SOLUTIONS OF VARIATIONS, PRACTICE TEST 2

\[ y' + xy(y + 2) = 0 \]
\[ y(0) = -1 \]

44-1. Let \( y \) be a real-valued function defined on the real line satisfying the initial value problem above. Compute \( \lim_{x \to -\infty} [y(x)] \).

**Solution:** Following the notation given in the problem, \( y \) and \( y(x) \) are used interchangeably. Also, \( y' \) and \( y'(x) \) are used interchangeably. For all \( x \in \mathbb{R} \), \( y'(x) = -xy(y + 2) \), and so \( y(x) \in \{0, -2\} \Rightarrow [y'(x) = 0] \).

So, by Picard-Lindelöf, exactly one of the following five possibilities holds:

- \( \forall x \in \mathbb{R}, y(x) < -2 \) or
- \( \forall x \in \mathbb{R}, y(x) = -2 \) or
- \( \forall x \in \mathbb{R}, -2 < y(x) < 0 \) or
- \( \forall x \in \mathbb{R}, 0 < y(x) \).

So, as \( y(0) = -1 \), we get \( \forall x \in \mathbb{R}, -2 < y(x) < 0 \). Then, for all \( x \in \mathbb{R} \),

\[ \frac{d}{dx} \left[ \frac{y'}{y + 2} - \frac{y'}{y} \right] = \frac{-2y'}{y(y + 2)} = 2x = \frac{d}{dx} [x^2]. \]

Then

\[ \frac{d}{dx} [(\ln(y + 2)) - (\ln y)] = \frac{d}{dx} \left[ \frac{y'}{y + 2} - \frac{y'}{y} \right] = \frac{d}{dx} [x^2]. \]

Choose \( C \in \mathbb{R} \) such that, for all \( x \in \mathbb{R} \),

\[ [\ln(y + 2)] - [\ln y] = x^2 + C. \]

Let \( K := e^C \). Then, for all \( x \in \mathbb{R} \), we have \( 1 + (2/y) = (y+2)/y = Ke^{x^2} \), so \( y(x) = y = 2/(Ke^{x^2} - 1) \). Then, because \( K = e^K \neq 0 \) and because we have \( \lim_{x \to -\infty} [e^{x^2/2}] = \infty \), we conclude that \( \lim_{x \to -\infty} [y(x)] = 0. \) \( \square \)
54-1. Choose a real number $x$ uniformly at random in the interval $[0, 3]$. Choose a real number $y$ independently of $x$, and uniformly at random in the interval $[0, 4]$. Find the probability that $y < x^2$.

Solution: Viewing this as a problem in measure theory, the answer is 

$$\frac{\text{area of } \{ (x, y) \in [0, 3] \times [0, 4] \mid y < x^2 \}}{\text{area of } [0, 3] \times [0, 4]},$$

Let $Q := \{ (x, y) \in [0, 3] \times [0, 4] \mid y < x^2 \}$. Then the answer is 

$$\frac{\text{area of } Q}{12}.$$

Let 

$$A := \{ (x, y) \in [0, 3] \times [0, 4] \mid y < x^2 \text{ and } x \leq 2 \}$$

$$B := \{ (x, y) \in [0, 3] \times [0, 4] \mid y < x^2 \text{ and } x > 2 \}.$$

Then $A \cap B = \emptyset$ and $Q = A \cup B$, so the area of $Q$ is the sum of the areas of $A$ and $B$. We have $A = \{ x, y \in [0, 2] \times [0, \infty) \mid y < x^2 \}$, so the area of $A$ is $\int_0^2 x^2 \, dx$. We compute $\int_0^2 x^2 \, dx = \left[ \frac{x^3}{3} \right]_{x=0}^{x=2} = \frac{8}{3}$. Thus the area of $A$ is $8/3$. We have $B = (2, 4] \times [0, 4]$, so the area of $B$ is $8$. Then the area of $Q$ is $(8/3) + 8 = 32/3$. Then the answer is: 

$$\frac{\text{area of } Q}{12} = \frac{32/3}{12} = \frac{8}{9}. \quad \square$$

61-1. A tank initially contains a salt solution of 35 ounces of salt dissolved in 50 gallons of water. Pure water is sprayed into the tank at a rate of 6 gallons per minute. The sprayed water is continually mixed with the salt solution in the tank, and the mixture flows out of the tank at a rate of 2 gallons per minute. If the mixing is instantaneous, how many ounces of salt are in the tank after 12 minutes have elapsed?

Solution: For all $t \geq 0$, let $s(t)$ denote the number ounces of salt in the tank at the $t$ minute mark, and let $L(t)$ denote the number of gallons of liquid in the tank at the $t$ minute mark. Then $s(0) = 35$ and $L(0) = 50$. We add 6 gallons per minute and remove 2 gallons per minute, so the net is 4 gallons per minute. Thus, for all $t \geq 0$, we have $L(t) = 50 + 4t$.

We use $s$ and $s(t)$ interchangeably. We also use $s'$ and $s'(t)$ interchangeably. We also use $L$ and $L(t)$ interchangeably. The water sprayed into the tank adds no salt. At any time $t \geq 0$, there are $L$ gallons of liquid in the tank, containing $s$ ounces of salt. So the density
of salt in the tank is $s/L$ ounces per gallon. The flow of water out of the tank therefore subtracts $2(s/L)$ ounces of salt per minute. Then, for all $t > 0$, we have $s'(t) = -2s/L = -2s/(50 + 4t) = -s/(25 + 2t)$. Then, for all $t > 0$, we have

$$\frac{d}{dt} \left[ \ln s \right] = \frac{s'}{s} = \frac{-1}{25 + 2t} = \frac{d}{dt} \left[ -\frac{\ln(25 + 2t)}{2} \right].$$

Choose $C \in \mathbb{R}$ such that, for all $t \geq 0$, $\ln(s(t)) = \left[ -\frac{1}{2} \right] \ln(25 + 2t) + C$. Let $K := e^C$. Then, for all $t \geq 0$, we have $s(t) = K(25 + 2t)^{-1/2}$. Then $35 = s(0) = K(25)^{-1/2} = K/5$, so $K = 35 \cdot 5$.

Then $s(12) = 35 \cdot 5 \cdot (25 + 2 \cdot 12)^{-1/2} = 35 \cdot 5 \cdot (49)^{-1/2} = 5 \cdot 5 = 25$. □

65-1. Let $g$ be a differentiable function of two real variables, and let $f$ be the function of a complex variable $z$ defined by

$$f(z) = e^{xy} + i \cdot (g(x, y)),$$

where $x$ and $y$ are the real and imaginary parts of $z$, respectively. If $f$ is an analytic function on the complex plane, then $(g(4, 2)) - (g(0, 1)) = \frac{\partial}{\partial y} g(4, 2) - \frac{\partial}{\partial y} g(0, 1)$.

Solution: We will compute $[g(4, 2)] - [g(4, 1)]$ and $[g(4, 1)] - [g(0, 1)]$ separately, and then add the results to get $(g(4, 2)) - (g(0, 1))$.

Define $Z : \mathbb{R}^2 \to \mathbb{C}$ by $Z(x, y) = x + iy$. Define $h : \mathbb{R}^2 \to \mathbb{R}^2$ by $h(x, y) = e^{xy}$. Then $f \circ Z = h + ig$.

According to the Cauchy-Riemann equations, a counterclockwise $90^\circ$ rotation of $(\partial h, \partial g)$ gives $(\partial_2 h, \partial_2 g)$. That is,

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \partial_1 h \\ \partial_1 g \end{bmatrix} = \begin{bmatrix} \partial_2 h \\ \partial_2 g \end{bmatrix}.$$

That is, $-\partial_1 g = \partial_2 h$ and $\partial_1 h = \partial_2 g$.

For all $x, y \in \mathbb{R}$, $h(x, y) = e^{xy}$. Computing partial derivatives, for all $x, y \in \mathbb{R}$, we get $(\partial_1 h)(x, y) = ye^{xy}$ and $(\partial_2 h)(x, y) = xe^{xy}$, and so

$$-(\partial_1 g)(x, y) = xe^{xy} \quad \text{and} \quad (\partial_2 g)(x, y) = ye^{xy}.$$

Multiplying the first equation by $-1$, and substituting $y \mapsto 1$, we see, for all $x \in \mathbb{R}$, that $(\partial_1 g)(x, 1) = -xe^x$. So, integrating this equation from $x = 0$ to $x = 4$, we get $[g(4, 1)] - [g(0, 1)] = \int_0^4 (-xe^x) \, dx$. We
compute the integral on the RHS by integration by parts (differentiating $-x$ w.r.t. $x$, and antidifferentiating $e^x$ w.r.t. $x$), and get

$$\int_0^4 (-xe^x) \, dx = \left[ -xe^x \bigg|_{x=0}^{x=4} \right] - \left[ \int_0^4 (-e^x) \, dx \right]$$

$$= \left[ -xe^x \bigg|_{x=0}^{x=4} \right] + \left[ \int_0^4 (e^x) \, dx \right]$$

$$= [-4e^4 - (-0)] + \left[ e^x \bigg|_{x=0}^{x=4} \right]$$

$$= [-4e^4] + [e^4 - 1] = -3e^4 - 1.$$

Then $[g(4, 1)] - [g(0, 1)] = -3e^4 - 1$.

Recall that, for all $x, y \in \mathbb{R}$, we have $(\partial_2 g)(x, y) = ye^{xy}$. Substituting $x :\to 4$, we see, for all $y \in \mathbb{R}$, that $(\partial_2 g)(4, y) = ye^{4y}$. So, integrating this equation from $y = 1$ to $y = 2$, we see that

$$[g(4, 2)] - [g(4, 1)] = \int_1^2 (ye^{4y}) \, dy.$$

We compute the integral on the RHS by integration by parts (differentiating $y$ w.r.t. $y$, and antidifferentiating $e^{4y}$ w.r.t. $y$), and get

$$\int_1^2 (ye^{4y}) \, dy = \left[ \left[ \frac{ye^{4y}}{4} \right]_{y=1}^{y=2} \right] - \left[ \frac{e^{4y}}{4} \right] \left|_{y=1}^{y=2} \right]$$

$$= \left[ 2e^8 - \frac{e^4}{4} \right] - \left[ \frac{e^8}{16} \right]$$

$$= \left[ 2e^8 - \frac{e^4}{4} \right] - \left[ \frac{e^8}{16} \right]$$

$$= \left[ \frac{8e^8 - 4e^4}{16} \right] - \left[ \frac{e^8 - e^4}{16} \right] = \frac{7e^8 - 3e^4}{16}. $$

Then, as $[g(4, 2)] - [g(0, 1)] = ([g(4, 2)] - [g(4, 1)]) + ([g(4, 1)] - [g(0, 1)])$,

$$[g(4, 2)] - [g(0, 1)] = \left( \frac{7e^8 - 3e^4}{16} \right) + (-3e^4 - 1)$$

$$= \frac{7e^8 - 3e^4}{16} + \frac{-48e^4 - 16}{16}$$

$$= \frac{7e^8 - 51e^4 - 16}{16}. \square$$