CALCULUS
Related rates
OLD2
0520-1. An isosceles right triangle is growing. At time $t$, its area is $A$ and its leg length is $s$, so $A$ and $s$ are expressions of $t$. The triangle’s hypotenuse length is $s\sqrt{2}$. Find a formula for $dA/dt$ in terms of $s$ and $ds/dt$.

**ANSWER:**

\[ A = \frac{s^2}{2} \]

\[ \frac{dA}{dt} = s \left[ \frac{ds}{dt} \right] \]
A regular pentagon is growing. At time $t$, its area is $A$ and its side length is $s$, so $A$ and $s$ are expressions of $t$. Find a formula for $dA/dt$ in terms of $s$ and $ds/dt$.

**ANSWER:**

Each triangle has area \[ \left[ \frac{\cot(\pi/5)}{4} \right] s^2. \]

\[
A = \left[ \frac{5(\cot(\pi/5))}{4} \right] s^2
\]

\[
\frac{dA}{dt} = \left[ \frac{5(\cot(\pi/5))}{2} \right] s \left[ \frac{ds}{dt} \right]
\]
A \textbf{\textit{J}_1 (Johnson_1) solid} is a pyramid whose base is a square and whose sides are equilateral triangles. A \textit{J}_1 solid is growing. At time \( t \), its volume is \( V \) and its edge length is \( s \), so \( V \) and \( s \) are expressions of \( t \). Find a formula for \( \frac{dV}{dt} \) in terms of \( s \) and \( \frac{ds}{dt} \).

\textbf{Hint:} \quad V = (\sqrt{2}/6)s^3.

\textbf{ANSWER:}

\[
V = \frac{\sqrt{2}}{6} s^3
\]

\[
\frac{dV}{dt} = \left[ \frac{\sqrt{2}}{2} s^2 \right] \left[ \frac{ds}{dt} \right]
\]
Suppose \( x^3 + y^3 + 27 = z^2 + z^3 \) and \( dx/dt = 6 \) and \( dy/dt = 8 \). Compute \( dz/dt \) at a certain moment when

\[ x = 1, \ y = 2 \text{ and } z = 3. \]

**ANSWER:**

\[
3x^2 \frac{dx}{dt} + 3y^2 \frac{dy}{dt} = 2z \frac{dz}{dt} + 3z^2 \frac{dz}{dt}
\]

\[
3[1^2][6] + 3[2^2][8] = 2[3][?] + 3[3^2][?]
\]

\[18 + 3 \cdot 32 = 6[?] + 27[?]
\]

\[18 + 96 = 33[?]
\]

\[? = \frac{18 + 96}{33} = \frac{114}{33} = \frac{38}{11}\]
0520-5. A streetlight is at the top of a 24 foot pole. A 6 foot tall man walks directly away from the light at a speed of 4 feet per second. How fast is his shadow growing?

**ANSWER:**

\[
\frac{dx}{dt} = 4
\]

\[
\frac{ds}{dt} = ?
\]
A streetlight is at the top of a 24 foot pole. A 6 foot tall man walks directly away from the light at a speed of 4 feet per second. How fast is his shadow growing?

**ANSWER:**

\[
\frac{dx}{dt} = 4
\]

\[
\frac{ds}{dt} = ?
\]

By similarity of the green triangles,

\[
\frac{s}{6} = \frac{s + x}{24}.
\]
0520-5. A streetlight is at the top of a 24 foot pole. A 6 foot tall man walks directly away from the light at a speed of 4 feet per second. How fast is his shadow growing?

**ANSWER:**

\[
\frac{ds}{dt} = \frac{(ds/dt) + (dx/dt)}{24}
\]

\[
\frac{dx}{dt} = 4
\]

\[
\frac{ds}{dt} = ?
\]

\[
\frac{?}{6} = \frac{? + 4}{24}
\]

\[
24? = 6? + 24
\]

\[
18? = 24
\]

By similarity of the green triangles,

\[
\frac{s}{6} = \frac{s + x}{24}
\]

\[
? = \frac{24}{18} = \frac{4}{3}
\]

feet/second
A jet flies in a straight line, with constant speed and altitude. It flies directly over a radar station, and, a few minutes later, the radar’s instruments show that the plane is 5 miles away, and that its distance from the radar station is increasing at 390 mph. **Assuming** that the altitude of the jet is 4 miles greater than that of the station, **find** the speed of the jet.

This distance is increasing at a rate of 390 mph.
A jet flies in a straight line, with constant speed and altitude. It flies directly over a radar station, and, a few minutes later, the radar’s instruments show that the plane is 5 miles away, and that its distance from the radar station is increasing at 390 mph. Assuming that the altitude of the jet is 4 miles greater than that of the station, find the speed of the jet.

**ANSWER:** \([\dot{z}]_{t=t_0} = 5\)

\[
\left[ \frac{dz}{dt} \right]_{t=t_0} = 390
\]

\[
\frac{dx}{dt} = ?
\]

\[
x^2 + 4^2 = z^2
\]
0520-6. A jet flies in a straight line, with constant speed and altitude. It flies directly over a radar station, and, a few minutes later, the radar’s instruments show that the plane is 5 miles away, and that its distance from the radar station is increasing at 390 mph. **Assuming** that the altitude of the jet is 4 miles greater than that of the station, **find** the speed of the jet.

**ANSWER:** \([z]_{t \rightarrow t_0} = 5\)

\[
\left[ \frac{dz}{dt} \right]_{t \rightarrow t_0} = 390
\]

\[
\frac{dx}{dt} = ?
\]

\[
x^2 + 4^2 = z^2
\]

\[
x = \sqrt{z^2 - 4^2}
\]

\[
[x]_{t \rightarrow t_0} = \sqrt{5^2 - 4^2} = 3
\]

\[
2x \frac{dx}{dt} + 0 = 2z \frac{dz}{dt}
\]

\[
2[3][?] + 0 = 2[5][390]
\]
A jet flies in a straight line, with constant speed and altitude. It flies directly over a radar station, and, a few minutes later, the radar’s instruments show that the plane is 5 miles away, and that its distance from the radar station is increasing at 390 mph. **Assuming** that the altitude of the jet is 4 miles greater than that of the station, **find** the speed of the jet.

**ANSWER:**

\[ 2[3][?] + 0 = 2[5][390] \]

\[ ? = \frac{2[5][390]}{2[3]} = 650 \text{ mph} \]
Water is being drained, at a rate of 4 cubic meters per minute, from a conical container of height 15 meters, whose top is a circle whose radius is 5 meters. When the water level is 7.5 meters, how fast is that level decreasing?
Water is being drained, at a rate of 4 cubic meters per minute, from a conical container of height 15 meters, whose top is a circle whose radius is 5 meters. When the water level is 7.5 meters, how fast is that level decreasing?

\[
\begin{align*}
V &= \text{volume of the water} \\
r &= \text{radius of the top of the water} \\
h &= \text{height of the water} \\
\frac{r}{h} &= \frac{5}{15} \\
\frac{dV}{dt} &= -4 \\
V &= \frac{1}{3}\pi r^2 h \\
[h]_{t\rightarrow t_0} &= 7.5 \\
\left[\frac{dh}{dt}\right]_{t\rightarrow t_0} &= -？
\end{align*}
\]

\[18\]
Water is being drained, at a rate of 4 cubic meters per minute, from a conical container of height 15 meters, whose top is a circle whose radius is 5 meters. When the water level is 7.5 meters, how fast is that level decreasing?

**ANSWER:**

\[ \frac{r}{h} = \frac{5}{15}, \text{ so } r = \frac{1}{3} h. \]

\[ V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi \left( \frac{1}{3} h \right)^2 h = \frac{1}{27} \pi h^3 \]

\[ [h]_{t \to t_0} = 7.5 \]

\[ \left[ \frac{dh}{dt} \right]_{t \to t_0} = -? \]

\[ \frac{dV}{dt} = -4 \]

\[ \frac{1}{27} \pi (3h^2) \frac{dh}{dt} \]

\[ -4 = \frac{1}{27} \pi (3(7.5)^2)(-?) \]
Water is being drained, at a rate of 4 cubic meters per minute, from a conical container of height 15 meters, whose top is a circle whose radius is 5 meters. When the water level is 7.5 meters, how fast is that level decreasing?

**ANSWER:**

\[-4 = \frac{1}{27} \pi (3(7.5)^2)(-)\]

\[-108 \quad \frac{3\pi(7.5)^2}{} = -?\]

? = \[\frac{108}{3\pi(7.5)^2} \div 0.2037 \text{ meters per second} \]

\[\quad\]
A camera at (3, 0) is following a UFO that strafes in from above, following the curve 
\( y = x^2 + 2 \) from left to right. At the moment when the UFO is at the point (2, 6), retreating back into outer space, the angle between the camera and the horizontal is increasing at 5 radians per second.

a. What is the rate of change in the \( x \)-coordinate of the UFO at that moment?

b. What is the rate of change in the \( y \)-coordinate of the UFO at that moment?
0520-8.

OLD2

ANSWER:

a. What is the rate of change in the $x$-coordinate of the UFO at that moment?

b. What is the rate of change in the $y$-coordinate of the UFO at that moment?

At time $t_0$, UFO at $(2, 6)$

$$[x]_{t: \to t_0} = 2$$
$$[y]_{t: \to t_0} = 6$$

$$a := [dx/dt]_{t: \to t_0}$$
$$b := [dy/dt]_{t: \to t_0}$$

Goal: Compute $a$ and $b$.

At time $t_0$, this angle is increasing at 5 radians per second.

$[d\theta/dt]_{t: \to t_0} = 5$
\[
\tan \theta = \frac{y}{3 - x}
\]

\[
[tan \theta]_{t \to t_0} = \frac{6}{3 - 2} = 6
\]

\[
[sec^2 \theta]_{t \to t_0} = [1 + tan^2 \theta]_{t \to t_0} = 1 + 6^2 = 37
\]

at time \(t_0\),

UFO at \((2, 6)\)

\[
[x]_{t \to t_0} = 2
\]

\[
[y]_{t \to t_0} = 6
\]

\[
a := [dx/dt]_{t \to t_0}
\]

\[
b := [dy/dt]_{t \to t_0}
\]

Goal: Compute \(a\) and \(b\).
\[
\tan \theta = \frac{y}{3-x}
\]

\[
\left(\sec^2 \theta\right) \left(\frac{d\theta}{dt}\right) = \frac{(3-x)(dy/dt) - (y)(-dx/dt)}{(3-x)^2}
\]

\[
(37)(5) = \left[\left(\sec^2 \theta\right) \left(\frac{d\theta}{dt}\right)\right]_{t\to t_0} = \frac{(3-2)(b) - (6)(-a)}{(3-2)^2}
\]

at time \(t_0\),

- UFO at (2, 6)
- \([x]_{t\to t_0} = 2\)
- \([y]_{t\to t_0} = 6\)

\(a := [dx/dt]_{t\to t_0}\)

\(b := [dy/dt]_{t\to t_0}\)

**Goal:** Compute \(a\) and \(b\).
\[ (37)(5) = \left[ \left( \sec^2 \theta \right) \left( \frac{d\theta}{dt} \right) \right]_{t \to t_0} = \frac{(3 - 2)(b) - (6)(-a)}{(3 - 2)^2} \]

\[ 185 = (3 - 2)(b) - (6)(-a) \]

\[ 6a + b = 185 \]

\[ \text{at time } t_0, \]

\[ \text{UFO at } (2, 6) \]

\[ [x]_{t \to t_0} = 2 \]

\[ [y]_{t \to t_0} = 6 \]

\[ a := [dx/dt]_{t \to t_0} \]

\[ b := [dy/dt]_{t \to t_0} \]

Goal: Compute \( a \) and \( b \).
\textbf{0520-8. ANSWER:}

\[ y = x^2 + 2 \]

\[ \frac{dy}{dt} = 2x \frac{dx}{dt} + 0 \]

\[ b = 2(2)a + 0 = 4a \]

\textbf{at time } t_0, \quad \text{UFO at } (2, 6)

\[ [x]_{t: \rightarrow t_0} = 2 \]

\[ [y]_{t: \rightarrow t_0} = 6 \]

\[ a := [dx/dt]_{t: \rightarrow t_0} \]

\[ b := [dy/dt]_{t: \rightarrow t_0} \]

\textbf{Goal: Compute } a \text{ and } b.\]

\[ 6a + b = 185 \]
0520-8. ANSWER:

\begin{align*}
  b &= 4a \\
  6a + b &= 185 \\
  10a &= 6a + 4a = 185 \\
  a &= 185/10 = 18.5 \\
  b &= 4a = 740/10 = 74 \\
  b &= 2(2)a + 0 = 4a
\end{align*}

at time \( t_0 \),

**UFO at (2, 6)**

\[
\begin{align*}
  [x]_{t \to t_0} &= 2 \\
  [y]_{t \to t_0} &= 6
\end{align*}
\]

\[
\begin{align*}
  a &= [dx/dt]_{t \to t_0} \\
  b &= [dy/dt]_{t \to t_0}
\end{align*}
\]

Goal: Compute \( a \) and \( b \).
a. What is the rate of change in the \( x \)-coordinate of the UFO at that moment?

**ANSWER:** \( a = 18.5 \)

b. What is the rate of change in the \( y \)-coordinate of the UFO at that moment?

**ANSWER:** \( b = 74 \)

At time \( t_0 \), UFO at \((2, 6)\):

\[
[x]_{t_0} = 2 \\
[y]_{t_0} = 6
\]

\[
a := [d x/d t]_{t_0} \\
b := [d y/d t]_{t_0}
\]

**Goal:** Compute \( a \) and \( b \).
0520-8.

a. What is the rate of change in the $x$-coordinate of the UFO at that moment?

**ANSWER:** $a = 18.5$

b. What is the rate of change in the $y$-coordinate of the UFO at that moment?

**ANSWER:** $b = 74$

**NOTES:**

(●) The speed of the UFO at time $t_0$ is

\[
\sqrt{(18.5)^2 + (74)^2} \\
= 76.277
\]

(●) No units of distance are given in the problem, so none can be given in the answer.
Sand is being poured, at a rate of 6 cubic meters per minute, into a conical pile that is always 5/8 as high as it is wide. How fast is the width of the pile increasing when the pile is 8 meters wide and 5 meters high?
Sand is being poured, at a rate of 6 cubic meters per minute, into a conical pile that is always 5/8 as high as it is wide. How fast is the width of the pile increasing when the pile 8 meters wide and 5 meters high?

\[ \text{ANS: } V := \text{volume of the pile} = \frac{1}{3} \pi \left( \frac{w}{2} \right)^2 \left( \frac{5w}{8} \right) \]

\[
\left[ \frac{dw}{dt} \right]_{t \to t_0} = ?
\]

\[
\frac{dV}{dt} = 6
\]

\[
[w]_{t \to t_0} = 8
\]
Sand is being poured, at a rate of 6 cubic meters per minute, into a conical pile that is always \(5/8\) as high as it is wide. How fast is the width of the pile increasing when the pile 8 meters wide and 5 meters high?

**ANS:**

\[
V := \text{volume of the pile} = \frac{1}{3} \pi \left(\frac{w}{2}\right)^2 \left(\frac{5w}{8}\right)
\]

\[
= \frac{5}{96} \pi w^3
\]

\[
\left[ \frac{dw}{dt} \right]_{t \to t_0} = \text{?}
\]

\[
\frac{dV}{dt} = \frac{d}{dt} \left[ \frac{5}{96} \pi w^3 \right]
\]

\[
\frac{dV}{dt} = 6
\]

\[
[w]_{t \to t_0} = 8
\]

\[
6 = \left[ \frac{5}{32} \pi w^2 \right] \left[ \frac{dw}{dt} \right]
\]

\[
6 = \left[ \frac{5}{32} \pi (8^2) \right] \text{[?]}\]
Sand is being poured, at a rate of 6 cubic meters per minute, into a conical pile that is always 5/8 as high as it is wide. How fast is the width of the pile increasing when the pile 8 meters wide and 5 meters high?

ANS:

\[ 6 = \left[ \frac{5}{32}\pi(8^2) \right] \text{ [?] } \]

\[ = [10\pi][?] \]

\[ \frac{6}{10\pi} \div 0.19099 \text{ meters per minute} \]