CALCULUS
The Mean Value Theorem
NEW
Let \( f(x) = x^2 - 2x - 3 \).

a. Check that \( f \) satisfies the conditions of Rolle’s Theorem on the interval \([-1, 3]\). That is, check

(i) that \( f \) is continuous on \([-1, 3]\),
(ii) that \( f \) is differentiable on \((-1, 3)\) and (iii) that \( f(-1) = f(3) \).

b. Find all solutions to the equation in the conclusion of Rolle’s Th’m for \( f \) on \([-1, 3]\). That is, find all \( c \in (-1, 3) \) s.t. \( f'(c) = 0 \).
0460-1. Let \( f(x) = x^2 - 2x - 3 \).

a. Check that \( f \) satisfies the conditions of Rolle’s Theorem on the interval \([-1, 3]\). That is, check

(i) that \( f \) is continuous on \([-1, 3]\),

(ii) that \( f \) is differentiable on \((-1, 3)\) and

(iii) that \( f(-1) = f(3) \).

\[ f(-1) = 1 + 2 - 3 = 0 \]
\[ f(3) = 9 - 6 - 3 = 0 \]
0460-1. Let $f(x) = x^2 - 2x - 3$.

b. Find all solutions to the equation in the conclusion of Rolle’s Th’m for $f$ on $[-1, 3]$. That is, find all $c \in (-1, 3)$ s.t. $f'(c) = 0$.

**ANSWER:** $f'(x) = 2x - 2$

$$f'(c) = 0 \text{ iff } 2c - 2 = 0$$

$$\text{iff } c = 1$$
0460-2. Let \( f(x) = x^2 + x - 3 \).

a. Check that \( f \) satisfies the conditions of the MVT on the interval \([-1, 3]\). That is, check

(i) that \( f \) is continuous on \([-1, 3]\) and (ii) that \( f \) is differentiable on \((-1, 3)\).

b. Find all solutions to the equation in the conclusion of the MVT for \( f \) on \([-1, 3]\). That is, find all \( c \in (-1, 3) \) s.t.

\[
f'(c) = \frac{[f(3)] - [f(-1)]}{3 - (-1)}.
\]
0460-2. Let $f(x) = x^2 + x - 3$.

a. Check that $f$ satisfies the conditions of the MVT on the interval $[-1, 3]$. That is, check

(i) that $f$ is continuous on $[-1, 3]$ and (ii) that $f$ is differentiable on $(-1, 3)$.

ANSWER:

(i) $f$ is a poly., so $f$ is continuous on $\mathbb{R}$, so $f$ is continuous on $[-1, 3]$.

(ii) $f$ is a poly., so $f$ is differentiable on $\mathbb{R}$, so $f$ is differentiable on $(-1, 3)$. 
0460-2. Let \( f(x) = x^2 + x - 3 \).

b. Find all solutions to the equation in the conclusion of the MVT for \( f \) on \([-1, 3]\). That is, find all \( c \in (-1, 3) \) s.t.

\[
f'(c) = \frac{[f(3)] - [f(-1)]}{3 - (-1)}.
\]

**ANSWER:** \( f'(x) = 2x + 1 \)

\[
\frac{[f(3)] - [f(-1)]}{3 - (-1)} = \frac{[9 + 3 - 3] - [1 - 1 - 3]}{4} = \frac{9 - (-3)}{4} = 3
\]

\[
f'(c) = \frac{[f(3)] - [f(-1)]}{3 - (-1)} \text{ iff } 2c + 1 = 3 \text{ iff } c = 1 \]
0460-3. Let $f(x) = 7 + |x - 5|$.

a. Show that $f$ is continuous on $[2, 8]$.

b. Show that $f(2) = f(8)$.

c. Show that the conclusion of Rolle’s Th’m, for $f$ on $[2, 8]$, fails. That is, show that there is no $c \in (2, 8)$ s.t. $f'(c) = 0$.

d. Explain why this does not contradict Rolle’s Theorem.
Let \( f(x) = 7 + |x - 5| \).

**a. Show** that \( f \) is continuous on \([2, 8]\).

**ANSWER:**

\( x - 5 \) is a polynomial in \( x \), so \( x - 5 \) is continuous in \( x \).

\( | \cdot | \) is continuous.

A composition of a continuous function and a continuous expression is continuous.

Then \( |x - 5| \) is continuous in \( x \).

\( 7 \) is a constant, so \( 7 \) is continuous in \( x \).

A sum of continuous expressions is continuous.

Then \( 7 + |x - 5| \) is continuous in \( x \).

That is, \( f \) is continuous. The domain of \( f \) is \( \mathbb{R} \).

So \( f \) is continuous on \( \mathbb{R} \), hence on \([2, 8]\).
Let \( f(x) = 7 + |x - 5| \).

b. Show that \( f(2) = f(8) \).

**ANSWER:** b.

\[
\begin{align*}
    f(2) & = 7 + |2 - 5| = 7 + |-3| = 7 + 3 = 10 \\
    f(8) & = 7 + |8 - 5| = 7 + |3| = 7 + 3 = 10
\end{align*}
\]
0460-3. Let \( f(x) = 7 + |x - 5| \).

c. Show that the conclusion of Rolle’s Th’m, for \( f \) on \([2, 8]\), fails. That is, show that there is no \( c \in (2, 8) \) s.t. \( f'(c) = 0 \).

**ANSWER:** c.

\[
f(x) = \begin{cases} 
7 - (x - 5), & \text{if } x \leq 5 \\
7 + (x - 5), & \text{if } x \geq 5
\end{cases}
\]

\[
f'(x) = \begin{cases} 
-1, & \text{if } x < 5 \\
1, & \text{if } x > 5
\end{cases}
\]

Therefore there is no \( c \in (2, 8) \) s.t. \( f'(c) = 0 \).
Let \( f(x) = 7 + |x - 5| \).

d. Explain why this does not contradict Rolle’s Theorem.

\[
f'(x) = \begin{cases} 
-1, & \text{if } x < 5 \\
1, & \text{if } x > 5 
\end{cases}
\]

\( f \) is not differentiable at 5, so \( f \) is not differentiable on \((2, 8)\).
Let \( f(x) = x + |x - 5| \).

a. Show that \( f \) is continuous on \([2, 8]\).

b. Show that the conclusion of the MVT, for \( f \) on \([2, 8]\), fails. That is, show that there is no \( c \in (2, 8) \) s.t.

\[
f'(c) = \frac{[f(8)] - [f(2)]}{8 - 2}.
\]

c. Explain why this does not contradict the MVT.
0460-4. Let \( f(x) = x + |x - 5| \).

a. Show that \( f \) is continuous on \([2, 8]\).

**ANSWER:** a. \( x - 5 \) is a polynomial in \( x \), so \( x - 5 \) is continuous in \( x \).

\( |\bullet| \) is continuous.

A composition of a continuous function and a continuous expression is continuous.

Then \( |x - 5| \) is continuous in \( x \).

\( x \) is a polynomial in \( x \), so \( x \) is continuous in \( x \).

A sum of continuous expressions is continuous.

Then \( x + |x - 5| \) is continuous in \( x \).

That is, \( f \) is continuous. The domain of \( f \) is \( \mathbb{R} \).

So \( f \) is continuous on \( \mathbb{R} \), hence on \([2, 8]\).
0460-4. Let \( f(x) = x + |x - 5| \).

b. Show that the conclusion of the MVT, for \( f \) on \([2, 8]\), fails. That is, show that there is no \( c \in (2, 8) \) s.t.

\[
f'(c) = \frac{[f(8)] - [f(2)]}{8 - 2}.
\]

**ANSWER:** b.

\[
\frac{[f(8)] - [f(2)]}{8 - 2} = \frac{[8 + |3|] - [2 + |1 - 3|]}{6}
\]

\[
= \frac{11 - 5}{6} = \frac{6}{6} = 1
\]
b. Show that the conclusion of the MVT, for \( f \) on \([2,8]\), fails. That is, **show** that there is no \( c \in (2,8) \) s.t.

\[
f'(c) = \frac{[f(8)] - [f(2)]}{8 - 2}.
\]

**ANSWER:**

\[
\frac{[f(8)] - [f(2)]}{8 - 2} = 1
\]

\[
f(x) = \begin{cases} 
    x - (x - 5), & \text{if } x \leq 5 \\
    x + (x - 5), & \text{if } x \geq 5
\end{cases}
\]

\[
f'(x) = \begin{cases} 
    1 - 1, & \text{if } x < 5 \\
    1 + 1, & \text{if } x > 5
\end{cases}
\]

Therefore there is no \( c \in (2,8) \) s.t. \( f'(c) = 1 \).
0460-4. Let \( f(x) = x + |x - 5| \).

c. Explain why this does not contradict the MVT.

**ANSWER:** c.

\[
f'(x) = \begin{cases} 
1 - 1, & \text{if } x < 5 \\
1 + 1, & \text{if } x > 5 
\end{cases}
\]

\( f \) is not differentiable at 5, so \( f \) is not differentiable on \((2, 8)\).
Let \( f(x) = \begin{cases} 
100, & \text{if } x = 2 \\
3x - 5, & \text{if } 2 < x < 8 \\
40, & \text{if } x = 8.
\end{cases} \)

a. Show that \( f \) is differentiable on \((2, 8)\).

b. Show that the conclusion of the MVT, for \( f \) on \([2, 8]\), fails. That is, show that there is no \( c \in (2, 8) \) s.t.

\[
f'(c) = \frac{[f(8)] - [f(2)]}{8 - 2}.
\]

c. Explain why this does not contradict the MVT.
Let \( f(x) = \begin{cases} 
100, & \text{if } x = 2 \\
3x - 5, & \text{if } 2 < x < 8 \\
40, & \text{if } x = 8.
\end{cases} \)

a. Show that \( f \) is differentiable on \((2, 8)\).

**ANSWER:**

\[ \forall x \in (2, 8), \quad f(x) = 3x - 5, \]

so,

\[ \forall x \in (2, 8), \quad f'(x) = 3, \]

so \( f \) is differentiable at \( x \).

Then \( f \) is differentiable on \((2, 8)\).
Let \( f(x) = \begin{cases} 
100, & \text{if } x = 2 \\
3x - 5, & \text{if } 2 < x < 8 \\
40, & \text{if } x = 8.
\end{cases}\)

b. Show that the conclusion of the MVT, for \( f \) on \([2,8]\), fails. That is, show that there is no \( c \in (2,8) \) s.t.

\[
\begin{align*}
f'(c) &= \frac{[f(8)] - [f(2)]}{8 - 2} \\
&= \frac{40 - 100}{6} = -10.
\end{align*}
\]

**ANSWER:**

\[
\begin{align*}
\frac{[f(8)] - [f(2)]}{8 - 2} &= \frac{40 - 100}{6} = -10 \\
\forall x \in (2,8), \quad f(x) &= 3x - 5, \\
\text{so,} \quad \forall x \in (2,8), \quad f'(x) &= 3, \\
\text{so,} \quad \forall c \in (2,8), \quad f'(c) &= 3 > -10.
\end{align*}
\]

Therefore there is no \( c \in (2,8) \) s.t. \( f'(c) = -10 \).
Let $f(x) = \begin{cases} 
100, & \text{if } x = 2 \\
3x - 5, & \text{if } 2 < x < 8 \\
40, & \text{if } x = 8. 
\end{cases}$

c. Explain why this does not contradict the MVT.

**ANSWER:**

\[
\lim_{x \to 8^-} [f(x)] = \lim_{x \to 8^-} [3x - 5] = 19
\]

\[
\lim_{x \to 2^+} [f(x)] = \lim_{x \to 2^+} [3x - 5] = 1
\]

\[
f(8) = 40
\]

\[
f(2) = 100
\]

$f$ is neither continuous from the right at 2 nor continuous from the left at 8, so $f$ is not continuous on $[2, 8]$. $\blacksquare$
0460-6. Show that \(3x + \cos(2x) = 100\) has exactly one real solution.

**ANS:** Let \(f(x) = 3x + \cos(2x)\).

\[
f(-1000) = -3000 + \cos(-2000) \leq -3000 + 1 < 100
\]

\[
f(1000) = 3000 + \cos(2000) \geq 3000 - 1 > 100
\]

Therefore, by the Intermediate Value Theorem, \(f(x) = 100\) has at least one real solution.

Want: \(f(x) = 100\) does not have two real solutions.
0460-6. Show that $3x + \cos(2x) = 100$ has exactly one real solution.

ANS: Let $f(x) = 3x + \cos(2x)$.

$f(x) = 100$ has at least one real solution.

Want: $f(x) = 100$ does not have two real sol’ns.

Suppose $s < u$ and

$$f(s) = 100 = f(u).$$

Want: Contradiction.

By Rolle’s Th’m, fix $t \in (s, u)$ s.t. $f’(t) = 0$.

$$f’(x) = 3 - 2\sin(2x)$$

Then $0 = f’(t) = 3 - 2\sin(2t) \geq 3 - 2 = 1$. Contradiction.
Let $c$ be any constant.

Show that $x^3 + x + c = 0$ has at most one real solution on $\mathbb{R}$.

**ANS:** Let $f(x) = x^3 + x + c$.

Suppose $s < u$ and

$$f(s) = 0 = f(u).$$

Want: Contradiction.

By Rolle’s Th’m, fix $t \in (s, u)$ s.t. $f'(t) = 0$.

$$f'(x) = 3x^2 + 1$$

Then $3t^2 + 1 = 0$.

However, $t^2 \geq 0$,

so $3t^2 + 1 \geq 1 > 0$. **Contradiction.**
At noon on some day, a certain car is at the 200 mile marker on some road. The speed limit on the road is 55 mph. A driver drives the car for seven hours, obeying the speed limit.

Let $f(t)$ denote the position of the car $t$ hours after noon; then

$$f(0) = 200$$

and

$$\forall t \in [0, 7], \quad f'(t) \leq 55.$$

With these constraints, what is the largest possible value for $f(7)$?
With these constraints, what is the largest possible value for $f(7)$?

**ANSWER:**

In the special case where $f(t) = 200 + 55t$,
we have $f(0) = 200$,
we have $\forall t \in [0, 7], \ f'(t) \leq 55$
and we have $f(7) = 585$.

Then 585 is a possible value for $f(7)$.

We will show that 585 is the largest possible value for $f(7)$. 
$f(0) = 200$  
and  
$\forall t \in [0, 7], \ f'(t) \leq 55$.  

With these constraints, what is the largest possible value for $f(7)$?

**ANSWER:** 585 is a possible value for $f(7)$.

We will show that 585 is the largest possible value for $f(7)$.

Suppose $f$ satisfies the two conditions above, i.e., suppose  
$f(0) = 200$  
and  
$\forall t \in [0, 7], \ f'(t) \leq 55$.  

**Want:** $f(7) \leq 585$
0460-8.

New

\[ f(0) = 200 \]

and

\[ \forall t \in [0, 7], \quad f'(t) \leq 55. \]

With these constraints, what is the largest possible value for \( f(7) \)?

**Answer:** Want: \( f(7) \leq 585 \)

By the MVT, fix \( c \in (0, 7) \) s.t.

\[
\frac{[f(7)] - [f(0)]}{7 - 0} = f'(c).
\]

Then

\[
\frac{[f(7)] - [200]}{7} = f'(c) \leq 55.
\]

Then

\[ [f(7)] - [200] \leq 7(55) = 385. \]

Then

\[ f(7) \leq 385 + 200 = 585. \]