52-1. Consider the following system of linear equations over the real numbers, where $x$, $y$ and $z$ are variables and $b$ is a real constant.

\[
\begin{align*}
  x + 2y + z &= 0 \\
  2x + 4y + 3z &= 0 \\
  x + 3y + bz &= 0
\end{align*}
\]

Which of the following statements are true?

I. There exists a value of $b$ for which the system has no solution.
II. There exists a value of $b$ for which the system has exactly one solution.
III. There exists a value of $b$ for which the system has more than one solution.

(A) II only  
(B) I and II only  
(C) I and III only  
(D) II and III only  
(E) I, II and III
52-2. Consider the following system of linear equations over the real numbers, where \( x, y \) and \( z \) are variables and \( b \) is a real constant.

\[
\begin{align*}
  x + 2y + z &= 0 \\
  2x + 4y + 3z &= 0 \\
  3x + 6y + bz &= 0
\end{align*}
\]

Which of the following statements are true?

I. There exists a value of \( b \) for which the system has no solution.

II. There exists a value of \( b \) for which the system has exactly one solution.

III. There exists a value of \( b \) for which the system has more than one solution.

(A) II only
(B) I and II only
(C) I and III only
(D) II and III only
(E) III only

53-1. In the complex plane, let \( C \) be the circle \(|z+2|=2\) with negative (clockwise) orientation. Compute \( \int_C \frac{dz}{(z-1)(z+3)^2} \).

53-2. In the complex plane, let \( C \) be the circle \(|z|=4\) with negative (clockwise) orientation. Compute \( \int_C \frac{dz}{(z-1)(z+3)^2} \).
54-1. Assume that, in a certain two-dimensional world, the wind velocity at any point \((x, y)\) is \((-11x + 10y, -10x + 14y)\). A small particle is simply pushed by the wind. Its position at any time \(t\) is given by \((f(t), g(t))\). Assume that its velocity at time \(t\) is
\[
\left( -11[f(t)] + 10[g(t)] , -10[f(t)] + 14[g(t)] \right).
\]
Because its velocity at time \(t\) is also given by \((f'(t), g'(t))\), its motion will satisfy the equations:
\[
f'(t) = -11[f(t)] + 10[g(t)], \quad g'(t) = -10[f(t)] + 14[g(t)].
\]
Assume that the initial position of the particle is \((f(0), g(0)) = (0, 1)\). We stand at the origin and watch the particle. Along what slope line will we look, asymptotically, as \(t \to \infty\)? That is, compute \(\lim_{t \to \infty} \frac{g(t)}{f(t)}\).

54-2. Assume that, in a certain two-dimensional world, the wind velocity at any point \((x, y)\) is \((-y, x)\). A small particle is simply pushed by the wind. Its position at any time \(t\) is given by \((f(t), g(t))\). Assume that its velocity at time \(t\) is \((-[g(t)], f(t))\). Because its velocity at time \(t\) is also given by \((f'(t), g'(t))\), its motion will satisfy the equations:
\[
f'(t) = -[g(t)], \quad g'(t) = f(t).
\]
Assume that the initial position of the particle is \((f(0), g(0)) = (2, 0)\). Find its position \((f(t), g(t))\) at any time \(t\).
55-1. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. True or False: If $f'(0) = 0$, then $f(x)$ has a local extremum at $x = 0$.

55-2. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. True or False: If $f(x)$ has a local extremum at $x = 0$, then $f'(0) = 0$.

55-3. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. True or False: If $f'(x)$ has a local extremum at $x = 0$, then $f(x)$ has a point of inflection at $x = 0$.

55-4. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. True or False: If $f(x)$ has a point of inflection at $x = 0$, then $f'(x)$ has a local extremum at $x = 0$.

55-5. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable. True or False: If $f''(0) = 0$, then $f(x)$ has a point of inflection at $x = 0$.

55-6. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable. True or False: If $f(x)$ has a point of inflection at $x = 0$, then $f''(0) = 0$.

56-1. True or false: For any metric $d$ on $\mathbb{R}$, there is a norm $\| \cdot \|$ on $\mathbb{R}$ such that, for all $x, y \in \mathbb{R}$, $d(x, y) = \|x - y\|$.

56-2. True or false: For every norm $\| \cdot \|$ on $\mathbb{R}$, there is an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}$ such that, for all $x \in \mathbb{R}$, we have $\|x\|^2 = \langle x, x \rangle$.

56-3. True or false: For every norm $\| \cdot \|$ on $\mathbb{R}^2$, there is an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^2$ such that, for all $v \in \mathbb{R}^2$, we have $\|v\|^2 = \langle v, v \rangle$.

57-1. Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}[x]$ the ring of polynomials in $x$ with coefficients in $\mathbb{R}$. Which of the following subsets of $\mathbb{R}[x]$ is a subring of $\mathbb{R}[x]$?

I. All polynomials in $\mathbb{R}[x]$ whose coefficient of $x^2$ is zero

II. All polynomials in $\mathbb{R}[x]$ all of whose terms have even degree, including the zero polynomial.

III. All polynomials in $\mathbb{R}[x]$ whose coefficients are nonnegative real numbers.
58-1. Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous and injective. Let \( U \) be an open subset of \( \mathbb{R} \). True or false: \( f(U) \) is necessarily an open subset of \( \mathbb{R} \).

58-2. Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous. Let \( U \) be an open subset of \( \mathbb{R} \). True or false: \( f(U) \) is necessarily an open subset of \( \mathbb{R} \).

58-3. Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous. Let \( U \) be an open subset of \( \mathbb{R} \). True or false: \( f^{-1}(U) \) is necessarily an open subset of \( \mathbb{R} \).

58-4. Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous. Let \( B \) be a bounded subset of \( \mathbb{R} \). True or false: \( f(B) \) is necessarily a bounded subset of \( \mathbb{R} \).

58-5. Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous. Let \( B \) be a bounded subset of \( \mathbb{R} \). True or false: \( f^{-1}(B) \) is necessarily a bounded subset of \( \mathbb{R} \).

58-6. Let \( f : (0,1) \to \mathbb{R} \) be continuous. Let \( B \) be a bounded subset of \( \mathbb{R} \). Assume that \( B \subseteq (0,1) \). True or false: \( f(B) \) is necessarily a bounded subset of \( \mathbb{R} \).