1. If $S$ is a plane in Euclidean 3-space containing $(0, 0, 0)$, $(2, 0, 0)$ and $(3, 1, 1)$, then $S$ is the
   (A) $xy$-plane
   (B) $xz$-plane
   (C) $yz$-plane
   (D) plane $y - z = 0$
   (E) plane $x + 2y - 2z = 0$

Solution: The $xy$-plane is $z = 0$ which does not contain $(0, 0, 1)$, so (A) is not correct. The $xz$-plane is $y = 0$ which does not contain $(3, 1, 1)$, so (B) is not correct. The $yz$-plane is $x = 0$ which does not contain $(2, 0, 0)$, so (C) is not correct.

The plane $x + 2y - 2z = 0$ is $(1, 2, -2) \cdot (x, y, z) = 0$, and we have

$$(1, 2, -2) \cdot (3, 1, 1) = 3 \neq 0,$$

so (E) is not correct.

The plane $y - z = 0$ is $(0, 1, -1) \cdot (x, y, z) = 0$, and this contains all three of the points $(0, 0, 0)$, $(2, 0, 0)$ and $(3, 1, 1)$. Answer: (D) $\square$
2. If $a$ and $b$ are real numbers, which of the following are necessarily true?

I. If $a < b$ and $ab > 0$, then $\frac{1}{a} > \frac{1}{b}$.
II. If $a < b$, then $ac < bc$, for all real numbers $c > 0$.
III. If $a < b$, then $a + c < b + c$, for all real numbers $c$.
IV. If $a < b$, then $-a > -b$.

Choose one of these answers:

(A) I only
(B) I and III only
(C) III and IV only
(D) II, III and IV only
(E) I, II, III and IV

Solution: If $a < b$ and $ab > 0$, then $\frac{a}{ab} < \frac{b}{ab}$, i.e., $\frac{1}{a} < \frac{1}{b}$, or, equivalently, $\frac{1}{a} > \frac{1}{b}$. Thus I is true.

Also, II, III and IV are all basic facts about the real number system; they are all true. Answer: (E)

3. Compute $\int_0^1 \int_0^y x^3 y^4 \, dx \, dy$.

Solution: We compute

$$\int_0^y x^3 y^4 \, dx = \left[ \left( \frac{x^4}{4} \right) y^4 \right]_{x=0}^{x=y} = \left[ \left( \frac{y^4}{4} \right) y^4 \right] - 0 = \frac{y^8}{4}.$$ 

Then

$$\int_0^1 \int_0^y x^3 y^4 \, dx \, dy = \int_0^1 \frac{y^8}{4} \, dy$$

$$= \left[ \frac{y^9}{36} \right]_{x=0}^{x=1} = \left[ \frac{1}{36} \right] - 0 = \frac{1}{36}.$$
4. For $x \geq 0$, compute $\frac{d}{dx} (x^{\pi} \cdot \pi^x)$.

**Solution:** By Logarithmic Differentiation,

$$
(d/dx)(x^{\pi}) = [\pi^x][(d/dx)(x \ln \pi)] = [\pi^x][\ln \pi].
$$

Then, using the Product Rule,

$$
\frac{d}{dx} (x^{\pi} \cdot \pi^x) = (\pi x^{\pi-1}) (\pi^x) + (x^{\pi}) (\pi^x)]\ln \pi] = x^{\pi-1} \cdot \pi^{x+1} + x^\pi \cdot \pi^x \cdot (\ln \pi). \quad \square
$$

5. Find all functions $f$ defined on the $xy$-plane such that

$$
\frac{\partial}{\partial x} [f(x, y)] = 2x - y \quad \text{and} \quad \frac{\partial}{\partial y} [f(x, y)] = x + 2y.
$$

**Solution:** If such a function $f$ were to exist, then we would have

$$
\frac{\partial}{\partial y} \frac{\partial}{\partial x} [f(x, y)] = \frac{\partial}{\partial x} \frac{\partial}{\partial y} [f(x, y)],
$$

yielding $-1 = 1$, a contradiction. Thus no such functions exist. \quad \square
6. Sketch the graph of an antiderivative of the function $f$ whose graph is shown in the figure above.

*Solution:* The graph of $f$ consists, piecewise, of

- a very negatively sloped half-line on the left, intersecting the horizontal axis,
- a horizontal line segment in the middle located a bit below the horizontal axis and
- a somewhat positively sloped half-line on the right, intersecting the horizontal axis.

Any antiderivative of $f$ will be, piecewise,

- a very concave down parabolic arc on the left, with a local maximum,
- a line segment in the middle, a bit negatively sloped and
- a somewhat concave up parabolic arc on the right, with a local minimum.

The graph of an antiderivative appears in red in the figure below.
7. Compute the shaded area shown above.

**Solution:** The line through (1, 2) and (6, 3) is \( y - 2 = (1/5)(x - 1) \), or \( y = (1/5)x + (9/5) \). The two line segments

- from (1, 2) to (3, 0)
- from (3, 0) to (6, 3)

are both on the graph of \( y = |x - 3| \). Thus we need to compute the area of the region between \( y = (1/5)x + (9/5) \) and \( y = |x - 3| \) from \( x = 1 \) to \( x = 6 \). This is \( \int_1^6 ((1/5)x + (9/5) - |x - 3|) \, dx \), or

\[
\begin{align*}
&= \left[ \int_1^3 \left( \frac{x}{5} + \frac{9}{5} - |x - 3| \right) \, dx \right] + \left[ \int_3^6 \left( \frac{x}{5} + \frac{9}{5} - |x - 3| \right) \, dx \right] \\
&= \left[ \int_1^3 \left( \frac{6x}{5} + \frac{-6}{5} \right) \, dx \right] + \left[ \int_3^6 \left( \frac{-4x}{5} + \frac{24}{5} \right) \, dx \right] \\
&= \left[ \frac{3x^2}{5} + \frac{-6x}{5} \right]_{x=1}^{x=3} + \left[ \frac{-2x^2}{5} + \frac{24x}{5} \right]_{x=3}^{x=6} \\
&= \left[ \frac{3 \cdot 8}{5} + \frac{-6 \cdot 2}{5} \right] + \left[ \frac{-2 \cdot 27}{5} + \frac{24 \cdot 3}{5} \right] = \frac{30}{5} = 6. \quad \square
\end{align*}
\]
8. Compute $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$.

Solution: For all integers $n \geq 1$, we have $1 \leq n^2$, so

$$\frac{n}{n^2 + 1} \geq \frac{n}{n^2 + n^2} = \frac{n}{2n^2} = \frac{1}{2n}.$$  

From the integral test for convergence, $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$. Multiplying this by $\frac{1}{2}$, we get $\sum_{n=1}^{\infty} \frac{1}{2n} = +\infty$. We conclude that $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1} = +\infty$. \qed