CALCULUS
Definite integration and Riemann sum problems
NEW
0590-1. Let \( f(x) = 2 + 2x^2 \).

a. Compute \( L_4 S^2_{2} f \).
   Sketch \( f \) over \([-2, 2]\) and add, into your sketch, the four rectangles represented by \( L_4 S^2_{2} f \).

b. Compute \( M_4 S^2_{2} f \).
   Sketch \( f \) over \([-2, 2]\) and add, into your sketch, the four rectangles represented by \( M_4 S^2_{2} f \).

c. Compute \( R_4 S^2_{2} f \).
   Sketch \( f \) over \([-2, 2]\) and add, into your sketch, the four rectangles represented by \( R_4 S^2_{2} f \).
0590-1. Let \( f(x) = 2 + 2x^2 \).

a. Compute \( L_4 S_2^2 f \).

Sketch \( f \) over \([-2, 2]\) and add, into your sketch, the four rectangles represented by \( L_4 S_2^2 f \).

ANS:

\[
L_4 S_2^2 f = 10 + 4 + 2 + 4 = 20
\]
0590-1. Let \( f(x) = 2 + 2x^2 \).

b. Compute \( M_4 S^2_{-2} f \).

Sketch \( f \) over \([-2, 2]\) and add, into your sketch, the four rectangles represented by \( M_4 S^2_{-2} f \).

ANS: \[ M_4 S^2_{-2} f = 6.5 + 2.5 + 2.5 + 6.5 = 18 \]
Let \( f(x) = 2 + 2x^2 \).

**c. Compute** \( R_4 S^2_{-2} f \).

Sketch \( f \) over \([-2, 2]\) and add, into your sketch, the four rectangles represented by \( R_4 S^2_{-2} f \).

\[
R_4 S^2_{-2} f = 4 + 2 + 4 + 10 = 20
\]
Let \( f(x) = e^x + 6 \).

a. Compute \( L_2 S_0^8 f \) to three decimal places.

b. Compute \( M_2 S_0^8 f \) to three decimal places.

c. Compute \( R_2 S_0^8 f \) to three decimal places.

**ANSWER:**

a. \( L_2 S_0^8 f = 4 \left[ (e^0 + 6) + (e^4 + 6) \right] \)
   \[ = 4[13 + e^4] \approx 270.393 \]

b. \( M_2 S_0^8 f = 4 \left[ (e^2 + 6) + (e^6 + 6) \right] \)
   \[ = 4[12 + e^2 + e^6] \approx 1691.271 \]

c. \( R_2 S_0^8 f = 4 \left[ (e^4 + 6) + (e^8 + 6) \right] \)
   \[ = 4[12 + e^4 + e^8] \approx 12910.225 \]
Let $f(x) = \sin^2 x$.

a. Compute $L_3 S^2 \pi f$ to three decimal places.

b. Compute $M_3 S^2 \pi f$ to three decimal places.

c. Compute $R_3 S^2 \pi f$ to three decimal places.
0590-3. Let $f(x) = \sin^2 x$.

a. Compute $L_3 S_{0}^{2\pi} f$ to three decimal places.

**ANSWER:**

$$a. \quad L_3 S_{0}^{2\pi} f = \frac{2\pi}{3} \left[ (\sin^2 (0)) + (\sin^2 \left( \frac{2\pi}{3} \right)) + (\sin^2 \left( \frac{4\pi}{3} \right)) \right]$$

$$= \frac{2\pi}{3} \left[ 0 + \frac{3}{4} + \frac{3}{4} \right]$$

$$= \pi \approx 3.142$$
Let \( f(x) = \sin^2 x \).

b. Compute \( M_3 S_{0}^{2\pi} f \) to three decimal places.

**ANSWER:**

\[
b. M_3 S_{0}^{2\pi} f = \frac{2\pi}{3} \left[ \left( \sin^2 \left( \frac{\pi}{3} \right) \right) + \left( \sin^2 \left( \frac{3\pi}{3} \right) \right) + \left( \sin^2 \left( \frac{5\pi}{3} \right) \right) \right]\]
\[
= \frac{2\pi}{3} \left[ \frac{3}{4} + 0 + \frac{3}{4} \right]
\]
\[
= \pi \approx 3.142
\]
c. Compute \( R_3 S_{0}^{2\pi} f \) to three decimal places.

\[ c. M_3 S_{0}^{2\pi} f = \frac{2\pi}{3} \left[ (\sin^2 \left( \frac{2\pi}{3} \right)) + (\sin^2 \left( \frac{4\pi}{3} \right)) + (\sin^2 \left( \frac{6\pi}{3} \right)) \right] = \frac{2\pi}{3} \left[ \frac{3}{4} + \frac{3}{4} + 0 \right] = \pi \approx 3.142 \]
A car’s acceleration is positive from time 0 to time 24 seconds, and its velocity at various times is given in the table below.

<table>
<thead>
<tr>
<th>time (secs)</th>
<th>0</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>velocity (ft/sec)</td>
<td>0</td>
<td>40</td>
<td>56</td>
<td>68</td>
<td>77</td>
<td>81</td>
<td>83</td>
</tr>
</tbody>
</table>

Find upper and lower estimates for the distance traveled by the car over these 24 seconds.

**ANSWER:**

upper estimate = \([4][40+56+68+77+81+83]\)

= \([4][405]\) = 1620 ft

lower estimate = \([4][0+40+56+68+77+81]\)

= \([4][322]\) = 1288 ft
The graph of a function $f$ appears below.

Estimate $\int_{0}^{10} f(x) \, dx$ by computing

(a) $L_{5} S_{0}^{10} f$,  \hspace{1cm} (b) $M_{5} S_{0}^{10} f$

and (c) $R_{5} S_{0}^{10} f$.  \hspace{1cm} 16
The gph of a function $f$ appears below.

**ANSWER:**

(a) $L_5 S_0^{10} f = 2[(-1) + 4 + 5 + 1 + 0]$

$= 18$
The gph of a function $f$ appears below.

**ANSWER:**

(b) $M_5S_0^{10}f = 2[2 + 5 + 3 + 0 + 1] = 22$
The gph of a function $f$ appears below.

**ANSWER:**

(c) $R_5S_0^{10} f = 2[4 + 5 + 1 + 0 + 2] = 22$
Express the area under \( y = e^{-x^2/5} \) from \( x = -2 \) to \( x = 0 \) as a limit of midpoint Riemann sums. (Don’t evaluate the limit.)

ANSWER:

\[
\lim_{n \to \infty} \frac{2}{n} \left[ \sum_{j=1}^{n} e^{-[-2+j(2/n)-(1/n)]^2/5} \right]
\]
Express the area under \( y = \sqrt{x^3 + x + 5} \) from \( x = 1 \) to \( x = 4 \) as a limit of left endpoint Riemann sums. (Don’t evaluate the limit.)

\[
\text{ANSWER:} \\
\lim_{n \to \infty} \frac{3}{n} \left[ \sum_{j=0}^{n-1} \sqrt{(1 + j(3/n))^3 + (1 + j(3/n)) + 5} \right]
\]
Express the area under \( y = \cos \left( x^4 - x \right) \) from \( x = 0 \) to \( x = 5 \) as a limit of right endpt Riemann sums. (Don’t evaluate the limit.)

\[
\lim_{n \to \infty} \frac{5}{n} \left[ \sum_{j=1}^{n} \cos \left( (0 + j(5/n))^4 - (0 + j(5/n)) \right) \right]
\]
0590-9. Express \( \int_{2}^{4} \frac{e^{-x^2}}{\sqrt{\pi}} \, dx \) as a limit of midpoint Riemann sums. (Don’t evaluate the limit.)

**ANSWER:**

\[
\lim_{n \to \infty} \frac{2}{n} \left[ \sum_{j=1}^{n} \frac{e^{-\left[2+j\left(2/n\right)-(1/n)\right]^2}}{\sqrt{\pi}} \right]
\]
0590-10. Let \( f(x) = 2x^3 \).

a. Write \( R_n S_0^2 f \) as a rational expression in \( n \) (i.e., as one polynomial in \( n \) divided by another).

b. Compute \( \lim_{n \to \infty} R_n S_0^2 f \).
0590-10. Let \( f(x) = 2x^3 \).

a. Write \( R_n S_0^2 f \) as a rational expression in \( n \) (i.e., as one polynomial in \( n \) divided by another).

ANSWER: a.

\[
R_n S_0^2 f = \frac{2}{n} \left[ \sum_{j=1}^{n} 2 \left(0 + j \left(\frac{2}{n}\right)\right)^3 \right]
\]

\[
= \frac{2}{n} \left[ \sum_{j=1}^{n} j^3 \left(\frac{16}{n^3}\right) \right]
\]

\[
= \frac{32}{n^4} \left[ \sum_{j=1}^{n} j^3 \right]
\]
0590-10. Let $f(x) = 2x^3$.

a. Write $R_n S_0^2 f$ as a rational expression in $n$ (i.e., as one polynomial in $n$ divided by another).

ANSWER: a.

$$R_n S_0^2 f = \frac{32}{n^4} \left[ \sum_{j=1}^{n} j^3 \right]$$

$$= \frac{32}{n^4} \left[ \frac{n^2(n + 1)^2}{4} \right]$$

$$= \frac{8(n + 1)^2}{n^2}$$
0590-10. Let $f(x) = 2x^3$.

a. Write $R_n S_0^2 f$ as a rational expression in $n$ (i.e., as one polynomial in $n$ divided by another).

**ANSWER:** a.

$$R_n S_0^2 f = \frac{8(n+1)^2}{n^2}$$

$$= \frac{8(n^2 + 2n + 1)}{n^2}$$

$$= \frac{8n^2 + 16n + 8}{n^2}$$
0590-10. Let \( f(x) = 2x^3 \).

b. Compute \( \lim_{n \to \infty} R_n S_0^2 f \).

\[
a. \quad R_n S_0^2 f = \frac{8n^2 + 16n + 8}{n^2}
\]

ANSWER: b.

\[
\lim_{n \to \infty} R_n S_0^2 f = \lim_{n \to \infty} \frac{8n^2 + 16n + 8}{n^2} = 8
\]
0590-11. The limit
\[
\lim_{n \to \infty} \left[ \frac{3}{n} \sum_{j=1}^{n} \left( \cos^2 (-4 + j(3/n)) \right) \right]
\]
represents the area under \( y = f(x) \) from \( x = a \) to \( x = b \), for some choice of \( f(x) \), \( a \) and \( b \).

a. Find \( f(x) \), \( a \) and \( b \).

b. Express the limit as a definite integral.

a. \( f(x) = \cos^2 x \), \( a = -4 \), \( b = -1 \)

b. \( \int_{-4}^{-1} \cos^2 x \, dx \)
0590-12. The limit

$$\lim_{n \to \infty} \left[ \frac{5}{n} \sum_{j=0}^{n-1} \cos \left( \frac{1}{3 + j(5/n)} \right) \right]$$

represents the area under $y = f(x)$ from $x = a$ to $x = b$, for some choice of $f(x)$, $a$ and $b$.

a. Find $f(x)$, $a$ and $b$.

b. Express the limit as a definite integral.

a. $f(x) = \cos \left( \frac{1}{x} \right)$, $a = 3$, $b = 8$

b. $\int_3^8 \cos \left( \frac{1}{x} \right) \, dx$
Let \( f(x) = 2 - \sqrt{4 - x^2} \).

a. Sketch the graph of \( y = f(x) \).

b. Compute \( \int_{-2}^{2} f(x) \, dx \), by interpreting this integral as an area.

\[
\begin{align*}
\text{b. } 2 \cdot 4 - \frac{\pi \cdot 2^2}{2} &= 8 - 2\pi \\
\end{align*}
\]