CALCULUS
Derivatives and rates of change
NEW
0270-1. Let $C$ be the curve $y = -x^2 + 2x + 3$. Let $L$ be the tangent line to $C$ at the point $(0, 3)$.

a. Find the slope of $L$, by computing a limit of slopes of secant lines.

b. Find an equation of $L$.

c. Graph $C$ and $L$ in the rectangle $-2 \leq x \leq 4, \quad -2 \leq y \leq 5$.

d. Graph $C$ and $L$ in the rectangle $-1 \leq x \leq 1, \quad 1 \leq y \leq 5$.

e. Graph $C$ and $L$ in the rectangle $-0.1 \leq x \leq 0.1, \quad 2.8 \leq y \leq 3.2$.

In c, d and e, note that, as you “zoom in”, the tangent line looks more and more like the curve.
Let $C$ be the curve $y = -x^2 + 2x + 3$. Let $L$ be the tangent line to $C$ at the point $(0, 3)$.

a. Find the slope of $L$ by...

**ANSWER:** a.

$$
\lim_{h \to 0} \frac{[-(0 + h)^2 + 2 \cdot (0 + h) + 3] - [-0^2 + 2 \cdot 0 + 3]}{h}
$$

$$
= - \left[ \lim_{h \to 0} \frac{(0 + h)^2 - 0^2}{h} \right] + 2 \left[ \lim_{h \to 0} \frac{(0 + h) - 0}{h} \right]
$$

$$
= - \left[ \lim_{h \to 0} \frac{h^2}{h} \right] + 2 \left[ \lim_{h \to 0} \frac{h}{h} \right]
$$
Let $C$ be the curve $y = -x^2 + 2x + 3$. Let $L$ be the tangent line to $C$ at the point $(0, 3)$.

a. Find the slope of $L$.

**ANSWER:** a.

\[
\begin{align*}
&= - \left[ \lim_{h \to 0} \frac{h^2}{h} \right] + 2 \left[ \lim_{h \to 0} \frac{1}{h} \right] \\
&= - \left[ \lim_{h \to 0} h \right] + 2 \left[ \lim_{h \to 0} 1 \right] \\
&= -0 + 2 \cdot 1 = 2
\end{align*}
\]
Let $C$ be the curve $y = -x^2 + 2x + 3$. Let $L$ be the tangent line to $C$ at the point $(0, 3)$.

b. Find an equation of $L$.

**ANSWER:** b. Slope of $L$: 2 (from Part a) a point on $L$: $(0, 3)$

equations of $L$: $y - 3 = 2(x - 0)$

or $y = 3 + 2(x - 0)$

or $y = 2x + 3$
Let $C$ be the curve $y = -x^2 + 2x + 3$. Let $L$ be the tangent line to $C$ at the point $(0,3)$.

c. Graph $C$ and $L$ in the rectangle $-2 \leq x \leq 4, \quad -2 \leq y \leq 5.$

**ANSWER:** c. equation of $L$: $y = 2x + 3$
Let $C$ be the curve $y = -x^2 + 2x + 3$. Let $L$ be the tangent line to $C$ at the point $(0, 3)$. Graph $C$ and $L$ in the rectangle $-1 \leq x \leq 1$, $1 \leq y \leq 5$. **ANSWER:** d. equation of $L$: $y = 2x + 3$
Let $C$ be the curve $y = -x^2 + 2x + 3$. Let $L$ be the tangent line to $C$ at the point $(0,3)$.

e. Graph $C$ and $L$ in the rectangle $-0.1 \leq x \leq 0.1, \quad 2.8 \leq y \leq 3.2$.

**ANSWER:** e. equation of $L$: $y = 2x + 3$
0270-2. a. Compute \( \lim_{h \to 0} \frac{\sqrt{4 + 3h} - 2}{h} \).

b. Find the slope of the secant line to \( y = \sqrt{3x + 1} \) through the points \( (1, 2) \) and \( (1 + h, \sqrt{4 + 3h}) \).

c. Find an equation of the tangent line to \( y = \sqrt{3x + 1} \) at the point \( (1, 2) \).
Compute \( \lim_{{h \to 0}} \frac{\sqrt{4 + 3h} - 2}{h} \). 

**ANSWER:** a.

\[
\lim_{{h \to 0}} \frac{\sqrt{4 + 3h} - 2}{h} = \lim_{{h \to 0}} \left[ \frac{\sqrt{4 + 3h} - 2}{h} \right] \left[ \frac{\sqrt{4 + 3h} + 2}{\sqrt{4 + 3h} + 2} \right] \\
= \lim_{{h \to 0}} \frac{(\sqrt{4 + 3h})^2 - 2^2}{h(\sqrt{4 + 3h} + 2)} = \lim_{{h \to 0}} \frac{(4 + 3h) - 4}{h(\sqrt{4 + 3h} + 2)} \\
= \lim_{{h \to 0}} \frac{3}{\sqrt{4 + 3h} + 2} = \frac{3}{\sqrt{4 + 0} + 2} \\
= \frac{3}{2 + 2} = \frac{3}{4}
\]
0270-2. a. Compute \( \lim_{{h \to 0}} \frac{\sqrt{4 + 3h} - 2}{h} \).

b. Find the slope of the secant line
to \( y = \sqrt{3x} + 1 \) through the points
\((1, 2)\) and \((1 + h, \sqrt{4 + 3h})\).

ANS: b. \[
\text{rise} = \sqrt{4 + 3h} - 2 \\
\text{run} = (1 + h) - 1 = h \\
\text{slope} = \frac{\sqrt{4 + 3h} - 2}{h}
\]

c. Find an equation of the tangent line
to \( y = \sqrt{3x} + 1 \) at the point \((1, 2)\).

ANS: c. \[
\text{slope} = \lim_{{h \to 0}} \frac{\sqrt{4 + 3h} - 2}{h} = \frac{3}{4}
\]

equation: \( y = 2 + \frac{3}{4}(x - 1) \)
A particle moves on a number line. Its position at any time \( t \) is \( \sqrt{3t + 1} \).

a. Find the average velocity between time \( t = 1 \) and time \( t = 1 + h \).

b. Find the instantaneous velocity at time \( t = 1 \).

**ANSWER:**

a. 
change in position \( = \sqrt{3(1 + h) + 1} - \sqrt{3 \cdot 1 + 1} \) 
\( = \sqrt{4 + 3h} - 2 \) 
change in time \( = (1 + h) - 1 = h \) 
average velocity \( = \frac{\sqrt{4 + 3h} - 2}{h} \)

b. 
instantaneous velocity \( = \lim_{h \to 0} \frac{\sqrt{4 + 3h} - 2}{h} \) 
\( = \frac{2}{3} \)
A heavy object is taken to the top of a building 200 feet high. At time $t = 0$, it is thrown upward at 10 feet/second. We engage the services of two Nobel prize-winning physicists who confer (i.e., yell and scream at one another). After several hours of scholarly study, followed by minor medical treatment for blunt trauma, lacerations and contusions, they hold a joint press conference, and inform their public that, $t$ seconds after release, the object will be located

$$200 + 10t - 16t^2$$

feet above the ground. Based on this, find the velocity of the object 0.4 seconds after release. Give your answer in feet per second.
... $t$ seconds after release, the object will be located
\[200 + 10t - 16t^2\] feet above the ground. Based on this, find the velocity of the object 0.4 seconds after release. Give your answer in feet per second.

**ANSWER:** [avg velocity between times 0.4 and 0.4 + $h$] =
\[
\frac{[200 + 10(0.4 + h) - 16(0.4 + h)^2] - [200 + 10(0.4) - 16(0.4)^2]}{h}
\]
\[
= 10 \left[ \frac{(0.4 + h) - 0.4}{h} \right] - 16 \left[ \frac{(0.4 + h)^2 - (0.4)^2}{h} \right]
\]
\[
= 10 - 16 \left[ \frac{(0.4)^2 + 2(0.4)h + h^2 - (0.4)^2}{h} \right]
\]
\[
= 10 - 16 \left[ \frac{2(0.4)h + h^2}{h} \right]
\]
... $t$ seconds after release, the object will be located

$$200 + 10t - 16t^2$$

feet above the ground. Based on this, find the velocity of the object 0.4 seconds after release. Give your answer in feet per second.

**ANSWER:**

[avg velocity between times 0.4 and 0.4 + $h$]

$$= 10 - 16 \left[ \frac{2(0.4)h + h^2}{h} \right]$$

$h \neq 0$

$$= 10 - 16 [2(0.4) + h]$$

$$= 10 - 16 [0.8 + h]$$

$$= 10 - 12.8 - 16h = -2.8 - 16h$$

[instantaneous velocity at time 0.4]

$$= \lim_{h \to 0} (-2.8 - 16h) = -2.8$$

That is, it’s traveling downward at 2.8 ft/sec.
Order these numbers, from smallest to largest:

\[ f'(-2), f'(-1), f'(0), f'(1), f'(2), f'(3) \]

Note that we are asking about \( f' \), not \( f \).

\textbf{ANSWER:}

\[ f'(-2) < f'(-1) < f'(0) < f'(1) < f'(2) < f'(3) \]
Let \( f(x) = \frac{7x + 5}{2x + 1} \).

a. Compute \( f'(2) \).

b. Compute \( f'(3) \).

c. Compute \( f'(4) \).

d. Compute \( f'(a) \), for an arbitrary number \( a \).
0270-6. Let \( f(x) = \frac{7x + 5}{2x + 1} \).

d. Compute \( f'(a) \), for an arbitrary number \( a \).

**ANSWER:** d. \[
\frac{1}{h \to 0} \lim h \left[ \frac{7(a + h) + 5}{2(a + h) + 1} - \frac{7a + 5}{2a + 1} \right] = \\
\frac{7a + 7h + 5}{2a + 2h + 1} - \frac{7a + 5}{2a + 1} = \\
\frac{[7h + 7a + 5][2a + 1] - [2h + 2a + 1][7a + 5]}{[2a + 2h + 1][2a + 1]} = \\
\frac{[7h][2a + 1] - [2h][7a + 5]}{[2a + 2h + 1][2a + 1]}.
\]
Let \( f(x) = \frac{7x + 5}{2x + 1} \).

d. Compute \( f'(a) \), for an arbitrary number \( a \).

\[
\text{ANSWER: d. } f'(a) = \lim_{h \to 0} \frac{1}{h} \left[ \frac{7(a + h) + 5}{2(a + h) + 1} - \frac{7a + 5}{2a + 1} \right]
\]

\[
= \lim_{h \to 0} \frac{[7(a + h) + 5][2a + 1] - [2h][7a + 5]}{[2a + 2h + 1][2a + 1]}
\]

\[
= \lim_{h \to 0} \frac{7h - 10h}{[2a + 2h + 1][2a + 1]}
\]

\[
= \lim_{h \to 0} \frac{-3h}{[2a + 2h + 1][2a + 1]}
\]

\[
= \frac{-3}{(2a + 1)^2}
\]
d. Compute \( f'(a) \), for an arbitrary number \( a \).

**ANSWER:**

\[
f'(a) = \lim_{h \to 0} \frac{1}{h} \left[ \frac{7(a + h) + 5}{2(a + h) + 1} - \frac{7a + 5}{2a + 1} \right] = \frac{-3}{[2a + 1]^2}
\]
Let \( f(x) = \frac{7x + 5}{2x + 1} \).

**d. Compute** \( f'(a) \), **for an arbitrary number** \( a \).

**ANSWER:**

d. \( f'(a) = \frac{-3}{(2a + 1)^2} \)

**a. Compute** \( f'(2) \).

**b. Compute** \( f'(3) \).

**c. Compute** \( f'(4) \).

**ANSWER:**

a. \( f'(2) = \frac{-3}{[2 \cdot 2 + 1]^2} = -\frac{3}{25} \)

b. \( f'(3) = \frac{-3}{[2 \cdot 3 + 1]^2} = -\frac{3}{49} \)

c. \( f'(4) = \frac{-3}{[2 \cdot 4 + 1]^2} = -\frac{3}{81} = -\frac{1}{27} \)
Find a function $f$ and a number $a$ s.t.

$$f'(a) = \lim_{h \to 0} \frac{[\ln(3 + h)] - [\ln(3)]}{h}.$$  

**ANSWER:**

$$f'(a) = \lim_{h \to 0} \frac{[f(a + h)] - [f(a)]}{h}$$

$$f = \ln, \quad a = 3$$