CALCULUS
Derivatives and rates of change
NEW
Let \( C \) be the curve \( y = -x^2 + 2x + 3 \).

Let \( L \) be the tangent line to \( C \) at the point \((1, 4)\).

a. Find the slope of \( L \), by computing a limit of slopes of secant lines.

b. Find an equation of \( L \).

c. Graph \( C \) and \( L \) in the rectangle \(-2 \leq x \leq 4, \ -2 \leq y \leq 5\).

d. Graph \( C \) and \( L \) in the rectangle \(0 \leq x \leq 2, \ 1 \leq y \leq 5\).

e. Graph \( C \) and \( L \) in the rectangle \(0.9 \leq x \leq 1.1, \ 3.8 \leq y \leq 4.2\).

In c, d and e, note that, as you “zoom in”, the tangent line looks more and more like the curve.
0270-1. Let \( C \) be the curve \( y = -x^2 + 2x + 3 \). Let \( L \) be the tangent line to \( C \) at the point \((1, 4)\).

\[
\text{a. Find the slope of } L \text{ by } \ldots
\]

**ANSWER:**

\[
\lim_{h \to 0} \frac{[-(1 + h)^2 + 2 \cdot (1 + h) + 3] - [-1^2 + 2 \cdot 1 + 3]}{h}
\]

\[
= - \left[ \lim_{h \to 0} \frac{(1 + h)^2 - 1^2}{h} \right] + 2 \left[ \lim_{h \to 0} \frac{(1 + h) - 1}{h} \right]
\]

\[
= - \left[ \lim_{h \to 0} \frac{(1 + 2h + h^2) - 1}{h} \right] + 2 \left[ \lim_{h \to 0} \frac{(1 + h) - 1}{h} \right]
\]

\[
= - \left[ \lim_{h \to 0} \frac{1}{h} \right] + 2 \left[ \lim_{h \to 0} \frac{1}{h} \right]
\]

\[
= -1 + 2 \cdot 1
\]

\[
= 1
\]
\[ 0 = -2 + 2 \cdot 1 = \]

\[ \left[ \begin{array}{c} 0 \\ \hline 1 \end{array} \right] + 2 \left[ \begin{array}{c} \hline \hline \end{array} \right] - = \]

\[ \left[ \begin{array}{c} \hline \hline \end{array} \right] + \left[ \begin{array}{c} \hline \hline \end{array} \right] - = \]

\[ \left[ \begin{array}{c} \hline \hline \end{array} \right] + 2 \left[ \begin{array}{c} \hline \hline \end{array} \right] - = \]

\[ \text{ANSWER: a. Find the slope of the tangent line at the point (1, 4).} \]

\[ y = x^2 + 2x + 3 \]
0270-1. Let $C$ be the curve $y = -x^2 + 2x + 3$.
Let $L$ be the tangent line to $C$ at the point $(1, 4)$.

b. Find an equation of $L$.

**ANSWER:** b. Slope of $L$: 0 (from Part a) a point on $L$: $(1, 4)$

Equations of $L$: $y - 4 = 0(x - 1)$

or $y = 4 + 0(x - 1)$

or $y = 4$
Let $C$ be the curve $y = -x^2 + 2x + 3$. Let $L$ be the tangent line to $C$ at the point $(1,4)$.

c. Graph $C$ and $L$ in the rectangle $-2 \leq x \leq 4$, $-2 \leq y \leq 5$.

**ANSWER:**
c. equation of $L$: $y = 4$
Let $C$ be the curve $y = -x^2 + 2x + 3$. Let $L$ be the tangent line to $C$ at the point $(1, 4)$.

**d.** Graph $C$ and $L$ in the rectangle $0 \leq x \leq 2, \quad 1 \leq y \leq 5$.

**ANSWER:** d. equation of $L$: $y = 4$
0270-1. Let $C$ be the curve $y = -x^2 + 2x + 3$. Let $L$ be the tangent line to $C$ at the point $(1,4)$.

e. Graph $C$ and $L$ in the rectangle $0.9 \leq x \leq 1.1, \quad 3.8 \leq y \leq 4.2$.

**ANSWER:** e. equation of $L$: $y = 4$
Compute \( \lim_{{h \to 0}} \frac{\sqrt{9 + 2h} - 3}{h} \).

b. Find the slope of the secant line to \( y = \sqrt{2x + 1} \) through the points \((4, 3)\) and \((4 + h, \sqrt{9 + 2h})\).

c. Find an equation of the tangent line to \( y = \sqrt{2x + 1} \) at the point \((4, 3)\).
Compute \( \lim_{{h \to 0}} \frac{\sqrt{9 + 2h} - 3}{h} \).

**ANSWER:**

\[
\lim_{{h \to 0}} \frac{\sqrt{9 + 2h} - 3}{h} = \lim_{{h \to 0}} \left[ \frac{\sqrt{9 + 2h} - 3}{h} \right] \left[ \frac{\sqrt{9 + 2h} + 3}{\sqrt{9 + 2h} + 3} \right] \\
= \lim_{{h \to 0}} \frac{(\sqrt{9 + 2h})^2 - 3^2}{h(\sqrt{9 + 2h} + 3)} \\
= \lim_{{h \to 0}} \frac{9 + 2h - 9}{h(\sqrt{9 + 2h} + 3)} \\
= \lim_{{h \to 0}} \frac{2}{\sqrt{9 + 2h} + 3} \\
= \frac{2}{\sqrt{9 + 0} + 3} \\
= \frac{2}{6} = \frac{1}{3}
\]
0270-2. a. Compute \( \lim_{h \to 0} \frac{\sqrt{9 + 2h} - 3}{h} \).

b. Find the slope of the secant line to \( y = \sqrt{2x + 1} \) through the points \((4, 3)\) and \((4 + h, \sqrt{9 + 2h})\).

\[
\text{rise} = \sqrt{9 + 2h} - 3
\]
\[
\text{run} = (4 + h) - 4 = h
\]
\[
\text{slope} = \frac{\sqrt{9 + 2h} - 3}{h}
\]

c. Find an equation of the tangent line to \( y = \sqrt{2x + 1} \) at the point \((4, 3)\).

\[
\text{ANS: c. slope} = \lim_{h \to 0} \frac{\sqrt{9 + 2h} - 3}{h} = \frac{1}{3}
\]

Equation: \( y = 3 + \frac{1}{3}(x - 4) \)
A particle moves on a number line. Its position at any time \( t \) is \( \sqrt{2t + 1} \).

a. Find the average velocity between time \( t = 4 \) and time \( t = 4 + h \).

b. Find the instantaneous velocity at time \( t = 4 \).

**ANSWER:**

a. 
   
   \[ \text{change in position} = \sqrt{2(4 + h) + 1} - \sqrt{2 \cdot 4 + 1} \]
   
   \[ = \sqrt{9 + 2h} - 3 \]
   
   \[ \text{change in time} = (4 + h) - 4 = h \]
   
   \[ \text{average velocity} = \frac{\sqrt{9 + 2h} - 3}{h} \]

b. \[ \text{instantaneous velocity} = \lim_{{h \to 0}} \frac{\sqrt{9 + 2h} - 3}{h} \]

0270-2. a. \[ \frac{1}{3} \]
A heavy object is taken to the top of a building 250 feet high. At time $t = 0$, it is thrown upward at 40 feet/second. We engage the services of two Nobel prize-winning physicists who confer (i.e., yell and scream at one another). After several hours of scholarly study, followed by minor medical treatment for blunt trauma, lacerations and contusions, they hold a joint press conference, and inform their public that, $t$ seconds after release, the object will be located

$$250 + 40t - 16t^2$$ feet above the ground. Based on this, find the velocity of the object 0.1 seconds after release. Give your answer in feet per second.
... $t$ seconds after release, the object will be located

$$250 + 40t - 16t^2$$

feet above the ground. Based on this, find the velocity of the object 0.1 seconds after release. Give your answer in feet per second.

**ANSWER:** [avg velocity between times 0.1 and 0.1 + $h$] =

$$\frac{250 + 40(0.1 + h) - 16(0.1 + h)^2}{h} - \frac{250 + 40(0.1) - 16(0.1)^2}{h}$$

$$= 40 \left[ \frac{(0.1 + h) - 0.1}{h} \right] - 16 \left[ \frac{(0.1 + h)^2 - (0.1)^2}{h} \right]$$

$$= 40 - 16 \left[ \frac{(0.1)^2 + 2(0.1)h + h^2}{h} - (0.1)^2 \right]$$

$$= 40 - 16 \left[ \frac{2(0.1)h + h^2}{h} \right]$$
... $t$ seconds after release, the object will be located

$$250 + 40t - 16t^2$$ feet above the ground. Based on this, find the velocity of the object 0.1 seconds after release. Give your answer in feet per second.

**ANSWER:**

[avg velocity between times 0.1 and 0.1 + $h$]

$$= 40 - 16 \left[ \frac{2(0.1)h + h^2}{h} \right]$$

$h \neq 0$

$$= 40 - 16 [2(0.1) + h]$$

$$= 40 - 16 [0.2 + h]$$

$$= 40 - 3.2 - 16h = 36.8 - 16h$$

[instantaneous velocity at time 0.1]

$$= \lim_{h \to 0} (36.8 - 16h) = 36.8$$
Order these numbers, from smallest to largest: 
\[ f'(-2), \ f'(-1), \ f'(0), \ f'(1), \ f'(2), \ f'(3) \]
Note that we are asking about \( f' \), not \( f \).

**ANSWER:**
\[ f'(-2) < f'(-1) < f'(0) < f'(1) < f'(2) < f'(3) \]
Let \( f(x) = \frac{9x - 2}{5x - 1} \).

a. Compute \( f'(2) \).

b. Compute \( f'(3) \).

c. Compute \( f'(4) \).

d. Compute \( f'(a) \), for an arbitrary number \( a \).
0270-6. Let \( f(x) = \frac{9x - 2}{5x - 1} \).

**d. Compute** \( f'(a) \), for an arbitrary number \( a \).

**ANSWER:** d. \[ f'(a) = \lim_{h \to 0} \frac{1}{h} \left[ \frac{9(a + h) - 2}{5(a + h) - 1} - \frac{9a - 2}{5a - 1} \right] \]

\[ \Rightarrow \frac{9a + 9h - 2}{5a + 5h - 1} - \frac{9a - 2}{5a - 1} \]

\[ \Rightarrow \frac{[9h + 9a - 2][5a - 1] - [5h + 5a - 1][9a - 2]}{[5a + 5h - 1][5a - 1]} \]

\[ \Rightarrow \frac{[9h][5a - 1] - [5h][9a - 2]}{[5a + 5h - 1][5a - 1]} \]
0270-6. Let \( f(x) = \frac{9x - 2}{5x - 1} \).

d. Compute \( f'(a) \), for an arbitrary number \( a \).

ANSWER: d. \[
f'(a) = \lim_{h \to 0} \frac{1}{h} \left[ \frac{9(a + h) - 2}{5(a + h) - 1} - \frac{9a - 2}{5a - 1} \right]
\] 

\[= \frac{[9h][5a - 1] - [5h][9a - 2]}{[5a + 5h - 1][5a - 1]}
\]

\[= \frac{-9h + 10h}{[5a + 5h - 1][5a - 1]}
\]

\[= \frac{h}{[5a + 5h - 1][5a - 1]}
\]
NEW 0270-6. Let \( f(x) = \frac{9x - 2}{5x - 1} \).

\[ f'(a) = \lim_{h \to 0} \frac{1}{h} \left[ \frac{9(a + h) - 2}{5(a + h) - 1} - \frac{9a - 2}{5a - 1} \right] \]

\[ = \lim_{h \to 0} \frac{1}{h} \frac{1}{[5a + 5h - 1][5a - 1]} \]

\[ = \lim_{h \to 0} \frac{1}{[5a + 5h - 1][5a - 1]} = \frac{1}{[5a - 1]^2} \]
Let \( f(x) = \frac{9x - 2}{5x - 1} \).

d. Compute \( f'(a) \), for an arbitrary number \( a \).

**ANSWER:**

d. \( f'(a) = \frac{1}{[5a - 1]^2} \)

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a. Compute \( f'(2) \).

b. Compute \( f'(3) \).

c. Compute \( f'(4) \).

**ANSWER:**

a. \( f'(2) = \frac{1}{[5 \cdot 2 - 1]^2} = \frac{1}{81} \)

b. \( f'(3) = \frac{1}{[5 \cdot 3 - 1]^2} = \frac{1}{196} \)

c. \( f'(4) = \frac{1}{[5 \cdot 4 - 1]^2} = \frac{1}{361} \)
0270-7. Find a function \( f \) and a number \( a \) s.t.

\[
f'(a) = \lim_{h \to 0} \frac{[\arctan(-3 + h)] - [\arctan(-3)]}{h}.
\]

**ANSWER:**

\[
f'(a) = \lim_{h \to 0} \frac{[f(a + h)] - [f(a)]}{h}
\]

\( f = \arctan, \ a = -3 \)