CALCULUS
Continuity
OLD2
0210-1. a. At which numbers is the function $f$, shown above, discontinuous?

b. For each of the numbers, given in Part a, where $f$ is discontinuous, state whether or not $f$ is continuous from the LEFT at that number.

c. For each of the numbers, given in Part a, where $f$ is discontinuous, state whether or not $f$ is continuous from the RIGHT at that number.
0210-1. a. At which numbers is the function $f$, shown above, discontinuous?

**ANSWER:** $-4, -2, 2, 4$
0210-1.b.

For each of the numbers, given in Part a, where \( f \) is discontinuous, state whether or not \( f \) is continuous from the LEFT at that number.

\[ \text{ANSWER:} \]

\( f \) is continuous from the LEFT at \(-2, 4\).
\( f \) is not continuous from the LEFT at \(-4, 2\).
0210-1. c.
OLD2
For each of the numbers, given in Part a, where \( f \) is discontinuous, state whether or not \( f \) is continuous from the RIGHT at that number.

**ANSWER:**
\( f \) is not continuous from the RIGHT at \(-4, -2, 2, 4.\)
0210-2. Display the graph of a function $f$

s.t. \( \lim_{x \to -1^-} f(x) = -3, \quad \lim_{x \to -1^+} f(x) = 1, \)

and s.t. \( f(-1) = 1, \)

and s.t. \( \lim_{x \to 1} f(x) = -\infty, \quad f(1) = 2, \)

and s.t. \( \lim_{x \to 2} f(x) = 1, \quad f(2) = 0, \)

and s.t. \( \lim_{x \to -\infty} f(x) = -1, \quad \lim_{x \to \infty} f(x) = -4. \)
\[ \lim_{x \to -1^-} f(x) = -3, \quad \lim_{x \to -1^+} f(x) = 1, \quad f(-1) = 1, \]
\[ \lim_{x \to 1} f(x) = -\infty, \quad f(1) = 2, \]
\[ \lim_{x \to 2} f(x) = 1, \quad f(2) = 0, \]
\[ \lim_{x \to -\infty} f(x) = -1, \quad \lim_{x \to \infty} f(x) = -4. \]

ANS:

Many, many other correct answers!
0210-3. Let \( f(t) = (4t^{2/3} + 3)^{85} \).

Using the properties of limits, show that \( f \) is continuous at 7.

**ANS:**

\[
\lim_{t \to 7} f(t) = \lim_{t \to 7} (4t^{2/3} + 3)^{85}
\]

- Limit commutes with powers

\[
= \left( \lim_{t \to 7} \left( 4t^{2/3} + 3 \right) \right)^{85}
\]

- Limit is linear

\[
= \left[ 4 \left( \lim_{t \to 7} t^{2/3} \right) + \left( \lim_{t \to 7} 3 \right) \right]^{85}
\]

- Limit commutes with powers

\[
= \left[ 4 \left( \lim_{t \to 7} t \right)^{2/3} + \left( \lim_{t \to 7} 3 \right) \right]^{85}
\]

- Polynomials are continuous

\[
= \left[ 4(7)^{2/3} + 3 \right]^{85} = f(7)
\]
0210-4. Let \( f(x) = \begin{cases} 
2x + 5, & \text{if } x < -1 \\
3, & \text{if } x = -1 \\
x^2 + 4, & \text{if } x > -1 
\end{cases} \)

a. Does \( \lim_{x \to -1} f(x) \) exist? If so, compute it.

b. Is \( f \) continuous from the left at \(-1\)?

**ANSWER:**

a. 
\[
\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (2x + 5) = 2(-1) + 5 = 3
\]

\[
\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (x^2 + 4) = (-1)^2 + 4 = 5
\]

Therefore the limit does not exist.

b. \( f(-1) = 3 = \lim_{x \to -1^-} f(x) \)

Therefore \( f \) is continuous from the left at \(-1\).
0210-5. Let \( g(x) = \begin{cases} 
\cos(2x), & \text{if } x < 0 \\
1, & \text{if } x = 0 \\
x^2 + 1, & \text{if } x > 0.
\end{cases} \)

a. Does \( \lim_{{x \to 0}} g(x) \) exist? If so, compute it.

b. Is \( g \) continuous at 0?

\textbf{ANSWER:} a.

\[
\lim_{{x \to 0^-}} g(x) = \lim_{{x \to 0^-}} \cos(2x) = \cos(2 \cdot 0) = 1 \\
\lim_{{x \to 0^+}} g(x) = \lim_{{x \to 0^+}} (x^2 + 1) = (0)^2 + 1 = 1
\]

Therefore \( \lim_{{x \to 0}} g(x) = 1. \)

b. \( g(0) = 1 = \lim_{{x \to 0}} g(x), \)

so \( g \) is continuous at 0.
0210-6. Let \( g(x) = \begin{cases} 
\cos(2x), & \text{if } x < 0 \\
1, & \text{if } x = 0 \\
x^2 + 1, & \text{if } x > 0.
\end{cases} \)

a. Does \( \lim_{x \to -1} g(x) \) exist? If so, compute it.

b. Is \( g \) continuous at \(-1\)?

**ANSWER:**

a. \( \lim_{x \to -1} g(x) = \lim_{x \to -1} \cos(2x) = \cos(-2) \);

in particular, the limit exists.

b. \( g(-1) = \cos(-2) = \lim_{x \to -1} g(x) \)

Therefore \( g \) is continuous at \(-1\). ■
Let \( f(x) = \sqrt[3]{x} \).

a. Is \( f \) continuous at 0?

b. Is \( f \) continuous on \([0, \infty)\)?

c. Is \( f \) continuous?

**ANS:**

a. \( \lim_{x \to 0} f(x) = \lim_{x \to 0} \sqrt[3]{x} = 0 = \sqrt[3]{0} = f(0) \),

so \( f \) is continuous at 0.

b. \( \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt[3]{x} = 0 = \sqrt[3]{0} = f(0) \),

so \( f \) is continuous from the right at 0.

\( \forall a \in (0, \infty), \ f \) is continuous at \( a \),

so \( f \) is continuous on \((0, \infty)\).

Then \( f \) is continuous on \([0, \infty)\).

c. \( \forall a \in \mathbb{R} = \text{dom}[f], \ f \) is continuous at \( a \),

so \( f \) is continuous.
0210-8. Let $g(x) = 1/\sqrt[3]{x}$.

a. Is $g$ continuous at 0?
b. Is $g$ continuous on $(0, \infty)$?
c. Is $g$ continuous?

**ANSWER:**

a. $g(0)$ DNE, so $g$ is **not** continuous at 0.

b. $\forall a \in (0, \infty)$, $g$ is continuous at $a$, so $g$ is continuous on $(0, \infty)$.

c. $\forall a \in \mathbb{R} \setminus \{0\} = \text{dom}[g]$, $g$ is continuous at $a$, so $g$ is continuous.
0210-9. Compute \( \lim_{x \to 27} \frac{x + \sqrt[3]{x}}{(x - 20)^2 - 2x + 6} \).

**ANSWER:** Let \( f(x) = \frac{x + \sqrt[3]{x}}{(x - 20)^2 - 2x + 6} \).

By the properties of limit, \( f \) is continuous at 27, i.e.,

\[
\lim_{x \to 27} \frac{x + \sqrt[3]{x}}{(x - 20)^2 - 2x + 6} = \left[ \frac{x + \sqrt[3]{x}}{(x - 20)^2 - 2x + 6} \right]_{x \to 27}
\]

\[
= \frac{27 + \sqrt[3]{27}}{(27 - 20)^2 - 2 \cdot 27 + 6} = \frac{27 + 3}{49 - 54 + 6} = \frac{30}{1} = 30
\]
Let \( f(x) = \begin{cases} 
  x^2 + 3, & \text{if } x < 2 \\
  2x + 2, & \text{if } 2 \leq x < 3 \\
  8[\cos(x - 3)], & \text{if } 3 \leq x.
\end{cases} \)

a. At which numbers is the function \( f \) discontinuous?

b. For each of the numbers, given in Part a, where \( f \) is discontinuous, state whether or not \( f \) is continuous from the LEFT at that number.

c. For each of the numbers, given in Part a, where \( f \) is discontinuous, state whether or not \( f \) is continuous from the RIGHT at that number.
0210-10. Let \( f(x) = \begin{cases} 
 x^2 + 3, & \text{if } x < 2 \\
 2x + 2, & \text{if } 2 \leq x < 3 \\
 8[\cos(x - 3)], & \text{if } 3 \leq x. 
\end{cases} \)

**a. At which numbers is the function \( f \) discontinuous?**

**ANS:**

\[
\lim_{{x \to 2^-}} f(x) = \lim_{{x \to 2^-}} x^2 + 3 = 7 \\
\lim_{{x \to 2^+}} f(x) = \lim_{{x \to 2^+}} 2x + 2 = 6 \\
\lim_{{x \to 3^-}} f(x) = \lim_{{x \to 3^-}} 2x + 2 = 8 \\
\lim_{{x \to 3^+}} f(x) = \lim_{{x \to 3^+}} 8[\cos(x - 3)] = 8 \\
\]

\( f(2) = [2x + 2]_{x \to 2} = 6 \)

\( f(3) = [8[\cos(x - 3)]]_{x \to 3} = 8 \)

\[a. \text{ } f \text{ is discontinuous only at } 2.\]
Let \( f(x) = \begin{cases} 
  x^2 + 3, & \text{if } x < 2 \\
  2x + 2, & \text{if } 2 \leq x < 3 \\
  8[\cos(x - 3)], & \text{if } 3 \leq x.
\end{cases} \)

b. For each of the numbers, given in Part a, where \( f \) is discontinuous, state whether or not \( f \) is continuous from the LEFT at that number.

\[ \text{ANS: } \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 + 3 = 7 \quad \quad f(2) = [2x + 2]_{x: \to 2} = 6 \]

\[ \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 2x + 2 = 6 \]

\[ \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} 2x + 2 = 8 \quad \quad f(3) = \lim_{x : \to 3^-} [8[\cos(x - 3)]] = 8 \]

\[ \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 8[\cos(x - 3)] = 8 \]

a. \( f \) is discontinuous only at 2.

b. \( f \) is not continuous from the LEFT at 2.
Let \( f(x) = \begin{cases} 
    x^2 + 3, & \text{if } x < 2 \\
    2x + 2, & \text{if } 2 \leq x < 3 \\
    8[\cos(x - 3)], & \text{if } 3 \leq x.
\end{cases} \)

c. For each of the numbers, given in Part a, where \( f \) is discontinuous, state whether or not \( f \) is continuous from the RIGHT at that number.

**ANS:**

\[
\lim_{{x \to 2^-}} f(x) = \lim_{{x \to 2^-}} x^2 + 3 = 7
\]

\[
f(2) = 2[2x + 2]_{{x \to 2}} = 6
\]

\[
\lim_{{x \to 2^+}} f(x) = \lim_{{x \to 2^+}} 2x + 2 = 6
\]

\[
f(3) = 8[\cos(x - 3)]_{{x \to 3}} = 8
\]

\[
\lim_{{x \to 3^-}} f(x) = \lim_{{x \to 3^-}} 2x + 2 = 8
\]

\[
\lim_{{x \to 3^+}} f(x) = \lim_{{x \to 3^+}} 8[\cos(x - 3)] = 8
\]

a. \( f \) is discontinuous only at 2.

b. \( f \) is continuous from the RIGHT at 2.
0210-11. Let \( g(x) = \begin{cases} 4e^x, & \text{if } x \leq 0 \\ (x + 2)^2, & \text{if } 0 < x < 1 \\ 7x + 2, & \text{if } 1 < x. \end{cases} \)

(a) At which numbers is the function \( g \) discontinuous?

(b) For each of the numbers, given in Part a, where \( g \) is discontinuous, state whether or not the discontinuity is removable.
0210-11. Let \( g(x) = \begin{cases} 4e^x, & \text{if } x \leq 0 \\ (x + 2)^2, & \text{if } 0 < x < 1 \\ 7x + 2, & \text{if } 1 < x. \end{cases} \)

a. At which numbers is the function \( g \) discontinuous?

\[
\begin{align*}
\text{ANS:} & \quad \lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} 4e^x = 4 \\
& \quad \lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x + 2)^2 = 4 \\
& \quad \lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} (x + 2)^2 = 9 \\
& \quad \lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} 7x + 2 = 9 \\
& \quad g(0) = [4e^x]_{x = 0} = 4 \\
& \quad g(1) \text{ DNE}
\end{align*}
\]

a. \( g \) is discontinuous only at 1.
0210-11. Let \( g(x) = \begin{cases} 4e^x, & \text{if } x \leq 0 \\ (x + 2)^2, & \text{if } 0 < x < 1 \\ 7x + 2, & \text{if } 1 < x. \end{cases} \)

b. For each of the numbers, given in Part a, where \( g \) is discontinuous, state whether or not the discontinuity is removable.

\[
\begin{align*}
\lim_{x \to 0^-} g(x) &= \lim_{x \to 0^-} 4e^x = 4 \\
\lim_{x \to 0^+} g(x) &= \lim_{x \to 0^+} (x + 2)^2 = 4 \\
\lim_{x \to 1^-} g(x) &= \lim_{x \to 1^-} (x + 2)^2 = 9 \\
\lim_{x \to 1^+} g(x) &= \lim_{x \to 1^+} 7x + 2 = 9 \\
\end{align*}
\]

\( g(0) = [4e^x]_{x \to 0} = 4 \)

\( g(1) \) DNE

a. \( g \) is discontinuous only at 1.

b. The discontinuity at 1 is removable.
Find a number $a$ s.t.

$$f(x) = \begin{cases} 
  a e^x, & \text{if } x \leq 0 \\
  a x^3 + 3a + 8, & \text{if } 0 < x
\end{cases}$$

is continuous at $x = 0$.

**ANSWER:**

$$\lim_{{x \to 0^-}} f(x) = \lim_{{x \to 0^-}} a e^x = a$$

$$f(0) = [a e^x]_{x: \to 0} = a$$

$$\lim_{{x \to 0^+}} f(x) = \lim_{{x \to 0^+}} a x^3 + 3a + 8 = 3a + 8$$

For continuity, we need $3a + 8 = a$.

$$2a = -8$$

$$a = -4$$
0210-13. Let \( h(s) = \frac{s^2 + 5s - 6}{s - 1} \).

Find a function \( p: \mathbb{R} \rightarrow \mathbb{R} \) such that \( p \) is continuous at 1 and such that, \( \forall s \in \mathbb{R} \setminus \{1\}, \ p(s) = h(s) \).

**Answer:**

\[
\begin{align*}
    h(s) &= \frac{s^2 + 5s - 6}{s - 1} \\
    &= s + 6 \quad (s \neq 1)
\end{align*}
\]

Let \( p(s) = s + 6 \). \( \square \)
Using the Intermediate Value Theorem, show that \( x^3 + 2x - 8 = 0 \) has a solution \( x = c \) that satisfies \(-2 < c < 2\).

**ANSWER:**

\[
\begin{align*}
[x^3 + 2x - 8]_{x \to -2} &= -8 - 4 - 8 < 0 \\
[x^3 + 2x - 8]_{x \to 2} &= 8 + 4 - 8 > 0
\end{align*}
\]

\( x^3 + 2x - 8 \) is continuous on \(-2 \leq x \leq 2\).

By the Intermediate Value Theorem, \( \exists c \in (-2, 2) \) s.t. \( [x^3 + 2x - 8]_{x \to c} = 0 \).
Using the Intermediate Value Theorem, show that \(4e^x + \cos x = x + 6\) has a sol’n \(x = c\) that satisfies \(-2 < c < 9\).

\[
\cos(-2) \leq 1
\]

\[
\begin{align*}
[4e^x + (\cos x) - x - 6]_{x \to -2} &\leq (4/e^2) + 1 - (-2) - 6 < 0 \\
[4e^x + (\cos x) - x - 6]_{x \to 9} &\geq 4 \cdot e^9 - 1 - 9 - 6 > 0
\end{align*}
\]

\(4e^x + (\cos x) - x - 6\) is continuous on \(-2 \leq x \leq 9\). By the Intermediate Value Theorem, \(\exists c \in (-2, 9)\) s.t.

\[
[4e^x + (\cos x) - x - 6]_{x \to c} = 0.
\]