CALCULUS
The Mean Value Theorem
NEW
Let \( f(x) = x^2 - 2x - 3 \).

a. Check that \( f \) satisfies the conditions of Rolle’s Theorem on the interval \([-1, 3]\). That is, check

(i) that \( f \) is continuous on \([-1, 3]\),
(ii) that \( f \) is differentiable on \((-1, 3)\) and
(iii) that \( f(-1) = f(3) \).

b. Find all solutions to the equation in the conclusion of Rolle’s Th’m for \( f \) on \([-1, 3]\). That is, find all \( c \in (-1, 3) \) s.t. \( f'(c) = 0 \).
0460-2. Let \( f(x) = x^2 + x - 3 \).

a. **Check** that \( f \) satisfies the conditions of the MVT on the interval \([-1, 3]\). That is, **check**

   (i) that \( f \) is continuous on \([-1, 3]\) and (ii) that \( f \) is differentiable on \((-1, 3)\).

b. **Find** all solutions to the equation in the conclusion of the MVT for \( f \) on \([-1, 3]\). That is, **find all** \( c \in (-1, 3) \) s.t.

\[
    f'(c) = \frac{[f(3)] - [f(-1)]}{3 - (-1)}.
\]
Let \( f(x) = 7 + |x - 5| \).

a. Show that \( f \) is continuous on \([2, 8]\).

b. Show that \( f(2) = f(8) \).

c. Show that the conclusion of Rolle’s Th’m, for \( f \) on \([2, 8]\), fails. That is, show that there is no \( c \in (2, 8) \) s.t. \( f'(c) = 0 \).

d. Explain why this does not contradict Rolle’s Theorem.
0460-4. Let \( f(x) = x + |x - 5| \).

a. Show that \( f \) is continuous on \([2, 8]\).

b. Show that the conclusion of the MVT, for \( f \) on \([2, 8]\), fails. That is, show that there is no \( c \in (2, 8) \) s.t.

\[
f'(c) = \frac{[f(8)] - [f(2)]}{8 - 2}.
\]

c. Explain why this does not contradict the MVT.
Let \( f(x) = \begin{cases} 
100, & \text{if } x = 2 \\
3x - 5, & \text{if } 2 < x < 8 \\
40, & \text{if } x = 8. 
\end{cases} \)

a. Show that \( f \) is differentiable on \((2, 8)\).

b. Show that the conclusion of the MVT, for \( f \) on \([2, 8]\), fails. That is, show that there is no \( c \in (2, 8) \) s.t.

\[
f'(c) = \frac{[f(8)] - [f(2)]}{8 - 2}.
\]

c. Explain why this does not contradict the MVT.
0460-6. Show that $3x + \cos(2x) = 100$ has exactly one real solution.

0460-7. Let $c$ be any constant. Show that $x^3 + x + c = 0$ has at most one real solution on $\mathbb{R}$. 
At noon on some day, a certain car is at the 200 mile marker on some road. The speed limit on the road is 55 mph. A driver drives the car for seven hours, obeying the speed limit.

Let \( f(t) \) denote the position of the car \( t \) hours after noon; then

\[
\begin{align*}
    f(0) &= 200 \\
    \text{and} \\
    \forall t \in [0, 7], \quad f'(t) &\leq 55.
\end{align*}
\]

With these constraints, what is the largest possible value for \( f(7) \)?