed is given by $\left[1^2 + (\tan t)^2\right]^{1/2} = \left[\sec^2 t\right]^{1/2} = \sec t$. The length of C is then

$$\begin{aligned} \int_0^{\pi/4} \sec t \, dt &= [\ln((\tan t) + (\sec t))]_{t:\to 0}^{t:\to \pi/4} \\ &= \left[\ln\left(1+\sqrt{2}\right)\right] - [\ln(0+1)] = \ln\left(1+\sqrt{2}\right). \end{aligned}$$ □

1-2. In xyz-space, let C be the curve with parametric equations $x = 2t$, $y = t^2$ and $z = t^3/3$, $0 \leq t \leq 1$. Find the length of C.

Solution: The velocity is given by $\dot{x} = 2$, $\dot{y} = 2t$ and $\dot{z} = t^2$. The speed is given by $\left[2^2 + (2t)^2 + (t^2)^2\right]^{1/2} = \left[4 + 4t^2 + t^4\right]^{1/2} = 2 + t^2$. The length of C is then $\displaystyle\int_0^1 (2+t^2)\,dt = \left[2t + \frac{t^3}{3}\right]_{t:\to 0}^{t:\to 1} = 2 + \frac{1}{3} = \frac{7}{3}$. □

2-1. Give an equation of the line tangent to the graph of $y = 5x + \sin x$ at $x = \pi$.

Solution: The slope is

$$[(d/dx)(5x + \sin x)]_{x:\to\pi} = [5 + \cos x]_{x:\to\pi} = 5 + (-1) = 4.$$

The y-coordinate of the point of tangency is

$$[5x + \sin x]_{x:\to\pi} = 5\pi + 0 = 5\pi.$$

so the point of tangency is $(\pi, 5\pi)$. An equation of the line is therefore $y - 5\pi = 4(x - \pi)$, or, equivalently, $y = 4x + \pi$. □

Date: Printout date: November 18, 2015.

3-1. If V is a 3-dimensional subspace of $\mathbb{R}^7$ and W is a 5-dimensional subspace of $\mathbb{R}^7$, what are the possible dimensions of $V \cap W$?

Solution: The minimum possible dimension of $V + W$ is

$$\max\{\dim V, \dim W\} \quad = \quad \max\{3,5\} \quad = \quad 5.$$

The maximum possible dimension of $V + W$ is

$$\min\{\,(\dim V) + (\dim W)\,,\, \dim \mathbb{R}^7\,\} \quad = \quad \min\{3+5,7\} \quad = \quad 7.$$

So the set of possible dimensions of $V + W$ is $\{5,6,7\}$. As

$$(\dim V) \;+\; (\dim W) \quad = \quad (\dim (V+W)) \;+\; (\dim (V \cap W)),$$

we see that $8 = 3 + 5 = (\dim (V+W)) + (\dim (V \cap W))$, so

$$\dim (V \cap W) \quad = \quad 8 \;-\; (\dim (V+W)).$$

Then the set of possible dimensions of $V \cap W$ is

$$\{\,8-5\,,\,8-6\,,\,8-7\,\} \quad = \quad \{3,2,1\}. \qquad \square$$

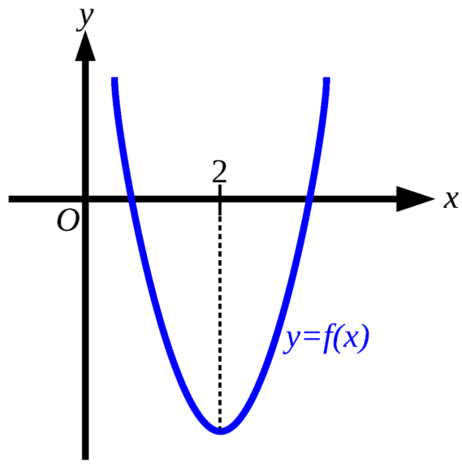
4-1. Let k be the number of real solutions of the equation $7 - x^5 - x = 0$ in the interval $[0,1]$, and let n be the number of real solutions that are <u>not</u> in $[0,1]$. Which of the following is true?

(A) $k = 0$ and $n = 1$
(B) $k = 1$ and $n = 0$
(C) $k = n = 1$
(D) $k > 1$
(E) $n > 1$

Solution: Since $[7 - x^5 - x]_{x:\to -1000} > 0$ and $[7 - x^5 - x]_{x:\to 1000} < 0$, it follows that the equation $7 - x^5 - x = 0$ has at least one solution in $\mathbb{R}$, so $k + n \geq 1$. For all $x \in \mathbb{R}$, $[d/dx][7 - x^5 - x] = -5x^4 - 1 < 0$, so $7 - x^5 - x$ is decreasing in x. Thus the equation $7 - x^5 - x = 0$ has at most solution in $\mathbb{R}$, so $k + n \leq 1$. Then $k + n = 1$.

Since $[7 - x^5 - x]_{x:\to 0} = 7 > 0$, since $[7 - x^5 - x]_{x:\to 1} = 7 - 1 - 1 > 0$ and since $7 - x^5 - x$ is decreasing in x, it follows that the equation $7 - x^5 - x = 0$ has no solutions in $[0,1]$. Then $k = 0$.

Then $n = (k+n) - k = 1 - 0 = 1$. Answer: (A) $\square$



5-1. Suppose b is a real number and $f(x) = 4x^2 + bx + 9$ defines a function on the real line, part of which is graphed above. Compute $f(5)$.

Solution: We have $f'(x) = 8x + b$, so $f'(2) = 16 + b$. From the graph, $f'(2) = 0$. Thus $16 + b = 0$, so $b = -16$. Then $f(x) = 4x^2 - 16x + 9$, so $f(5) = 4 \cdot 5^2 - 16 \cdot 5 + 9 = 100 - 80 + 9 = 29$. □

6-1. For what values of b does the curve $4x^2 + (y - b)^2 = 1$ have exactly one intersection point with $y = 2x$?

Solution: Let $b \in \mathbb{R}$. For all $x, y \in \mathbb{R}$,

$$[\ (\ 4x^2 + (y-b)^2 = 1\ ) \text{ and } (\ y = 2x\ )\ ] \quad \text{iff}$$
$$[\ (\ y^2 + (y-b)^2 = 1\ ) \text{ and } (\ y = 2x\ )\ ] \quad \text{iff}$$
$$[\ (\ y^2 + y^2 - 2yb + b^2 = 1\ ) \text{ and } (\ x = y/2\ )\ ] \quad \text{iff}$$
$$[\ (\ 2y^2 - 2by + (b^2 - 1) = 0\ ) \text{ and } (\ x = y/2\ )\ ] \quad \text{iff}$$

The expression $2y^2 - 2by + (b^2 - 1)$ is a quadratic in y with coefficients

$$2, \qquad -2b, \qquad b^2 - 1.$$

The discriminant of this quadratic is

$$(-2b)^2 - 4 \cdot 2 \cdot (b^2 - 1) \quad = \quad 4b^2 - 8(b^2 - 1) \quad = \quad -4b^2 + 8.$$

Thus

$$[\ \text{there is one intersection point}\ ] \quad \text{iff}$$
$$[\ -4b^2 + 8 = 0\ ] \quad \text{iff} \quad [\ b^2 = 2\ ] \quad \text{iff}$$
$$[\ (\ b = \sqrt{2}\ ) \text{ or } (\ b = -\sqrt{2}\ )\ ].$$
□

7-1. Compute $\displaystyle\int_{-3}^{3} e^{|x+1|}\,dx$.

Solution: The integral is equal to

$$\begin{aligned} & \left[\int_{-3}^{-1} e^{|x+1|}\,dx\right] + \left[\int_{-1}^{3} e^{|x+1|}\,dx\right] \\ = & \left[\int_{-3}^{-1} e^{-x-1}\,dx\right] + \left[\int_{-1}^{3} e^{x+1}\,dx\right] \\ = & \left[-e^{-x-1}\right]_{x:\to-3}^{x:\to-1} + \left[e^{x+1}\right]_{x:\to-1}^{x:\to3} \\ = & \left[-e^0 - \left(-e^2\right)\right] + \left[e^4 - e^0\right] \\ = & -1 + e^2 + e^4 - 1 \quad = \quad e^4 + e^2 - 2. \qquad \square \end{aligned}$$

8-1. Let R be a rectangle whose vertices are (x,y), $(-x,y)$, $(-x,0)$ and $(x,0)$. Assume that $0 < x < 3$, that $0 < y < 3$ and that $x^4 + y^4 = 1$. What is the maximum possible area inside such a rectangle R?

Solution: The rectangle has width $2x$ and has height y. Therefore, since $y = (1-x^4)^{1/4}$, the area is $2xy = 2x(1-x^4)^{1/4}$. We have

$$\begin{aligned} f'(x) &= \left[2(1-x^4)^{1/4}\right] + \left[2x(1/4)(1-x^4)^{-3/4}(-4x^3)\right] \\ &= \left[\frac{2(1-x^4)}{(1-x^4)^{3/4}}\right] + \left[\frac{2x(1/4)(-4x^3)}{(1-x^4)^{3/4}}\right] \\ &= \frac{[2(1-x^4)] + [-2x^4]}{(1-x^4)^{3/4}} \quad = \quad \frac{2(1-2x^4)}{(1-x^4)^{3/4}}. \end{aligned}$$

Then $f'(x) > 0$ on $0 < x < 2^{-1/4}$. Also, $f'(x) < 0$ on $2^{-1/4} < x < 1$. Thus $f(x)$ is increasing on $0 < x < 2^{-1/4}$. Also, $f(x)$ is decreasing on $2^{-1/4} < x < 1$. Then $f(x)$ attains a global maximum at $x = 2^{-1/4}$. So, since $f(x) = 2x(1-x^4)^{1/4}$, the maximum value is

$$\begin{aligned} f\left(2^{-1/4}\right) &= 2\cdot 2^{-1/4}\cdot\left(1-2^{-1}\right)^{1/4} = 2\cdot 2^{-1/4}\cdot(1/2)^{1/4} \\ &= 2^{1-(1/4)-(1/4)} = 2^{1/2} = \sqrt{2}. \qquad \square \end{aligned}$$

9-1. Define

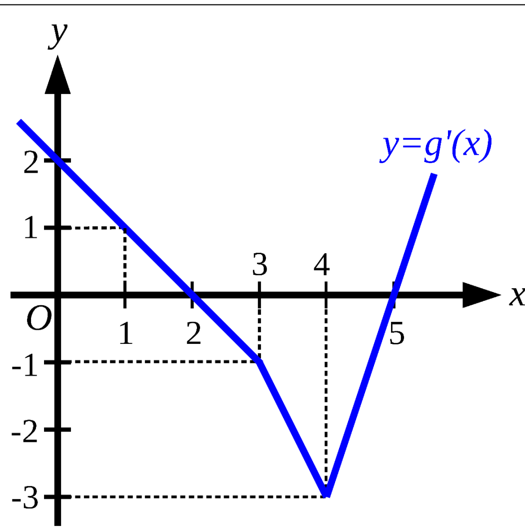
$$\begin{aligned} J &:= \int_1^2 \sqrt{256 - x^4}\, dx \\ K &:= \int_1^2 \sqrt{256 + x^4}\, dx \\ L &:= \int_1^2 \sqrt{256 - x^8}\, dx \end{aligned}$$

Order 16, J, K, L from smallest to largest.

Solution: On $1 < x < 2$, we have:

$$\begin{aligned} & -256 < -x^8 < -x^4 \quad < 0 < x^4, \\ \text{which implies} \quad & 0 < 256 - x^8 < 256 - x^4 < 256 < 256 + x^4, \\ \text{which implies} \quad & \sqrt{256 - x^8} < \sqrt{256 - x^4} < 16 < \sqrt{256 + x^4}. \end{aligned}$$

Thus $L < J < 16 < K$. □



10-1. Let g be a function whose derivative g' is continuous and has the graph shown above. On $0 < x < 5$, what are the maximal open intervals of concavity for $g(x)$?

Solution: Since $g'(x)$ is decreasing on $0 < x < 4$, it follows that $g(x)$ is concave down on $0 < x < 4$. Since $g'(x)$ is increasing on $4 < x < 5$, it follows that $g'(x)$ is concave up on $4 < x < 5$.

It follows, on $0 < x < 5$, that the only maximal open interval of concave down for $g(x)$ is $0 < x < 4$. It also follows, on $0 < x < 5$, that the only maximal open interval of concave up for $g(x)$ is $4 < x < 5$. $\square$

11-1. Approximate $[3.59]\left[(10)^{5/2}\right]$.

Solution: We have:

$$\begin{aligned}
\left[\sqrt{3.59}\right]\left[(10)^{5/2}\right] &= \left[\sqrt{3.59}\right]\left[\sqrt{(10)^5}\right] \\
&= \left[\sqrt{3.59}\right]\left[\sqrt{10}\right]\left[\sqrt{(10)^4}\right] \\
&= \left[\sqrt{(3.59)(10)}\right]\left[10^2\right] \\
&= \left[\sqrt{35.9}\right][100] \\
&\approx< \left[\sqrt{36}\right][100] \\
&= [6][100] \\
&= 600 \quad \square
\end{aligned}$$

NOTE: Two two decimals, the exact answer is 599.17.

12-1. Let A be a 5×5 matrix such that the entries in each row add up to 10. Let $B := 6A^3 + 4A^2 + 7A$. True or False: The entries any row of B will add up to 6470.

Solution: True. Proof: Let v be the row vector $\left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \end{array}\right]$. Let $w := v^t$ be the transpose of v, so that w is a 5×1 column vector, with all entries equal to 1. Since sum of the entries in each row of A is 10, we get $Aw = 10w$. Then $A^2w = 100w$ and $A^3w = 1000w$. Then

$$Bw = 6A^3w + 4A^2w + 7Aw = 6000w + 400w + 70w = 6470w.$$

Then the entries in any row of B add up to 6470. $\square$

13-1. We have available 75 square feet of material, and wish to use it to form the sides and bottom of an open-topped rectangular box. What is the maximum volume of the box?

Solution: Let x and y be the dimensions of the base of an open-topped rectangular box constructed from 75 square feet of material. Let h denote the height of the box. The total of the surface areas of the bottom and the four sides is $xy+xh+yh+xh+yh$, or $xy+(2x+2y)h$. Then $xy + (2x + 2y)h = 75$. Then $h = (75 - xy)/(2x + 2y)$. Also, because $x, y, h > 0$, it follows that $xy < 75$. Let

$$D \quad := \quad \{ \ (x,y) \in \mathbb{R}^2 \mid x,y > 0, \ xy < 75 \ \}.$$

For all $(x, y) \in D$, let $V(x, y)$ denote the volume of the box. Then $V(x, y) = xyh = xy(75-xy)/(2x+2y)$. We wish to maximize V on D.

For all $s \in (0, \infty)$, let $D_s := (0, \sqrt{75/s}\,)$ and define $V_s : D_s \to \mathbb{R}$ by $V_s(x) = V(x, sx)$. Then, because $D = \bigcup\limits_{s>0} \{(x, sx) \,|\, x \in D_s\}$, it follows that $V(D) = \bigcup\limits_{s>0} \{V(x, sx) \,|\, x \in D_s\} = \bigcup\limits_{s>0} [V_s(D_s)]$.

We will show, for all $s > 0$, that, on D_s, the function V_s attains its global maximum at $5/\sqrt{s}$. We will also show that the function $s \mapsto V_s(5/\sqrt{s}) : (0, \infty) \to \mathbb{R}$ attains its global maximum at $s = 1$. It will then follow that the global maximum value of V is $V_1(5/\sqrt{1})$.

We first show that the function $s \mapsto V_s(5/\sqrt{s}) : (0, \infty) \to \mathbb{R}$ attains its global maximum at 1, as follows: For all $s > 0$, we have

$$\begin{aligned} V_s(5/\sqrt{s}) &= V(5/\sqrt{s}, s \cdot (5/\sqrt{s})) \quad = \quad V(5/\sqrt{s}, 5 \cdot \sqrt{s}) \\ &= \frac{(5/\sqrt{s})(5 \cdot \sqrt{s})(75 - (5/\sqrt{s})(5 \cdot \sqrt{s}))}{2 \cdot (5/\sqrt{s}) + 2 \cdot (5 \cdot \sqrt{s})} \\ &= \frac{25 \cdot (75 - 25)}{(10/\sqrt{s}) + 10 \cdot \sqrt{s}} \\ &= \frac{25 \cdot 50}{10 \cdot [(1/\sqrt{s}) + \sqrt{s}]} \cdot \frac{\sqrt{s}}{\sqrt{s}} \quad = \quad 125 \frac{\sqrt{s}}{1+s}. \end{aligned}$$

Differentiating with respect to s, we find, on $s > 0$, that

$$\begin{aligned} \frac{d}{ds}[V(5/\sqrt{s})] &= 125 \frac{(1+s)((1/2)s^{-1/2}) - \sqrt{s}}{(1+s)^2} \cdot \frac{2\sqrt{s}}{2\sqrt{s}} \\ &= \frac{125}{2} \frac{(1+s) - 2s}{(1+s)^2\sqrt{s}} \quad = \quad \frac{125}{2} \frac{1-s}{(1+s)^2\sqrt{s}}. \end{aligned}$$

Thus $V_s(5/\sqrt{s})$ is increasing on $0 < s \leq 1$ and decreasing on $1 \leq s$. It follows that the function $s \mapsto V_s(5/\sqrt{s}) : (0, \infty) \to \mathbb{R}$ attains its global maximum at 1. It remains to show, for all $s > 0$, that, on D_s, the function V_s attains its global maximum at $5/\sqrt{s}$. Given $s > 0$. We wish show, on D_s, that V_s attains its global maximum at $5/\sqrt{s}$.

By definition, for all $x \in D_s$, we have

$$V_s(x) = V(x, sx) = \frac{x \cdot (sx) \cdot (75 - x \cdot (sx))}{2x + 2 \cdot (sx)},$$

so

$$V_s(x) = \frac{sx^2(75 - sx^2)}{(2 + 2s)x} = \left[\frac{s}{2 + 2s}\right]\left[\frac{x^2(75 - sx^2)}{x}\right],$$

so $V_s(x) = \left[\frac{s}{2 + 2s}\right][75x - sx^3]$. Differentiating with respect to x, we find, on $0 < x < \sqrt{75/s}$, that

$$\frac{d}{dx}[V_s(x)] = \left[\frac{s}{2 + 2s}\right][75 - 3sx^2] = \left[\frac{3s}{2 + 2s}\right][25 - sx^2].$$

Thus $V_s(x)$ is increasing on the interval $0 < x < 5/\sqrt{s}$ and $V_s(x)$ is decreasing on the interval $5/\sqrt{s} < x < \sqrt{75/s}$. Thus V_s attains its global maximum at $5/\sqrt{s}$.

Thus the global maximum value of V is $V_1(5/\sqrt{1})$, which is equal to

$$V_1(5) = V(5, 5) = \frac{5 \cdot 5 \cdot (75 - 5 \cdot 5)}{2 \cdot 5 + 2 \cdot 5} = \frac{25 \cdot 50}{20} = \frac{125}{2}. \quad \square$$

NOTE: To simplify the problem we could specify, in advance, that the rectangular box is to have a square base. That would result in a problem in one-variable calculus, instead of multi-variable calculus.

14-1. What is the hundreds digit in the standard decimal expansion of the number 7^{26}?

Solution: In this solution, congruences ($\equiv$) are all mod 100. We have

$$\begin{aligned} 7^2 = 7 \cdot 7 &= 49, \qquad \text{so} \\ 7^3 = 7 \cdot 7^2 \equiv 7 \cdot 49 = 343 &\equiv 43, \qquad \text{so} \\ 7^4 = 7 \cdot 7^3 \equiv 7 \cdot 43 = 301 &\equiv 1. \end{aligned}$$

Then $7^{26} = 7^2 \cdot (7^4)^6 \equiv 49 \cdot 1^6 = 49$. Thus the hundreds digit of 7^{25} is the same as that of 49, namely 4. $\square$

15-1. True or False: Let f be a continuous real-valued function defined on the open interval $(-2,3)$. Then f is bounded.

Solution: False. Counterexample: Let $f:(-2,3)\to\mathbb{R}$ be defined by $f(x)=1/(x+2)$. Then $\lim\limits_{x\to-2^+}(f(x))=\infty$, so f is not bounded above, so f is not bounded. □

15-2. True or False: Let f be a continuous real-valued function defined on the closed interval $[-2,3]$. There exists $c\in(-2,3)$ such that f is differentiable at c and such that $5\cdot[f'(c)]=[f(3)]-[f(-2)]$.

Solution: False. Counterexample: Let $f:[-2,3]\to\mathbb{R}$ be defined by $f(x)=|x|$. Then $[f(3)]-[f(-2)]=|3|-|-2|=1$.

On the other hand,

- for all $c\in(-2,0)$, $\qquad 5\cdot[f'(c)]=5\cdot[-1]=-5$,
- f is not differentiable at 0, and
- for all $c\in(0,3)$, $\qquad 5\cdot[f'(c)]=5\cdot 1=5$.

Thus there is no $c\in(-2,3)$ such that f is differentiable at c and such that $5\cdot[f'(c)]=1$. □

15-3. True or False: Let f be a continuous real-valued function defined on the closed interval $[-2,3]$. Assume that f is differentiable at 0 and that $f'(0)=0$. Then f has a local extremum at 0.

Solution: False. Counterexample: Let $f:[-2,3]\to\mathbb{R}$ be defined by $f(x)=x^3$. Then f is differentiable at 0 and $f'(0)=3\cdot 0^2=0$, but f is increasing on $[-2,3]$, so f has no local extremum at 0. □

15-4. True or False: Let f be a continuous real-valued function defined on the closed interval $[-2,3]$. Assume that all of the following are true:

- f is twice-differentiable at 0,
- $f'(0)=0$ and
- $f''(0)\neq 0$.

Then f has a local extremum at 0.

Solution: True. Proof: By the Second Derivative Test, if $f''(0)>0$, then f has a local minimum at 0. Also, by the Second Derivative Test, if $f''(0)<0$, then f has a local maximum at 0. In either case, f has a local extremum at 0. □

16-1. What is the volume of the solid formed by revolving, about the x-axis, the region in the first quadrant of the xy-plane bounded by: the coordinate axes and the graph of the equation $y = \sqrt{\frac{x}{1+x^4}}$?

Solution: By the disk method, with $u = x^2$, $du = 2x\,dx$, the volume is

$$\begin{aligned}\int_0^\infty \pi \left[\sqrt{\frac{x}{1+x^4}}\right]^2 dx &= \pi \int_0^\infty \frac{x\,dx}{1+x^4} \\ &= \pi \int_0^\infty \frac{du/2}{1+u^2} \\ &= \frac{\pi}{2}\left[\arctan u\right]_{u:\to 0}^{u:\to\infty} \\ &= \frac{\pi}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{4}. \quad \square\end{aligned}$$

16-2. What is the volume of the solid formed by revolving, about the y-axis, the region in the first quadrant of the xy-plane bounded by: the coordinate axes and the graph of the equation $y = \frac{x^2}{(1+x^4)^{3/2}}$?

Solution: By the shell method, with $u = 1 + x^4$, $du = 3x^2\,dx$, the volume is

$$\begin{aligned}\int_0^\infty 2\pi x \left[\frac{x^2}{(1+x^4)^{3/2}}\right] dx &= 2\pi \int_0^\infty \frac{x^3\,dx}{(1+x^4)^{3/2}} \\ &= 2\pi \int_1^\infty \frac{du/3}{u^{3/2}} \\ &= \frac{2\pi}{3}\int_1^\infty u^{-3/2}\,du \\ &= \frac{2\pi}{3}\left[\frac{u^{-1/2}}{-1/2}\right]_{u:\to 1}^{u:\to\infty} \\ &= \frac{2\pi}{3}\left[0 - \left(\frac{1}{-1/2}\right)\right] = \frac{4\pi}{3}. \quad \square\end{aligned}$$

17-1. How many real roots does the polynomial $x^5 - 5x + 3$ have?

Solution: Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^5 - 5x + 3$. We want to know how many real roots f has.

For all $x \in \mathbb{R}$, we have $f'(x) = 5x^4 - 5 = 5(x-1)(x+1)(x^2+1)$. Thus f' is positive on $(1, \infty)$, is negative on $(-1, 1)$ and is positive on $(-\infty, -1)$. Thus, by the Increasing Test and the Decreasing Test, f is increasing on $(1, \infty)$, is decreasing on $(-1, 1)$ and is increasing on $(-\infty, -1)$. Thus f has at most one root on $(1, \infty)$, has at most one root on $(-1, 1)$ and has at most one root on $(-\infty, -1)$.

We have $\lim\limits_{x \to -\infty} [f(x)] = \lim\limits_{x \to -\infty} x^5 = -\infty$ and $f(-1) = -1+5+3 > 0$ and $f(1) = 1 - 5 + 3 < 0$ and $\lim\limits_{x \to \infty} [f(x)] = \lim\limits_{x \to \infty} x^5 = \infty$.

Because $\lim\limits_{x \to -\infty} [f(x)] = -\infty$ and $f(-1) > 0$, it follows, from the Intermediate Value Theorem, that f has at least one root on $(-\infty, -1)$. So, since f has at most one root on $(-\infty, -1)$, we conclude that f has *exactly* one root on $(-\infty, -1)$.

Because $f(-1) > 0$ and $f(1) < 0$, it follows, from the Intermediate Value Theorem, that f has at least one root on $(-1, 1)$. So, since f has at most one root on $(-1, 1)$, we conclude that f has *exactly* one root on $(-1, 1)$.

Because $f(1) = 1-5+3 < 0$ and $\lim\limits_{x \to \infty} [f(x)] = \infty$, it follows, from the Intermediate Value Theorem, that f has at least one root on $(1, \infty)$. So, since f has at most one root on $(1, \infty)$, we conclude that f has *exactly* one root on $(1, \infty)$.

Because f has exactly one root on $(-\infty, -1)$ and on $(-1, 1)$ and on $(1, \infty)$, and because $f(-1) \neq 0$ and $f(1) \neq 0$, we conclude that f has exactly three real roots. $\square$

18-1. Let V be the real vector space of all real homogeneous polynomials in x and y of degree 7 (together with the zero polynomial). Let W be the real vector space of all real polynomials in x of degree ≤ 3 (together with the zero polynomial). If T is a linear transformation from V onto W, what is the dimension of the subspace $\{\mathbf{v} \in V \,|\, T(\mathbf{v}) = \mathbf{0}\}$ of V?

Solution: By definition of kernel, we have

$$\ker[T] \quad = \quad \{\mathbf{v} \in V \,|\, T(\mathbf{v}) = \mathbf{0}\}.$$

We therefore wish to calculate $\dim(\ker[T])$. A basis for V is

$$\{\, x^7,\ x^6y,\ x^5y^2,\ x^4y^3,\ x^3y^4,\ x^2y^5,\ xy^6,\ y^7 \,\},$$

so $\dim(V) = 8$. A basis for W is $\{1, x, x^2, x^3\}$, so $\dim(W) = 4$. Then

- $\dim(\mathrm{dom}\,[T]) = \dim V = 8$ and
- $\dim(\mathrm{im}\,[T]) = \dim W = 4$.

So, since

$$\dim(\ker[T]) + \dim(\mathrm{im}\,[T]) = \dim(\mathrm{dom}\,[T]),$$

we conclude that $\dim(\ker[T]) + 4 = 8$, and so $\dim(\ker[T]) = 4$. □

18-2. Let V be the real vector space of all real polynomials in x and y of degree ≤ 7 (together with the zero polynomial). Let W be the real vector space of all real polynomials in x of degree ≤ 3 (together with the zero polynomial). If T is a linear transformation from V onto W, what is the dimension of the subspace $\{\mathbf{v} \in V \,|\, T(\mathbf{v}) = \mathbf{0}\}$ of V?

Solution: By definition of kernel, we have

$$\ker[T] \quad = \quad \{\mathbf{v} \in V \,|\, T(\mathbf{v}) = \mathbf{0}\}.$$

We therefore wish to calculate $\dim(\ker[T])$. A basis for V is

$$\begin{aligned} \{ \quad & x^7,\ x^6y,\ x^5y^2,\ x^4y^3,\ x^3y^4,\ x^2y^5,\ xy^6,\ y^7, \\ & x^6,\ x^5y,\ x^4y^2,\ x^3y^3,\ x^2y^4,\ xy^5,\ y^6, \\ & x^5,\ x^4y,\ x^3y^2,\ x^2y^3,\ xy^4,\ y^5, \\ & x^4,\ x^3y,\ x^2y^2,\ xy^3,\ y^4, \\ & x^3,\ x^2y,\ xy^2,\ y^3, \\ & x^2,\ xy,\ y^2, \\ & x,\ y, \\ & 1 \quad \}, \end{aligned}$$

so $\dim(V) = 8+7+\cdots+1 = 8\cdot 9/2 = 36$. A basis for W is $\{1, x, x^2, x^3\}$, so $\dim(W) = 4$. Then

- $\dim(\operatorname{dom}[T]) = \dim V = 36$ and
- $\dim(\operatorname{im}[T]) = \dim W = 4$.

So, since

$$\dim(\ker[T]) + \dim(\operatorname{im}[T]) = \dim(\operatorname{dom}[T]),$$

we conclude that $\dim(\ker[T]) + 4 = 36$, and so $\dim(\ker[T]) = 32$. $\square$

19-1. True or False: Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that, for all $x \in \mathbb{R}$, we have $-x^2 \le f(x) \le x^2$. Then, for all $x \in \mathbb{R}$, we have $-2x \le f'(x) \le 2x$.

Solution: False. Counterexample: Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \cdot (\sin(1/x)), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Then, for all $x \in \mathbb{R}$, we have $-x^2 \le f(x) \le x^2$. Also, f is differentiable, and, for all $x \in \mathbb{R}\backslash\{0\}$, we have

$$\begin{aligned} f'(x) &= 2x \cdot (\sin(1/x)) + x^2 \cdot (\cos(1/x)) \cdot (-1/x^2) \\ &= 2x \cdot (\sin(1/x)) - (\cos(1/x)). \end{aligned}$$

In particular, $f'(1/\pi) = (2/\pi) \cdot 0 - (-1) = 1 > 2/\pi$. So it is NOT true, for all $x \in \mathbb{R}$, that $f'(x) \le 2x$. □

19-2. True or False: Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that, for all $x \in \mathbb{R}$, we have $-x^2 \le f(x) \le x^2$. Then $f'(0) = 0$.

Solution: True. Proof: We have $-0^2 \le f(0) \le 0^2$, so $f(0) = 0$. Then, for all $h \in \mathbb{R}\backslash\{0\}$, we have

$$\frac{[f(0+h)] - [f(0)]}{h} \quad = \quad \frac{f(h)}{h}.$$

We therefore wish to show that $\lim_{h \to 0} \left[\frac{f(h)}{h} \right] = 0$. It therefore suffices to show both that $\lim_{h \to 0^+} \left[\frac{f(h)}{h} \right] = 0$ and that $\lim_{h \to 0^-} \left[\frac{f(h)}{h} \right] = 0$.

For all $h > 0$, we have

$$-h \quad = \quad \frac{-h^2}{h} \quad \le \quad \frac{f(h)}{h} \quad \le \quad \frac{h^2}{h} \quad = \quad h.$$

It follows, from the Squeeze Theorem, that $\lim_{h \to 0^+} \left[\frac{f(h)}{h} \right] = 0$. It remains to show that $\lim_{h \to 0^-} \left[\frac{f(h)}{h} \right] = 0$.

For all $h < 0$, we have

$$-h \quad = \quad \frac{-h^2}{h} \quad \ge \quad \frac{f(h)}{h} \quad \ge \quad \frac{h^2}{h} \quad = \quad h.$$

It follows, from the Squeeze Theorem, that $\lim_{h\to 0^-}\left[\frac{f(h)}{h}\right] = 0.$ $\square$

19-3. True or False: Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(1) = 5$ and $f'(3) = 9$. Then $\exists c \in (1, 3)$ such that $f'(c) = 7$.

Solution: For every $h \in \mathbb{R}\backslash\{0\}$, define $g_h : \mathbb{R} \to \mathbb{R}$ by

$$g_h(x) \qquad = \qquad \frac{[f(x+h)] - [f(x)]}{h}.$$

We have

$$\lim_{h\to 0}[g_h(1)] = f'(1) = 5 \qquad \text{and} \qquad \lim_{h\to 0}[g_{-h}(3)] = f'(3) = 9.$$

Choose $b > 0$ so small that $g_b(1) < 6$ and $g_{-b}(3) > 8$. Then

$$\begin{aligned} g_b(3-b) &= \frac{[f((3-b)+b)] - [f(3-b)]}{b} \\ &= \frac{[f(3)] - [f(3-b)]}{b} = \frac{[f(3-b)] - [f(3)]}{-b} \\ &= \frac{[f(3+(-b))] - [f(3)]}{-b} = g_{-b}(3) > 8. \end{aligned}$$

Since $g_b(1) < 6$, since $g_b(3-b) > 8$ and since $g_b : \mathbb{R} \to \mathbb{R}$ is continuous, by the Intermediate Value Theorem, we choose $a \in (1, 3-b)$ such that $g_b(a) = 7$. Then $\frac{[f(a+b)] - [f(a)]}{b} = g_b(a) = 7$. Then, by the Mean Value Theorem, we choose $c \in (a, a+b)$ such that $f'(c) = 7$. It remains to show that $c \in (1, 3)$.

Since $a \in (1, 3-b)$, it follows that $1 \le a$ and that $a + b \le 3$. Then $c \in (a, a+b) \subseteq (1, 3)$, as desired. $\square$

NOTE: It is a general fact that if a real-valued function is differentiable on $\mathbb{R}$, then the function satisfies the Intermediate Value Theorem. Problem 19-3 simply asks whether this is true in a particular case, using the interval $[1, 3]$. The solution given above is based on a proof of that general fact.

19-4. True or False: Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then there exists $c \in \mathbb{R}$ such that f' is continuous at c.

NOTE: Let's talk about measure theory ...

In measure theory, one thinks of a property as holding "almost everywhere" (or, sometimes, "generically") if it holds except on a set of measure zero. An old joke has it that doing measure theory is torture because you have to say "almost everywhere" almost everywhere.

A set is **null** if it has measure zero. A set is **conull** if its complement is null. Then a generic property is one that holds on a conull set.

Now we move from measure theory to topology ...

For any $A, B \subseteq \mathbb{R}$, "$B$ is **dense in** A" means that A is a subset of the closure in $\mathbb{R}$ of B. A subset of $\mathbb{R}$ is **somewhere dense**, or **somewhere dense in** $\mathbb{R}$, if it is dense in a nonempty open subset of $\mathbb{R}$. So, for example, $\mathbb{Q} \cap (0, 1)$ is not dense in $\mathbb{R}$, but it *is* dense in $(0, 1)$, and, consequently, is somewhere dense in $\mathbb{R}$. A subset of $\mathbb{R}$ is

- **nowhere dense** if it is not somewhere dense,
- **meager** if it is a countable union of nowhere dense sets and
- **comeager** if its complement in $\mathbb{R}$ is meager.

Warning: In older books, instead of "meager" you'll see "of first category", and, instead of "comeager", you'll see "residual".

Next, let's compare topology and measure theory ...

We use Lebesgue measure on $\mathbb{R}$, and so a subset of $\mathbb{R}$ is null iff it has Lebesgue measure zero. While

- meager does not imply null, and
- null does not imply meager,

it's work to actually *find* a meager set that isn't null, or a null set that isn't meager. More importantly, the intuition for meager sets is about the same as for null sets, and, in some sense, "meager" is a topological analogue of "null". So, for example, topologists typically say that a property is "generic" if it happens on a comeager set.

Finally, let's talk about Problem 19-4 ...

If a real-valued function is differentiable on $\mathbb{R}$, then its derivative is a pointwise limit of its difference quotients, each of which is continuous. While a limit of continuous functions need not be continuous on all of $\mathbb{R}$, it is a general fact that it must be "generically continuous", *i.e.*, continuous on a *comeager* subset of $\mathbb{R}$. By the Baire Category Theorem, any comeager subset of $\mathbb{R}$ is nonempty, and it follows that the answer to Problem 19-4 is "true". The solution given below simply follows a detailed proof of that general fact, where $g := f'$ is the pointwise limit of continuous difference quotients $g_1, g_2, \ldots$ of f.

Solution: True. Proof: Let $g := f'$. For all $h > 0$, let $\gamma_h : \mathbb{R} \to \mathbb{R}$ be defined by $\gamma_h(x) = \dfrac{[f(x+h)] - [f(x)]}{h}$. Then, for all $x \in \mathbb{R}$, we have $\gamma_h(x) \to g(x)$, as $h \to 0$. Since f is differentiable, f is continuous. Then, for all $h > 0$, the function γ_h is continuous. For all integers $j \geq 1$, let $g_j := \gamma_{1/j}$; then g_j is continuous. Also, for all $x \in \mathbb{R}$, we have $g_j(x) \to g(x)$, as $j \to \infty$.

Let D be the set of $x \in \mathbb{R}$ such that g is discontinuous at x. We wish to show that $D \neq \mathbb{R}$. By the Baire Category Theorem, it suffices to show that D is meager. Assume that D is nonmeager. We aim for a contradiction.

For all $\varepsilon > 0$, let D_ε denote the set of all $x \in \mathbb{R}$ such that

- for all $\delta > 0$, $\quad g((x-\delta, x+\delta)) \not\subseteq ([g(x)] - \varepsilon, [g(x)] + \varepsilon)$.

Then $D = D_1 \cup D_{1/2} \cup D_{1/3} \cup \cdots$. Fix an integer $m \geq 1$ such that $D_{1/m}$ is nonmeager. Let $\varepsilon := 1/m$. Then D_ε is nonmeager.

We define $\eta := \varepsilon/5$ and $S := \{\ell\eta \,|\, \ell \in \mathbb{Z}\}$. For all $s \in S$, we define $I_s := [s - \eta, s + \eta]$. Then $\bigcup_{s \in S} I_s = \mathbb{R}$, so $\bigcup_{s \in S} [g^{-1}(I_s)] = g^{-1}(\mathbb{R})$. Then $\bigcup_{s \in S} [(g^{-1}(I_s)) \cap D_\varepsilon] = [g^{-1}(\mathbb{R})] \cap D_\varepsilon = \mathbb{R} \cap D_\varepsilon = D_\varepsilon$. As S is countable and D_ε is nonmeager, fix $s \in S$ such that $(g^{-1}(I_s)) \cap D_\varepsilon$ is nonmeager. Let $A := (g^{-1}(I_s)) \cap D_\varepsilon$. Then A is nonmeager.

For all integers $k \geq 1$, let L_k be the set of all $x \in \mathbb{R}$ such that

- for all integers $j \geq k$, $\quad |\,[g_j(x)] - [g(x)]\,| < \eta$.

Recall, for all $x \in \mathbb{R}$, that $g_j(x) \to g(x)$, as $j \to \infty$. It follows that $\bigcup_{k=1}^{\infty} L_k = \mathbb{R}$. Then $\bigcup_{k=1}^{\infty} [L_k \cap A] = \mathbb{R} \cap A = A$. Choose an integer $k \geq 1$ such that $L_k \cap A$ is nonmeager.

For any function $\phi : \mathbb{R} \to \mathbb{R}$, for any $\Omega \subseteq \mathbb{R}$, for any $\rho \geq 0$, let's agree that "ϕ is ρ-constant on Ω" means:

- for all $\mu, \nu \in \Omega$, $\qquad |\,(\phi(\mu)) - (\phi(\nu))\,| \leq \rho$.

Because $I_s = [s - \eta, s + \eta]$, we see that g is (2η)-constant on $g^{-1}(I_s)$. Since $A = (g^{-1}(I_s)) \cap D_\varepsilon$, we get $A \subseteq D_\varepsilon$. Then g is (2η)-constant on A. Then, by construction of L_k, we see, for all integers $j \geq k$, that g_j is (4η)-constant on $L_k \cap A$.

Every nowhere dense set is meager, while $L_k \cap A$ is *non*meager. Thus $L_k \cap A$ somewhere dense. Let C denote the closure in $\mathbb{R}$ of $L_k \cap A$. Let U be a open subset of $\mathbb{R}$ s.t. $\emptyset \neq U \subseteq C$. For all integers $j \geq k$, g_j is continuous and (4η)-constant on $L_k \cap A$, and so g_j is (4η)-constant on C, and so g_j is (4η)-constant on U. So, as $g_j \to g$ pointwise, as $j \to \infty$, we conclude that g is (4η)-constant on U. On the other hand, since $4\eta < \varepsilon$, by definition of D_ε, we see that g is *not* (4η)-constant on any open neighborhood of any point of D_ε. Then $U \cap D_\varepsilon = \emptyset$. So, since $L_k \cap A \subseteq A = (g^{-1}(I_s)) \cap D_\varepsilon \subseteq D_\varepsilon$, we conclude that $U \cap (L_k \cap A) = \emptyset$. That is, $L_k \cap A \subseteq \mathbb{R} \backslash U$. So, since $\mathbb{R} \backslash U$ is closed in $\mathbb{R}$, it follows that $C \subseteq \mathbb{R} \backslash U$, and so $U \cap C = \emptyset$. Because $U \subseteq C$, we see that $U = U \cap C$. Then $\emptyset \neq U = U \cap C = \emptyset$, contradiction. $\square$

20-1. Let f be the function defined on the real line by

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational;} \\ 2x, & \text{if } x \text{ is irrational.} \end{cases}$$

Compute the set of points of discontinuity of f.

Solution: Because the rationals and irrationals are both dense in $\mathbb{R}$, we see, for all $c > 2$, that $\liminf_{x \to c} [f(x)] = 2c$ and that $\limsup_{x \to c} [f(x)] = c^2$, and, therefore, that f is not continuous at c. Because the rationals and irrationals are both dense in $\mathbb{R}$, we see, for all $c < 0$, that $\liminf_{x \to c} [f(x)] = 2c$ and that $\limsup_{x \to c} [f(x)] = c^2$, and, therefore, that f is not continuous at c. Because the rationals and irrationals are both dense in $\mathbb{R}$, we see, for all $c \in (0, 2)$, that $\liminf_{x \to c} [f(x)] = c^2$ and that $\limsup_{x \to c} [f(x)] = 2c$, and, therefore, that f is not continuous at c.

We have $\liminf_{x \to 0} [f(x)] \geq 0$ and $\limsup_{x \to 0} [f(x)] \leq 0$ and $f(0) = 0$, and so f is continuous at 0. Finally, $\liminf_{x \to 2} [f(x)] \geq 4$ and $\limsup_{x \to 2} [f(x)] \leq 4$ and $f(2) = 4$, and so f is continuous at 2.

Thus the set of points of discontinuity of f is $\mathbb{R} \backslash \{0, 2\}$. $\square$

21-1. Let $p := 7919$, which is a prime number. Let $Q := \{p, 2p, 3p, \ldots\}$ be the set of multiples of p. Let $K := \{0, 1, \ldots, p\}$ denote the set of integers from 0 to p. For all $k \in K$, let C_k^p be the binomial coefficient "p choose k". Let $S := \{k \in K \,|\, C_1^p, \ldots, C_k^p \in Q\}$. So, for example, because $C_1^p = p \in Q$ and $C_2^p = [(p-1)/2]p = 3959p \in Q$, we get $2 \in S$. Compute the maximum element of S.

Solution: Claim: $\forall k \in \{1, \ldots, p-1\}$, we have $C_k^p \in Q$. *Proof of claim:* Given $k \in \{1, \ldots, p\}$. Let $c := C_k^p$. We wish to show that $c \in Q$.

Let $\ell := p - k$. Then $c = [p!]/[(k!)(\ell!)]$. Since p is a prime, it follows, for all integers $m, n \geq 1$, that:

$(*)$ $[mn \in Q] \Rightarrow [(m \in Q) \text{ or } (n \in Q)]$.

Taking the contrapositive, for any positive integers m and n, we have:

$(**)$ $[(m \notin Q) \text{ and } (n \notin Q)] \Rightarrow [mn \notin Q]$.

Since $1, \ldots, k < p$, it follows that $1, \ldots, k \notin Q$. Repeatedly applying $(**)$, we see that $(1)(2)\cdots(k) \notin Q$, *i.e.*, that $k! \notin Q$. Since $1, \ldots, \ell < p$, it follows that $1, \ldots, \ell \notin Q$. Repeatedly applying $(**)$, we see that $(1)(2)\cdots(\ell) \notin Q$, *i.e.*, that $\ell! \notin Q$.

We have $c(k!)(\ell!) = p! \in Q$. Then either $c \in Q$ or $(k!)(\ell!) \in Q$. Since both $k! \notin Q$ and $\ell! \notin Q$, it follows, from $(**)$, that $(k!)(\ell!) \notin Q$. Then $c \in Q$, as desired. *End of proof of claim.*

By the claim, $C_1^p, C_2^p, \ldots, C_{p-1}^p \in Q$. Then $p - 1 \in S$. On the other hand, $C_p^p = 1 \notin Q$, so $p \notin S$. So, since $S \subseteq K = \{0, \ldots, p\}$, it follows that $\max S = p - 1 = 7918$. □

22-1. Let $C(\mathbb{R})$ be the collection of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Then $C(\mathbb{R})$ is a real vector space with vector addition defined by

$$\forall f, g \in C(\mathbb{R}), \forall x \in \mathbb{R}, \qquad (f+g)(x) \quad = \quad [f(x)] \ + \ [g(x)],$$

and with scalar multiplication defined by

$$\forall f \in C(\mathbb{R}), \forall r, x \in \mathbb{R}, \qquad (rf)(x) \quad = \quad r \cdot [f(x)].$$

Let S denote the set of $f \in C(\mathbb{R})$ such that all of the following hold:

- f is twice differentiable,
- for all $x \in \mathbb{R}$, $f(x + 2\pi) = f(x)$.
- $f'' = -f$.

True or False: S is a subspace of $C(\mathbb{R})$.

Solution: True. Proof: Let V denote the set of all $\phi \in C(\mathbb{R})$ such that ϕ is twice differentiable. Then V is a subspace of $C(\mathbb{R})$.

For all $f \in C(\mathbb{R})$, let $f^+ \in C(\mathbb{R})$ be defined by $f^+(x) = f(x + 2\pi)$. Define $\Gamma : C(R) \to C(R)$ by $\Gamma(f) = f - f^+$. Then Γ is a linear map. Then $\ker[\Gamma]$ is a subspace of $C(\mathbb{R})$. Define $\Delta : V \to C(\mathbb{R})$ by $\Gamma(f) = f'' + f$. Then Δ is a linear map. Then $\ker[\Delta]$ is a subspace of V, and, therefore, is a subspace of $C(\mathbb{R})$. Then, because $S = (\ker[\Gamma]) \cap (\ker[\Delta])$, we see that S is a subspace of $C(\mathbb{R})$. □

23-1. True or False: There exists a real number b such that the line $y = 10x$ tangent to the curve $y = bx^2 + 10x + 1$ at some point in the xy-plane.

Solution: False. Proof: Let $b \in \mathbb{R}$, and assume that the line $y = 10x$ is tangent to the curve $y = bx^2 + 10x + 1$. We aim for a contradiction.

Let $x \in \mathbb{R}$ be the first coordinate of the point of tangency. Then

$$10x = bx^2 + 10x + 1 \qquad \text{and} \qquad 10 = 2bx + 10.$$

By the second equation, we see that $bx = 0$. By the first equation, we see that $x \neq 0$. Since $bx = 0$ and $x \neq 0$, we get $b = 0$. Then the first equation says $10x = 10x + 1$, and so $0 = 1$, contradiction □

24-1. Let h be the function defined by $h(x) = \displaystyle\int_0^{x^2} e^{(x+t)^2}\,dt$, for all real numbers x. Compute $h'(1)$.

Solution: Claim: For all $x \in \mathbb{R}$, we have $h(x) = \displaystyle\int_x^{x^2+x} e^{s^2}\,ds$. *Proof of claim:* Given $x \in \mathbb{R}$. We wish to prove that $h(x) = \displaystyle\int_x^{x^2+x} e^{s^2}\,ds$.

Make the change of variables $s = x + t$ and $ds = dt$, in the definition of $h(x)$. This yields $h(x) = \displaystyle\int_x^{x^2+x} e^{s^2}\,ds$. *End of proof of claim.*

Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = \displaystyle\int_0^{x} e^{s^2}\,ds$. By the claim, for all $x \in \mathbb{R}$, we have $f(x) = [F(x^2 + x)] - [F(x)]$. Differentiating, for all $x \in \mathbb{R}$, we have $f'(x) = [F'(x^2 + x)][2x + 1] - [F'(x)]$. Evaluating this at $x :\to 1$ yields $f'(1) = [F'(2)][3] - [F'(1)]$. By the Fundamental Theorem of Calculus, for all $x \in \mathbb{R}$, $F'(x) = e^{x^2}$. Then $F'(2) = e^4$ and $F'(1) = e$. Then $f'(1) = [F'(2)][3] - [F'(1)] = 3e^4 - e$. □

25-1. Let $\{a_n\}_{n=1}^{\infty}$ be defined recursively by $a_1 = 7$ and

$$\text{for all integers } n \geq 1, \qquad a_{n+1} = \left(\frac{n}{n+3}\right) a_n.$$

Compute a_{25}.

Solution: We have

$$\begin{aligned} a_{25} &= \left(\frac{24}{27}\right) a_{24} \\ &= \left(\frac{24}{27}\right)\left(\frac{23}{26}\right) a_{23} \\ &= \left(\frac{24}{27}\right)\left(\frac{23}{26}\right)\left(\frac{22}{25}\right) a_{22} \\ &= \cdots \\ &= \left(\frac{24}{27}\right)\left(\frac{23}{26}\right)\left(\frac{22}{25}\right)\cdots\left(\frac{1}{4}\right) a_1 \\ &= \frac{24!}{(27!)/(3!)} = \frac{(3)(2)(1)}{(27)(26)(25)} \\ &= \frac{1}{(9)(13)(25)} = \frac{1}{2925}. \quad \square \end{aligned}$$

26-1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = 2x^2 - 4xy + y^4$. Find all the absolute extreme values of f, and where they occur.

Solution: For all $x, y \in \mathbb{R}$,

$$f(x, y) = 2(x - y)^2 + y^4 - 2y^2 = 2(x - y)^2 + (y^2 - 1)^2 - 1,$$

so -1 is an absolute minimum value for f. Also, for all $x, y \in \mathbb{R}$, we have: $[\ f(x, y) = -1\ ]$ iff $[\ (x - y = 0)$ and $(y^2 - 1 = 0)\ ]$. Thus f attains its absolute minimum value at, and only at, $(1, 1)$ and $(-1, -1)$.

Since $\lim\limits_{x\to\infty} [f(x, 0)] = \infty$, f has no absolute maximum value. $\square$

27-1. Find the dimension of the solution space, in $\mathbb{R}^4$, of

$$\begin{array}{rcrcrcrcr} 3w & + & 4x & - & 2y & - & 3z & = & 1 \\ 2w & + & x & - & y & & & = & 2 \\ -\ w & + & 7x & - & y & - & 9z & = & -7. \end{array}$$

Solution: Multiply the first equation by -3 and add to the last:

$$\begin{array}{rcrcrcrcr} 3w & + & 4x & - & 2y & - & 3z & = & 1 \\ 2w & + & x & - & y & & & = & 2 \\ -\ 10w & - & 5x & + & 5y & & & = & -10. \end{array}$$

The third equation is -5 times the second equation; we eliminate it:

$$\begin{array}{rcrcrcrcl} 3w & + & 4x & - & 2y & - & 3z & = & 1 \\ 2w & + & x & - & y & & & = & 2. \end{array}$$

For any $w, x \in \mathbb{R}$, there is a unique solution of this system, and all solutions are obtained this way.

So the dimension of the solution space is 2. □

27-2. Find the dimension of the solution space, in $\mathbb{R}^4$, of

$$\begin{array}{rcrcrcrcr} 3w & + & 4x & - & 2y & - & 3z & = & 1 \\ 2w & + & 2x & - & y & & & = & 2 \\ -\ w & + & 7x & - & y & - & 9z & = & -7. \end{array}$$

Solution: Multiply the first equation by -3 and add to the last:

$$\begin{array}{rcrcrcrcr} 3w & + & 4x & - & 2y & - & 3z & = & 1 \\ 2w & + & 2x & - & y & & & = & 2 \\ -\ 10w & - & 5x & + & 5y & & & = & -10. \end{array}$$

Multiply the second equation by 5 and add to the last:

$$\begin{array}{rcrcrcrcl} 3w & + & 4x & - & 2y & - & 3z & = & 1 \\ 2w & + & 2x & - & y & & & = & 2 \\ & & 5x & & & & & = & 0. \end{array}$$

Divide the last equation by 5:

$$\begin{array}{rcrcrcrcl} 3w & + & 4x & - & 2y & - & 3z & = & 1 \\ 2w & + & 2x & - & y & & & = & 2 \\ & & x & & & & & = & 0. \end{array}$$

Multiply the last equation by -4 and add to the first:

$$\begin{array}{rcrcrcrcl} 3w & & & - & 2y & - & 3z & = & 1 \\ 2w & + & 2x & - & y & & & = & 2 \\ & & x & & & & & = & 0. \end{array}$$

Multiply the last equation by -2 and add to the second:

$$\begin{array}{rcrcrcrcl} 3w & & & - & 2y & - & 3z & = & 1 \\ 2w & & & - & y & & & = & 2 \\ & & x & & & & & = & 0. \end{array}$$

For any $w \in \mathbb{R}$, there is a unique solution of this system, and all solutions are obtained this way.

So the dimension of the solution space is 1. □

27-3. Find the solution space, in $\mathbb{R}^4$, of

$$\begin{array}{rrcrcrcrcl} & 3w & + & 4x & - & 2y & - & 3z & = & 1 \\ & 2w & + & x & - & y & & & = & 2 \\ - & w & + & 7x & - & y & - & 9z & = & 5. \end{array}$$

Solution: Multiply the first equation by -3 and add to the last:

$$\begin{array}{rrcrcrcrcl} & 3w & + & 4x & - & 2y & - & 3z & = & 1 \\ & 2w & + & x & - & y & & & = & 2 \\ - & 10w & - & 5x & + & 5y & & & = & 2. \end{array}$$

Multiply the second equation by 5 and add to the last:

$$\begin{array}{rcrcrcrcl} 3w & + & 4x & - & 2y & - & 3z & = & 1 \\ 2w & + & x & - & y & & & = & 2 \\ & & & & & & 0 & = & 12. \end{array}$$

Because of the inconsistency of the last equation $(0 = 12)$, we see that the solution space is the empty set, $\emptyset$. □

28-1. Let T be a graph with 378 vertices. Assume T is a tree, which is a connected graph with no cycles. How many edges does T have?

Solution: Let V be the set of vertices of T and let E be the set of edges of T. We know that $\#V = 378$, and we wish to compute $\#E$.

An induction proof shows that any tree has one more vertex than it has edges. Then $\#V = 1 + (\#E)$.

Then $\#E = (\#V) - 1 = 378 - 1 = 377$. □

29-1. For all positive functions f and g of the real variable x, let $\sim$ be a relation defined by

$$f \sim g \qquad \text{if and only if} \qquad \lim_{x\to\infty}\left[\frac{f(x)}{g(x)}\right] = 1.$$

True or False: Let f, g, ϕ, ψ be positive functions of x. Assume that $f \sim g$ and that $\phi \sim \psi$. Then $f + \phi \sim g + \psi$.

Solution: True. Proof: Given $\varepsilon > 0$. We wish to show that there exists $M \in \mathbb{R}$ such that, for all $x > M$, we have

$$\left|\frac{[f(x)] + [\phi(x)]}{[g(x)] + [\psi(x)]} - 1\right| \quad < \quad \varepsilon.$$

Choose $L > 0$ such that, for all $x > A$, we have $\left|\dfrac{f(x)}{g(x)} - 1\right| < \varepsilon$. Choose $\Lambda > 0$ such that, for all $x > \Lambda$, we have $\left|\dfrac{\phi(x)}{\psi(x)} - 1\right| < \varepsilon$.

Let $M := \max\{L, \Lambda\}$. Given $x > M$. We wish to prove that

$$\left|\frac{[f(x)] + [\phi(x)]}{[g(x)] + [\psi(x)]} - 1\right| \quad < \quad \varepsilon.$$

Let $s := f(x)$, $t := g(x)$, $\sigma := \phi(x)$, $\tau := \psi(x)$. We wish to prove that

$$\left|\frac{s+\sigma}{t+\tau} - 1\right| \quad < \quad \varepsilon.$$

Since $x > M \geq \Lambda$, it follows that $\left|\dfrac{\sigma}{\tau} - 1\right| < \varepsilon$, and so

$$1 - \varepsilon \quad < \quad \frac{\sigma}{\tau} \quad < \quad 1 + \varepsilon.$$

Multiplying by the positive number τ, we get $(1-\varepsilon)\tau < \sigma < (1+\varepsilon)\tau$.

Since $x > M \geq L$, it follows that $\left|\dfrac{s}{t} - 1\right| < \varepsilon$, and so

$$1 - \varepsilon \quad < \quad \frac{s}{t} \quad < \quad 1 + \varepsilon.$$

Multiplying by the positive number t, we get $(1-\varepsilon)t < s < (1+\varepsilon)t$. Adding this to $(1-\varepsilon)\tau < \sigma < (1+\varepsilon)\tau$ yields

$$(1-\varepsilon)(t+\tau) \quad < \quad s + \sigma \quad < \quad (1+\varepsilon)(t+\tau).$$

Dividing by the positive number $t + \tau$, we get

$$1 - \varepsilon \quad < \quad \frac{s+\sigma}{t+\tau} \quad < \quad 1 + \varepsilon,$$

and so $\left|\dfrac{s+\sigma}{t+\tau}-1\right|<\varepsilon$, as desired. □

30-1. Let S and T be sets and assume that there exists a function $f:S\to T$ such that f is onto T. True or False: There must exist a function $g:T\to S$ such that g is one-to-one.

Solution: True. Proof: By the Axiom of Choice, for all $y\in T$, choose $x_y\in S$ such that $f(x_y)=y$. Define $g:T\to S$ by $g(y)=x_y$. We wish to show that g is one-to-one. Given $t,\tau\in T$. Assume $g(t)=g(\tau)$. We wish to prove that $t=\tau$.

We have $x_t=g(t)=g(\tau)=x_\tau$, and so $f(x_t)=f(x_\tau)$. So, since $f(x_t)=t$ and $f(x_\tau)=\tau$, we get $t=f(x_t)=f(x_\tau)=\tau$, as desired. □

30-2. Let S and T be sets. Assume that there does NOT exist a function $f:S\to T$ such that f is one-to-one. True or False: There must exist a function $g:T\to S$ such that g is one-to-one.

Solution: True. Proof: Let $\mathcal{I}$ denote the set of functions $f:S_0\to T$ such that $S_0\subseteq S$ and such that f is one-to-one. We define a partial ordering $\leq$ on $\mathcal{I}$ by $[\,f\leq g\,]\ \Leftrightarrow\ [\,\exists A\subseteq \operatorname{dom}[g] \text{ s.t. } g|A=f\,]$. Then every $\leq$-chain has an upper bound, so, by Zorn, let $f\in\mathcal{I}$ be a maximal element with respect to $\leq$.

Let $S_0\subseteq S$ be the domain of f and let T_0 be the image of f. By assumption, $S_0\neq S$. Then $T_0=T$; otherwise, we could extend f, contradicting maximality of f. Then f is a bijection from S_0 onto T. Let g be the inverse of f. Then g is an bijection from T onto S_0, so $g:T\to S$ is one-to-one. □

31-1. True or False: There exists a solution $y:\mathbb{R}\to\mathbb{R}$ to the differential equation $y'=x^4+2x^2y^2+y^4$ with the property that, for every $x\in\mathbb{R}$, we have $-1000<y(x)<1000$.

Solution: False. Proof: Following the notation given in the problem, y and $y(x)$ are used interchangeably. Also, y' and $y'(x)$ are used interchangeably. Let $y:\mathbb{R}\to\mathbb{R}$ be a solution to the ODE $y'=x^4+2x^2y^2+y^4$. We will show that there exists $x\in\mathbb{R}$ such that $y(x)\geq 1000$.

For all $x\geq 1$, we have $y'(x)=x^4+2x^2y^2+y^4\geq x^4\geq 1$. So, by the Mean Value Theorem, for all $x\geq 1$, we have $[y(x)]-[y(1)]\geq x-1$.

Let $x := \max\{1, 1001 - [y(1)]\}$. Then $[y(x)] - [y(1)] \geq x - 1$. Then $y(x) \geq [y(1)] + x - 1 \geq [y(1)] + 1001 - [y(1)] - 1 = 1000$. $\square$

32-1. True or False: Let G be a group. Assume, for all $a, b \in G$, for all integers $n \geq 1$, that $(ab)^n = a^n b^n$. Then G is Abelian.

Solution: True. Proof: Given $a, b \in G$. We wish to show that $ab = ba$.

For all integers $n \geq 1$, $(ab)^n = a^n b^n$. In particular, $(ab)^2 = a^2 b^2$. Then $abab = aabb$, so $a^{-1}[abab]b^{-1} = a^{-1}[aabb]b^{-1}$, so $ba = ab$. $\square$

33-1. True or False: Let p and q be prime numbers, and let n be an integer. Assume that $p \neq q$. Then there exist integers k and ℓ such that $\dfrac{n}{p^2 q} = \dfrac{k}{p^2} + \dfrac{\ell}{q}$.

Solution: True. Proof: By the Euclidean algorithm, choose $a, b \in \mathbb{Z}$ such that $ap^2 + bq = 1$. Let $k := nb$ and let $\ell := na$. We wish to prove that $\dfrac{n}{p^2 q} = \dfrac{k}{p^2} + \dfrac{\ell}{q}$. Equivalently, we wish to prove that $n = kq + \ell p^2$.

We have $n = n \cdot 1 = n(ap^2 + bq) = (na)p^2 + (nb)q = \ell p^2 + kq$. $\square$

33-2. True or False: Let p and q be prime numbers, and let n be an integer. Assume that $p \neq q$. Then there exist integers r, s, t, u such that $0 \leq s < p$ and $0 \leq t < p$ and $0 \leq u < q$ and $\dfrac{n}{p^2 q} = r + \dfrac{s}{p} + \dfrac{t}{p^2} + \dfrac{u}{q}$.

Solution: True. Proof: By 33-1, choose integers k and ℓ such that

$$\frac{n}{p^2 q} \qquad = \qquad \frac{k}{p^2} + \frac{\ell}{q}.$$

By the Division Algorithm, choose integers a and $t \in [0, p)$ such that $k = ap + t$, then choose integers b and $s \in [0, p)$ such that $a = bp + s$, then choose integers c and $u \in [0, q)$ such that $\ell = cq + u$. We then define $r := b + c$. We wish to prove that $\dfrac{n}{p^2 q} = r + \dfrac{s}{p} + \dfrac{t}{p^2} + \dfrac{u}{q}$.

We have $\dfrac{n}{p^2q} = \dfrac{k}{p^2} + \dfrac{\ell}{q} = \dfrac{ap+t}{p^2} + \dfrac{cq+u}{q}$, and so

$$\begin{aligned} \frac{n}{p^2q} &= \frac{a}{p} + \frac{t}{p^2} + c + \frac{u}{q} \\ &= \frac{bp+s}{p} + \frac{t}{p^2} + c + \frac{u}{q} \\ &= b + \frac{s}{p} + \frac{t}{p^2} + c + \frac{u}{q} \\ &= r + \frac{s}{p} + \frac{t}{p^2} + \frac{u}{q}. \quad \square \end{aligned}$$

33-3. True or False: Let $\mathbb{R}[x]$ denote the ring of polynomials, with real coefficients, in the indeterminate x. Let $p, q \in \mathbb{R}[x]$ be irreducible polynomials, and let $f \in \mathbb{R}[x]$. Assume that $p \neq q$. Then there exist $r, s, t, u \in \mathbb{R}[x]$ such that $\deg[s] < \deg[p]$ and $\deg[t] < \deg[p]$ and $\deg[u] < \deg[q]$ and $\dfrac{f}{p^2q} = r + \dfrac{s}{p} + \dfrac{t}{p^2} + \dfrac{u}{q}$.

Solution: True. The proof is the same as for 33-2, except: We are using "f" instead of "n", and we must follow the Euclidean Algorithm and the Division Algorithm in $\mathbb{R}[x]$, rather than in $\mathbb{Z}$. $\square$

NOTE: This is an example of a partial fractions decomposition, often taught as a technique of integration in first year Calculus.

34-1. Define $N : \mathbb{R}^2 \to [0, \infty)$ by $N(x, y) = [x^4 + y^4]^{1/4}$. (This is sometimes called the L^4-norm on $\mathbb{R}^2$.) Let $C := (1, 2) \in \mathbb{R}^2$ and let $D := (3, 5) \in \mathbb{R}^2$. Let

$$\begin{aligned} S &:= \{A \in \mathbb{R}^2 \,|\, N(A - C) = 1\} \\ T &:= \{B \in \mathbb{R}^2 \,|\, N(B - D) = 2\} \end{aligned}$$

(These are two L^4-spheres in $\mathbb{R}^2$.) Minimize $N(A - B)$ subject to the constraints $A \in S$ and $B \in T$. (That is, compute how close the one L^4-sphere gets to the other.)

Solution: In this problem dist denotes L^4-distance, so, for all $P, Q \in \mathbb{R}^2$, we define $\text{dist}(P, Q) := N(P - Q)$. The L^4-triangle inequality asserts: For all $P, Q, R \in \mathbb{R}^2$, $\text{dist}(P, R) \leq [\text{dist}(P, Q)] + [\text{dist}(Q, R)]$. Also, note: For all $P, Q, R \in \mathbb{R}^2$, if Q is on the line segment from P to R, then $\text{dist}(P, R) = [\text{dist}(P, Q)] + [\text{dist}(Q, R)]$. (These facts hold for any norm on $\mathbb{R}^2$, and, in particular, for the L^4-norm.) We define the length of a line segment in $\mathbb{R}^2$ to be the L^4-distance between its endpoints. We define the length of a polygonal path in $\mathbb{R}^2$ to be the sum of the lengths of its line segments.

Let L be the line segment from C to D. Let A be the point of intersection of S and L. Let B be the point of intersection of T and L. By the L^4-triangle inequality (and mathematical induction), there is no polygonal path from C to D whose total L^4-length is $< \mathrm{dist}(C, D)$. If $\exists A_1 \in S, B_1 \in T$ s.t. $\mathrm{dist}(A_1, B_1) < \mathrm{dist}(A, B)$, then, concatenating

- the line segment from C to A_1 (which has length 1),
- the line segment from A_1 to B_1,
- the line segment from B_1 to D (which has length 2),

we would arrive at a polygonal path from C to D whose total length is $< 1 + [\mathrm{dist}(A, B)] + 2 = \mathrm{dist}(C, D)$, which, as we just pointed out, is impossible. Thus the minimal distance between any point on S and any point on T is $\mathrm{dist}(A, B)$, and this is the number we seek. Because A and B are points on L, we have

$$\mathrm{dist}(A, B) \;=\; [\mathrm{length}(L)] \;-\; [\mathrm{dist}(C, A)] \;-\; [\mathrm{dist}(D, B)].$$

We have $C = (1, 2)$ and $D = (3, 5)$, so the L^4-length of L is

$$\begin{aligned} \mathrm{dist}(C, D) \;&=\; N(C - D) \;=\; [(1-3)^4 + (2-5)^4]^{1/4} \\ &=\; [(-2)^4 + (-3)^4]^{1/4} \;=\; [16 + 81]^{1/4} \;=\; \sqrt[4]{97}. \end{aligned}$$

The distance from C to A is the radius of S, which is 1. The distance from D to B is the radius of T, which is 2.

Thus $\mathrm{dist}(A, B) = \sqrt[4]{97} - 1 - 2 = \sqrt[4]{97} - 3$. □

Alternate Solution: The gradient of $(p - s)^4 + (q - t)^4$ w.r.t. p, q, s, t is

$$(\, 4(p - s)^3 \,,\, 4(q - t)^3 \,,\, -4(p - s)^3 \,,\, -4(q - t)^3 \,).$$

The gradients of $(p-1)^4+(q-2)^4$ and $(s-3)^4+(t-5)^4$, w.r.t. p, q, s, t are

$$\begin{aligned} &(\, 4(p - 1)^3 \,,\, 4(q - 2)^3 \,,\, 0 \,,\, 0 \,) \qquad \text{and} \\ &(\, 0 \,,\, 0 \,,\, 4(s - 3)^3 \,,\, 4(t - 5)^3 \,) \end{aligned}$$

Choose p, q, s, t so as to minimize

$$(p - s)^4 + (q - t)^4$$

subject to

$$\begin{aligned} (p-1)^4 + (q-2)^4 &= 1 \qquad \text{and} \\ (s-3)^4 + (t-5)^4 &= 16 \end{aligned}$$

We wish to compute

$$\sqrt[4]{(p-s)^4 + (q-t)^4}.$$

By Lagrange Multipliers, choose $\lambda_1, \mu_1 \in \mathbb{R}$ such that

$$(\, 4(p-s)^3 \,,\, 4(q-t)^3 \,,\, -4(p-s)^3 \,,\, -4(q-t)^3 \,).$$

is equal to the λ_1, μ_1 linear combination of

$$\begin{aligned} &(\, 4(p-1)^3 \,,\, 4(q-2)^3 \,,\, 0 \,,\, 0 \,) \qquad \text{and} \\ &(\, 0 \,,\, 0 \,,\, 4(s-3)^3 \,,\, 4(t-5)^3 \,) \end{aligned}$$

Let $\lambda := \sqrt[3]{\lambda_1}$ and let $\mu := \sqrt[3]{\mu_1}$. Then

$$\begin{aligned} (\, p-s \,,\, q-t \,) &= \lambda \cdot (\, p-1 \,,\, q-2 \,) \qquad \text{and} \\ (\, s-p \,,\, t-q \,) &= \mu \cdot (\, s-3 \,,\, t-5 \,). \end{aligned}$$

Throughout this problem dist is L^4-distance, so, for all $V, W \in \mathbb{R}^2$, we define $\text{dist}(V,W) := N(V-W)$. Let $A := (p,q)$ and let $B := (s,t)$. Let $C := (1,2)$ and let $D := (3,5)$. Then $A - B = \lambda(A-C)$ and $B - A = \mu(B-D)$. Let L be the line in $\mathbb{R}^2$ through C and D. Because

$$\begin{aligned} (p-1)^4 + (q-2)^4 &= 1 \qquad \text{and} \\ (s-3)^4 + (t-5)^4 &= 16, \end{aligned}$$

it follows that $\text{dist}(A,C) = 1$ and $\text{dist}(B,D) = 2$. Then

$$\begin{aligned} \text{dist}(C,D) &= \sqrt[4]{(1-3)^4 + (2-5)^4} = \sqrt[4]{97} \\ &> 3 = [\text{dist}(A,C)] + [\text{dist}(B,D)], \end{aligned}$$

Thus

$$\text{dist}(C,D) \qquad > \qquad \text{dist}(C,A) + \text{dist}(B,D).$$

On the other hand, by the L^4-triangle inequality, we have

$$\text{dist}(C,D) \qquad \leq \qquad \text{dist}(C,A) + \text{dist}(A,D).$$

Therefore $A \neq B$. Let L be the line through A and B.

Because $A - B = \lambda(A-C)$, it follows that $B = \lambda C + (1-\lambda)A$, and so B is on the line in $\mathbb{R}^3$ through C and A. This line, which passes through A and B, must be L. Then $C \in L$.

Because $B - A = \mu(B - D)$, it follows that $A = \mu D + (1 - \mu)B$, and so A is on the line in $\mathbb{R}^3$ through D and B. This line, which passes through A and B, must be L. Then $D \in L$.

Since $C, D \in L$, we conclude that L is the line through C and D. So, since $A \in L$, choose $v \in (0, 1)$ such that $A = (1 - v)C + vD$. Similarly, since $B \in L$, choose $w \in (0, 1)$ such that $B = wC + (1 - w)D$.

Then $A - C = v(D - C)$ and $B - D = w(C - D)$. That is,

$$\begin{aligned}(p-1, q-2) &= v(-2, -3) \qquad \text{and}\\ (s-3, t-5) &= w(2, 3).\end{aligned}$$

Recall that

$$\begin{aligned}(p-1)^4 + (q-2)^4 &= 1 \qquad \text{and}\\ (s-3)^4 + (t-5)^4 &= 16.\end{aligned}$$

Then

$$\begin{aligned}v^4((-2)^4 + (-3)^4) &= 1 \qquad \text{and}\\ w^4(2^4 + 3^4) &= 16.\end{aligned}$$

Then $v = \pm\sqrt[4]{1/97}$ and $w = \pm\sqrt[4]{16/97}$. Let $z := 1/\sqrt[4]{97}$. Then $v = \pm z$ and $w = \pm 2z$. Since $A = (1 - v)C + vD$, we see that

$$(p, q) \quad = \quad A \quad \in \quad \{\ (1 - z)C + zD,\ (1 + z)C - zD\ \}.$$

Since $B = wC + (1 - w)D$, we see that

$$(s, t) \quad = \quad B \quad \in \quad \{\ 2zC + (1 - 2z)D,\ -2zC + (1 + 2z)D\ \}.$$

Thus there are two possibilities for (p, q) and there are two possibilities for (s, t). This gives four possibilities for (p, q, s, t), each of which satisfies the constraints.

We evaluate the objective $(p - s)^4 + (q - t)^4 = [N(A - B)]^4$ assuming $(p, q) = A = (1 - z)C + zD$ and $(s, t) = B = 2zC + (1 - 2z)D$, and obtain $[\ N(\ [(1 - z)C + zD] - [2zC + (1 - 2z)D]\ )\ ]^4$, which equals

$$[\ N(\ (1 - 3z)C + (3z - 1)D\ )\ ]^4 \quad = \quad [\ N(\ (1 - 3z)(C - D)\ )\ ]^4,$$

which equals

$$(1 - 3z)^4[N(C - D)]^4 \ = \ (1 - 3z)^4[N((-2, -3))]^4 \ = \ 97(1 - 3z)^4.$$

Evaluating the objective assuming the other three possibilities yields

$$97(1 + z)^4 \qquad \text{and} \qquad 97(1 - z)^4 \qquad \text{and} \qquad 97(1 + 3z)^4.$$

Recall: $z = 1/\sqrt[4]{97}$. Among these four possibilities, the smallest value of the objective occurs at the first, so $(p, q) = A = (1 - z)C + zD$ and $(s, t) = B = 2zC + (1 - 2z)D$. Then $[N(A - B)]^4 = 97(1 - 3z)^4$.

Recall that we wish to compute

$$\sqrt[4]{(p - s)^4 + (q - t)^4},$$

which equals $N(A - B)$. Thus our final answer is

$$N(A - B) = \sqrt[4]{[N(A - B)]^4} = \sqrt[4]{97(1 - 3z)^4} = \sqrt[4]{97}(1 - 3z),$$

which equals

$$(1/z)(1 - 3z) = (1/z) - 3 = \sqrt[4]{97} - 3. \quad \square$$

42-1. Let $p : \mathbb{R} \to \mathbb{R}$ be defined by $p(x) = [e^{-x^2/2}]/[\sqrt{2\pi}]$. Let X and Y be independent random variables. Assume that X and Y are both standard normal, *i.e.*, that both X and Y have probability density function p. Compute the probability that $X < 9Y$.

Solution: Let $Z := (X, Y)$, which is an $\mathbb{R}^2$-valued random variable. Let $v := (-1, 9) \in \mathbb{R}^2$. Then $v \cdot Z = (-1, 9) \cdot (X, Y) = -X + 9Y$. We therefore wish to calculate $\Pr[v \cdot Z > 0]$

Let $v_0 := (\sqrt{82}, 0)$. Then v and v_0 have the same length. Denote by $R : \mathbb{R}^2 \to \mathbb{R}^2$ the rotation such that $R(v) = v_0$. Let $Z_0 := R(Z)$. Then $v \cdot Z = v_0 \cdot Z_0$. We therefore wish to calculate $\Pr[v_0 \cdot Z_0 > 0]$.

Define $P : \mathbb{R}^2 \to \mathbb{R}$ by $P(x, y) = [p(x)][p(y)]$. Then P is the probability density function of Z, and so $P \circ R^{-1}$ is the probability density function of Z_0. For all $x, y \in \mathbb{R}$, we have $P(x, y) = [e^{(-x^2 - y^2)/2}]/[2\pi]$. Because $(x, y) \mapsto x^2 + y^2 : \mathbb{R}^2 \to \mathbb{R}$ is rotationally invariant, it follows that $P \circ R = P$. Thus Z and Z_0 have the same distribution. We therefore wish to calculate $\Pr[v_0 \cdot Z > 0]$.

We have $v_0 \cdot Z = (\sqrt{82}, 0) \cdot (X, Y) = \sqrt{82}X$. Then

$$[\ v_0 \cdot Z > 0\ ] \qquad \Leftrightarrow \qquad [\ X > 0\ ].$$

We therefore wish to calculate $\Pr[X > 0]$.

Because p is the PDF of X, it follows both that $\int_{-\infty}^{\infty} p = 1$ and that $\Pr[X > 0] = \int_0^{\infty} p$. Since p is even, we have $\int_0^{\infty} p = \int_{-\infty}^{0} p$. Then

$$2\left[\int_0^{\infty} p\right] \quad = \quad \left[\int_{-\infty}^{0} p\right] + \left[\int_0^{\infty} p\right] \quad = \quad \int_{-\infty}^{\infty} p \quad = \quad 1,$$

and so $\int_0^{\infty} p = \frac{1}{2}$. Then $\Pr[X > 0] = \int_0^{\infty} p = \frac{1}{2}$. □

46-1. TRUE OR FALSE: For any cyclic group G, for any homomorphism $f : G \to G$, there exists an integer n such that, for all $x \in G$, we have $f(x) = x^n$.

Solution: True. Proof: Given a cyclic group G and a homomorphism $f : G \to G$. We wish to show that there exists an integer n such that, for all $x \in G$, we have $f(x) = x^n$.

Since G is cyclic, choose a generator a of G. Then, for all $g \in G$, there exists an integer k such that $g = a^k$. Choose an integer n such that $f(a) = a^n$. Given $x \in G$. We wish to show that $f(x) = x^n$.

Choose an integer m such that $x = a^m$. Then

$$f(x) \;=\; f(a^m) \;=\; [f(a)]^m \;=\; [a^n]^m \;=\; [a^m]^n \;=\; x^n,$$

as desired. □

46-2. TRUE OR FALSE: For any Abelian group G, for any homomorphism $f : G \to G$, there exists an integer n such that, for all $x \in G$, we have $f(x) = x^n$.

Solution: False. Counterexample: Let C denote the multiplicative group $\{-1, 1\}$. Let $G := C \oplus C$. Then G is Abelian. Let $e := (1, 1)$, $a := (1, -1)$, $b := (-1, 1)$, $c := (-1, -1)$. Then $G = \{e, a, b, c\}$. Also, $ab = c$, $bc = a$ and $ca = b$. Let $f : G \to G$ be the function defined by:

$$f(e) = e, \qquad f(a) = b, \qquad f(b) = c \qquad f(c) = a.$$

Let n be an integer, and assume, for all $x \in G$, that $f(x) = x^n$. We aim for a contradiction.

We have $f(a) = a^n$. Then

$$\begin{aligned} b &= f(a) = a^n \\ &\in \{a^k \,|\, k \text{ is an even integer}\} \cup \{a^k \,|\, k \text{ is an odd integer}\} \\ &= \{e\} \cup \{a\} = \{e, a\}. \end{aligned}$$

However, $b \notin \{e, a\}$, contradiction. □

49-1. Up to isomorphism, how many additive Abelian groups are there of order 12?

Solution: For any integer $n \geq 1$, let $C_n := \mathbb{Z}/(n\mathbb{Z})$ be the finite additive cyclic group of order n. By the Structure Theorem for Finite Abelian Groups, any additive Abelian group is isomorphic to a direct sum of additive cyclic groups of prime power order. The prime powers that divide 12 are 1, 2, 3 and 4. Consequently, up to isomorphism, the only additive Abelian groups of order 12 are

$$C_4 \oplus C_3, \qquad C_2 \oplus C_2 \oplus C_3.$$

Thus the answer is: two. □

49-2. Up to isomorphism, how many additive Abelian groups G of order 12 have the property that, for all $x \in G$, $x+x+x+x+x+x = 0$?

Solution: For all $x \in G$, the condition [$x + x + x + x + x + x = 0$] is equivalent to [the order of x is a divisor of 6], and this, in turn, is equivalent to [the order of x is 1 or 2 or 3 or 6]. By 49-1, we need only check $C_4 \oplus C_3$ and $C_2 \oplus C_2 \oplus C_3$. In $C_4 \oplus C_3$, the element $(1, 1)$ has order 12, so G cannot be isomorphic to $C_4 \oplus C_3$. In $C_2 \oplus C_2 \oplus C_3$, every element has order 1 or 2 or 3 or 6. Thus G can be isomorphic to $C_2 \oplus C_2 \oplus C_3$. Thus the answer is: one. □

49-3. Up to isomorphism, how many additive Abelian groups are there of order 24?

Solution: For any integer $n \geq 1$, let $C_n := \mathbb{Z}/(n\mathbb{Z})$ be the finite additive cyclic group of order n. By the Structure Theorem for Finite Abelian Groups, any additive Abelian group is isomorphic to a direct sum of additive cyclic groups of prime power order. The prime powers that divide 24 are 1, 2, 3, 4 and 8. Consequently, up to isomorphism, the only additive Abelian groups of order 12 are

$$C_8 \oplus C_3, \quad C_4 \oplus C_2 \oplus C_3, \quad C_2 \oplus C_2 \oplus C_2 \oplus C_3.$$

Thus the answer is: three. □

49-4. Up to isomorphism, how many additive Abelian groups G of order 24 have the property that, for all $x \in G$, $x + x + x + x + x = 0$?

Solution: We claim that no such group G exists. Let G be a group of order 24 such that, for all $x \in G$, $x + x + x + x + x = 0$. We aim for a contradiction.

Choose $x \in G \backslash \{0\}$. Let n denote the order of x. Because

$$x \ + \ x \ + \ x \ + \ x \ + \ x \qquad = \qquad 0,$$

it follows that n is a divisor of 5, so $n \in \{1, 5\}$. Because $\#G = 24$, it follows that n is a divisor of 24, so $n \in \{1, 2, 3, 4, 6, 8, 12, 24\}$. Then $n \in \{1, 5\} \cap \{1, 2, 3, 4, 6, 8, 12, 24\} = \{1\}$, so $n = 1$. That is, the order of x is 1, and it follows that $x = 0$. However, $x \in G \backslash \{0\}$, so $x \neq 0$, contradiction, completing the proof of the claim.

Since no such group G exists, the answer is: zero. □

49-5. Up to isomorphism, how many additive Abelian groups G of order 24 have the property that, for all $x \in G$, $x + x + x + x = 0$?

Solution: For all $x \in G$, the condition [$x + x + x + x = 0$] is equivalent to [the order of x is a divisor of 4], and this, in turn, is equivalent to [the order of x is 1 or 2 or 4]. By 49-3, we need only check $C_8 \oplus C_3$ and $C_4 \oplus C_2 \oplus C_3$ and $C_2 \oplus C_2 \oplus C_2 \oplus C_3$. In $C_8 \oplus C_3$, the element $(1, 1)$ has order 24, so G cannot be isomorphic to $C_8 \oplus C_3$. In $C_4 \oplus C_2 \oplus C_3$, the element $(1, 1, 1)$ has order 12, so G cannot be isomorphic to $C_4 \oplus C_2 \oplus C_3$. In $C_2 \oplus C_2 \oplus C_2 \oplus C_3$, the element $(1, 1, 1, 1)$ has order 6, so G cannot be isomorphic to $C_2 \oplus C_2 \oplus C_2 \oplus C_3$. Thus the answer is: zero. □

59-1. Let f be an analytic function of a complex variable $z = x + iy$ given by

$$f(z) = (3x + 5y) + i \cdot (g(x, y)),$$

where $g(x, y)$ is a real-valued function of the real variables x and y. If $g(0, 0) = 1$, then $g(7, 3) =$

Solution: We will compute $[g(7, 3)] - [g(7, 0)]$ and $[g(7, 0)] - [g(0, 0)]$ separately, and then add the results to get $[g(7, 3)] - [g(0, 0)]$. We will then add $g(0, 0)$, which is given in the problem as 1, and obtain $g(7, 3)$.

Define $Z : \mathbb{R}^2 \to \mathbb{C}$ by $Z(x, y) = x + iy$. Define $h : \mathbb{R}^2 \to \mathbb{R}^2$ by $h(x, y) = 3x + 5y$. Then $f \circ Z = h + ig$.

According to the Cauchy-Riemann equations, a counterclockwise 90° rotation of $(\partial_1 h, \partial_1 g)$ gives $(\partial_2 h, \partial_2 g)$. That is,

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \partial_1 h \\ \partial_1 g \end{bmatrix} = \begin{bmatrix} \partial_2 h \\ \partial_2 g \end{bmatrix}.$$

That is, $-\partial_1 g = \partial_2 h$ and $\partial_1 h = \partial_2 g$.

For all $x, y \in \mathbb{R}$, $h(x, y) = 3x + 5y$. Computing partial derivatives, for all $x, y \in \mathbb{R}$, we get $(\partial_1 h)(x, y) = 3$ and $(\partial_2 h)(x, y) = 5$, and so

$$-(\partial_1 g)(x, y) \;=\; 5 \qquad \text{and} \qquad (\partial_2 g)(x, y) \;=\; 3.$$

Multiplying the first equation by -1, and substituting $y :\to 0$, we see, for all $x \in \mathbb{R}$, that $(\partial_1 g)(x, 0) = -5$. Integrating this equation from $x = 0$ to $x = 7$, we see that $[g(7, 0)] - [g(0, 0)] = \displaystyle\int_0^7 (-5)\, dx$. Then $[g(7, 0)] - [g(0, 0)] = (-5)(7) = -35$. Recall that, for all $x, y \in \mathbb{R}$, $(\partial_2 g)(x, y) = 3$. Substituting $x :\to 7$, we see, for all $y \in \mathbb{R}$, that $(\partial_2 g)(7, y) = 3$. Integrating this equation from $y = 0$ to $y = 3$ yields $[g(7, 3)] - [g(7, 0)] = \displaystyle\int_0^3 3\, dx = (3)(3) = 9$. Then

$$\begin{aligned} [g(7, 3)] - [g(0, 0)] &= ([g(7, 3)] - [g(7, 0)]) + ([g(7, 0)] - [g(0, 0)]) \\ &= 9 + (-35) = -26. \end{aligned}$$

Then $g(7, 3) = [g(0, 0)] + (-26) = 1 + (-26) = -25$. □