CALCULUS
Implicit differentiation
NEW
Let an expression $y$ of $x$ be given, implicitly, by the formula $x^2y + \pi x - \sqrt{2}y = 3$.

a. Find $dy/dx$ by implicit differentiation.

b. Solve for $y$ as an explicit expression of $x$.

c. Differentiate your answer to Part b, writing $dy/dx$ as an explicit expression of $x$.

d. Substitute your answer for Part b into every $y$ appearing in your answer to Part a, writing $dy/dx$ as an explicit expression of $x$.

e. Verify that your answers to Part c and Part d are the same.
Let an expression $y$ of $x$ be given, implicitly, by the formula $x^2y + \pi x - \sqrt{2}y = 3$.

a. Find $dy/dx$ by implicit differentiation.

**Answer:** a. $2xy + x^2y' + \pi - \sqrt{2}y' = 0$

$$(x^2 - \sqrt{2})y' = -2xy - \pi$$

$$y' = \frac{-2xy - \pi}{x^2 - \sqrt{2}}$$
Let an expression \( y \) of \( x \) be given, implicitly, by the formula \( x^2y + \pi x - \sqrt{2}y = 3 \).

b. Solve for \( y \) as an explicit expression of \( x \).

c. Differentiate your answer to Part b, writing \( dy/dx \) as an explicit expression of \( x \).

**ANSWER:**

b. \((x^2 - \sqrt{2})y = 3 - \pi x\)

\[
y = \frac{3 - \pi x}{x^2 - \sqrt{2}}
\]

c. \[
y' = \frac{(x^2 - \sqrt{2})(-\pi) - (3 - \pi x)(2x)}{(x^2 - \sqrt{2})^2}
\]

\[
= \frac{-\pi x^2 + \pi \sqrt{2} - 6x + 2\pi x^2}{(x^2 - \sqrt{2})^2}
\]

\[
= \frac{\pi x^2 - 6x + \pi \sqrt{2}}{(x^2 - \sqrt{2})^2}
\]
Let an expression $y$ of $x$ be given, implicitly, by the formula $x^2y + \pi x - \sqrt{2}y = 3$.

**d.** Substitute your answer for Part b into every $y$ appearing in your answer to Part a, writing $dy/dx$ as an explicit expression of $x$.

**ANSWER:**

b. \[ y = \frac{3 - \pi x}{x^2 - \sqrt{2}} \]

a. \[ y' = \frac{-2xy - \pi}{x^2 - \sqrt{2}} \]

d. \[ y' = \frac{-2x \left( \frac{3 - \pi x}{x^2 - \sqrt{2}} \right) - \pi}{x^2 - \sqrt{2}} \]
Let an expression $y$ of $x$ be given, implicitly, by the formula $x^2y + \pi x - \sqrt{2}y = 3$.

e. Verify that your answers to Part c and Part d are the same.

**ANSWER:**

d. $y' = \frac{-2x \left( \frac{3-\pi x}{x^2-\sqrt{2}} \right) - \pi}{x^2 - \sqrt{2}} \cdot \frac{x^2 - \sqrt{2}}{x^2 - \sqrt{2}}$

e. $y' = \frac{-2x(3 - \pi x) - \pi(x^2 - \sqrt{2})}{(x^2 - \sqrt{2})^2}$

$$= -\frac{6x + 2\pi x^2 - \pi x^2 + \pi \sqrt{2}}{(x^2 - \sqrt{2})^2}$$

$$= \frac{\pi x^2 - 6x + \pi \sqrt{2}}{(x^2 - \sqrt{2})^2}$$

C. $y' = \frac{\pi x^2 - 6x + \pi \sqrt{2}}{(x^2 - \sqrt{2})^2}$
Let an expression \( y \) of \( x \) be given, implicitly, by the formula \( x^4 + y^3 = 1 \).

a. Find \( dy/dx \) by implicit differentiation.

b. Solve for \( y \) as an explicit expression of \( x \).

c. Differentiate your answer to Part b, writing \( dy/dx \) as an explicit expression of \( x \).

d. Substitute your answer for Part b into every \( y \) appearing in your answer to Part a, writing \( dy/dx \) as an explicit expression of \( x \).

e. Verify that your answers to Part c and Part d are the same.
Let an expression $y$ of $x$ be given, implicitly, by the formula $x^4 + y^3 = 1$.

a. Find $dy/dx$ by implicit differentiation.

**ANSWER:**

a. $4x^3 + 3y^2 y' = 0$

$$3y^2 y' = -4x^3$$

$$y' = -\frac{4x^3}{3y^2}$$
Let an expression $y$ of $x$ be given, implicitly, by the formula $x^4 + y^3 = 1$.

b. Solve for $y$ as an explicit expression of $x$.

c. Differentiate your answer to Part b, writing $dy/dx$ as an explicit expression of $x$.

**ANSWER:**

b. $y^3 = 1 - x^4$

$$y = (1 - x^4)^{1/3}$$

c. $y' = \frac{1}{3}(1 - x^4)^{-2/3}(-4x^3)$

$$= -\frac{4}{3}x^3(1 - x^4)^{-2/3}$$
Let an expression \( y \) of \( x \) be given, implicitly, by the formula \( x^4 + y^3 = 1 \).

Substitute your answer for Part b into every \( y \) appearing in your answer to Part a, writing \( dy/dx \) as an explicit expression of \( x \).

**ANSWER:**

b. \( y = (1 - x^4)^{1/3} \)

a. \( y' = -\frac{4x^3}{3y^2} \)

d. \( y' = -\frac{4x^3}{3((1 - x^4)^{1/3})^2} \)
Let an expression $y$ of $x$ be given, implicitly, by the formula $x^4 + y^3 = 1$.

e. Verify that your answers to Part c and Part d are the same.

**ANSWER:**

d. $y' = -\frac{4x^3}{3((1 - x^4)^{1/3})^2}$

e. $y' = -\frac{4}{3} \cdot \frac{x^3}{(1 - x^4)^{2/3}}$

$= -\frac{4}{3} x^3 (1 - x^4)^{-2/3}$

C. $y' = -\frac{4}{3} x^3 (1 - x^4)^{-2/3}$
Let an expression $y$ of $x$ be given, implicitly, by the formula

$$ye^x - \sqrt{2}e^2x + e^y \csc x = 2.$$ 

Find $dy/dx$ by implicit differentiation.

**ANSWER:**

$$y'e^x + ye^x - \sqrt{2}e^2 + (e^y y')(\csc x) - (e^y)(\csc x)(\cot x) = 0$$

$$e^x y' + e^y (\csc x) y' = - ye^x + \sqrt{2}e^2 + e^y(\csc x)(\cot x)$$

$$[e^x + e^y(\csc x)]y'$$

$$y' = \frac{- ye^x + \sqrt{2}e^2 + e^y(\csc x)(\cot x)}{e^x + e^y(\csc x)}$$
Let an expression \( y \) of \( x \) be given, implicitly, by the formula
\[
\cos^2 y = -x + 3\sqrt{7}y.
\]
Find \( dy/dx \) by implicit differentiation.

**ANSWER:**

\[
2(\cos y)(-\sin y)y' = -1 + 3\sqrt{7}y' \\
2(\cos y)(\sin y)y' = 1 - 3\sqrt{7}y' \\
2(\cos y)(\sin y)y' + 3\sqrt{7}y' = 1 \\
\| \\
[2(\cos y)(\sin y) + 3\sqrt{7}]y' \\
y' = \frac{1}{2(\cos y)(\sin y) + 3\sqrt{7}}
\]
0430-5. Let an expression \( y \) of \( x \) be given, implicitly, by the formula
\[ x^4 + y^3 = 15. \]
Find the equation of the tangent line to the graph of this equation at the point \((-2, -1)\).

**ANSWER:** \( 4x^3 + 3y^2y' = 0 \)

\[ 3y^2y' = -4x^3 \]
\[ y' = -\frac{4x^3}{3y^2} \]

\[ y' \bigg|_{x=-2, y=-1} = -\frac{4 \cdot (-2)^3}{3 \cdot (-1)^2} = \frac{32}{3} \]
\[ y = -1 + \frac{32}{3}(x + 2) \]
Let an expression $y$ of $x$ be given, implicitly, by the formula
\[3y^2 = x^6 - \sqrt[3]{16}x.\]
Find the equation of the tangent line to the graph of this equation at the point \((\sqrt[3]{4}, 2)\).

**ANSWER:**

\[
6yy' = 6x^5 - \sqrt[3]{16} \quad \Rightarrow \quad y' = \frac{6x^5 - \sqrt[3]{16}}{6y}
\]

\[
[y']_{x \rightarrow \sqrt[3]{4}, y = 2} = \frac{6 \cdot (\sqrt[3]{4})^5 - \sqrt[3]{16}}{6 \cdot 2} = \frac{6 \cdot 4 \cdot (\sqrt[3]{4^2}) - \sqrt[3]{16}}{12} = \frac{24 \cdot (\sqrt[16]{16}) - \sqrt[3]{16}}{12} = \frac{23 \cdot (\sqrt[16]{16})}{12}
\]

\[
y = 2 + \left(\frac{23}{6} \sqrt[3]{2}\right) (x - \sqrt[3]{4})
\]
Let an expression $y$ of $x$ be given, implicitly, by the formula

$$-x^4 + \sqrt[7]{2}y^3 + e^2y = 2.$$ 

Find $d^2y/dx^2$ by implicit differentiation.

**ANSWER:** $-4x^3 + 3\sqrt[7]{2}y^2y' + e^2y' = 0$,

so $y' = \frac{4x^3}{3\sqrt[7]{2}y^2 + e^2}$.

$$-12x^2 + 6\sqrt[7]{2}y(y')^2 + 3\sqrt[7]{2}y^2y'' + e^2y'' = 0$$

$$y'' = \frac{12x^2 - 6\sqrt[7]{2}y(y')^2}{3\sqrt[7]{2}y^2 + e^2}$$

$$= \frac{12x^2 - 6\sqrt[7]{2}y \left(\frac{4x^3}{3\sqrt[7]{2}y^2 + e^2}\right)^2}{3\sqrt[7]{2}y^2 + e^2}$$
0430-8. Let an expression $y$ of $x$ be given, implicitly, by the formula

$$-x^7 + 4\pi y^3 = 8 + xy.$$ 

Find $d^2y/dx^2$ by implicit differentiation.

**ANS:** $-7x^6 + 12\pi y^2 y' = y + xy'$, so $y' = \frac{7x^6 + y}{12\pi y^2 - x}$.

$$-42x^5 + 24\pi y(y')^2 + 12\pi y^2 y'' = y' + y' + xy''$$

$$y'' = \frac{42x^5 - 24\pi y(y')^2 + 2y'}{12\pi y^2 - x}$$

$$= \frac{42x^5 - 24\pi y \left( \frac{7x^6 + y}{12\pi y^2 - x} \right)^2 + 2 \left( \frac{7x^6 + y}{12\pi y^2 - x} \right)}{12\pi y^2 - x}$$
For every $a \in \mathbb{R}$, for every $b > 0$, let $G_a$ be graph of $15x^3 - 8y^2 = ax^3y^2$ and let $H_b$ be graph of $x^5 + y^4 = b$.

(a) Let $p$ be the point $(1, 1)$, which lies both on $G_7$ and on $H_2$. Show that the tangent lines to $G_7$ and $H_2$ at $p$ are perpendicular.

(b) Let $a$ and $b$ be any two real numbers, with $b > 0$. Let $q$ be any point which lies both on $G_a$ and on $H_b$. Show that the tangent lines to $G_a$ and $H_b$ at $q$ are perpendicular.
For every \( a \in \mathbb{R} \), for every \( b > 0 \), let \( G_a \) be graph of \( 15x^3 - 8y^2 = ax^3y^2 \) and let \( H_b \) be graph of \( x^5 + y^4 = b \). 

Let \( p \) be the point \((1, 1)\), which lies both on \( G_7 \) and on \( H_2 \). Show that the tangent lines to \( G_7 \) and \( H_2 \) at \( p \) are perpendicular.

**ANSWER:**

For \( G_7 \):

\[
15(3x^2) - 8(2yy') = 7((3x^2)(y^2) + x^3(2yy'))
\]

\[
45x^2 - 16yy' = 21x^2y^2 + 14x^3yy'
\]

\[
y' = \frac{-45x^2 + 21x^2y^2}{-16y - 14x^3y}
\]

(slope at \( p \)) \[
= \left[ y' \right]_{x: \rightarrow 1, y = 1} = \frac{-45 + 21}{-16 - 14} = \frac{4}{5}
\]
For every $a \in \mathbb{R}$, for every $b > 0$, let $G_a$ be graph of $15x^3 - 8y^2 = ax^3y^2$ and let $H_b$ be graph of $x^5 + y^4 = b$.

(a) Let $p$ be the point $(1, 1)$, which lies both on $G_7$ and on $H_2$. Show that the tangent lines to $G_7$ and $H_2$ at $p$ are perpendicular.

**ANSWER:**

For $G_7$: 

\[
\text{[slope at } p] = \frac{4}{5}
\]

For $H_2$: 

\[
x^5 + y^4 = 2
\]

\[
5x^4 + 4y^3y' = 0
\]

\[
y' = -\frac{5x^4}{4y^3}
\]

\[
\text{[slope at } p] = \left[y'\right]_{x \to 1, y = 1} = -\frac{5}{4}
\]
NEW 0430-9. For every $a \in \mathbb{R}$, for every $b > 0$, let $G_a$ be graph of $15x^3 - 8y^2 = ax^3y^2$ and let $H_b$ be graph of $x^5 + y^4 = b$.

a. Let $p$ be the point $(1, 1)$, which lies both on $G_7$ and on $H_2$. Show that the tangent lines to $G_7$ and $H_2$ at $p$ are perpendicular.

**ANSWER:** a. For $G_7$: [slope at $p$] = $\frac{4}{5}$

For $H_2$: [slope at $p$] = $-\frac{5}{4}$

$$\left(\frac{4}{5}\right) \left(-\frac{5}{4}\right) = -1,$$ so the lines are perpendicular.
\[ G_a: \ 15x^3 - 8y^2 = ax^3y^2; \quad H_b: \ x^5 + y^4 = b \]

Show that the tangent lines to \( G_a \) and \( H_b \) at \( q \) are perpendicular.

\textbf{ANSWER:} \( b. \) Write \( q = (s, t). \)

For \( G_a: \)
\[ 15x^3 - 8y^2 = ax^3y^2 \]
\[ 15(3x^2) - 8(2yy') = a((3x^2)(y^2) + (x^3)(2yy')) \]
\[ 45x^2 - 16yy' = 3ax^2y^2 + 2ax^3yy' \]

\[ y' = \frac{-45x^2 + 3ax^2y^2}{-16y - 2ax^3y} \]

\[ [\text{slope at } q] = \begin{bmatrix} y' \end{bmatrix}_{x:s \to s,y:t} = \frac{-45s^2 + 3as^2t^2}{-16t - 2as^3t} \]
0430-9. \( G_a: \ 15x^3 - 8y^2 = ax^3y^2; \ H_b: \ x^5 + y^4 = b \)

q on \( G_a \) and on \( H_b \).

b. Show that the tangent lines to \( G_a \) and \( H_b \) at \( q \) are perpendicular.

ANSWER: b. Write \( q = (s, t) \).

\[
15s^3 - 8t^2 = as^3t^2
\]

For \( G_a \): [slope at \( q \)] = \[
\frac{-45s^2 + 3as^2t^2}{-16t - 2as^3t}
\]

For \( H_b \): \( x^5 + y^4 = b \)

\[
5x^4 + 4y^3y' = 0
\]

\[
y' = -\frac{5x^4}{4y^3}
\]

[slope at \( q \)] = \[
\left[ y' \right]_{x \rightarrow s, y \rightarrow t} = -\frac{5s^4}{4t^3}
\]
0430-9. \( G_a: 15x^3 - 8y^2 = ax^3y^2; \)  \( H_b: x^5 + y^4 = b \)

q on \( G_a \) and on \( H_b \).

b. Show that the tangent lines to \( G_a \) and \( H_b \) at \( q \) are perpendicular.

**ANSWER:**

b. Write \( q = (s, t) \). \( 15s^3 - 8t^2 = as^3t^2 \)

For \( G_a \): [slope at \( q \)] = \( \frac{-45s^2 + 3as^2t^2}{-16t - 2as^3t} \)

For \( H_b \): [slope at \( q \)] = \( -\frac{5s^4}{4t^3} \)

Want: \( \left[ \frac{-45s^2 + 3as^2t^2}{-16t - 2as^3t} \right] \left[ -\frac{5s^4}{4t^3} \right] = -1 \)
0430-9. \( G_a: \ 15x^3 - 8y^2 = ax^3y^2; \ H_b: \ x^5 + y^4 = b \)

q on \( G_a \) and on \( H_b \).

b. Show that the tangent lines to \( G_a \) and \( H_b \) at \( q \) are perpendicular.

**ANSWER:** b. Write \( q = (s, t) \).

\[
15s^3 - 8t^2 = as^3 t^2
\]

Want:

\[
\begin{bmatrix}
-45s^2 + 3as^2 t^2 \\
-16t - 2as^3 t
\end{bmatrix}
\begin{bmatrix}
5s^4 \\
4t^3
\end{bmatrix}
= -1
\]

\[
\begin{bmatrix}
-45s^2 + 3as^2 t^2 \\
-16t - 2as^3 t
\end{bmatrix}
\begin{bmatrix}
5s^4 \\
4t^3
\end{bmatrix}
= -\frac{(-45s^3 + 3as^3 t^2)5s^3}{(-16t^2 - 2as^3 t^2)4t^2}
\]

\[
= -\frac{(-45s^3 + 3(15s^3 - 8t^2))5s^3}{(-16t^2 - 2(15s^3 - 8t^2))4t^2}
\]

\[
= -\frac{(-24t^2)5s^3}{(-30s^3)4t^2}
\]

\[
= -\frac{120s^3 t^2}{120s^3 t^2} = -1
\]
Challenge problem (not assigned):

For every \( a, b \in \mathbb{R} \),

let \( G_a \) be graph of \( e^{-x} - e^{-y} = a \) and

let \( H_b \) be graph of \( e^x + e^y = 2b \).

A. Let \( p \) be the point \((1, 1)\), which lies both on \( G_0 \) and on \( H_e \).

Show that the tangent lines to \( G_0 \) and \( H_e \) at \( p \) are perpendicular.

B. Let \( a \) and \( b \) be any two real numbers.

Let \( q \) be any point which lies both on \( G_a \) and on \( H_b \).

Show that the tangent lines to \( G_a \) and \( H_b \) at \( q \) are perpendicular.
Ch. prb: $G_a: e^{-x} - e^{-y} = a; \quad H_b: e^x + e^y = 2b$

a. Let $p$ be the point $(1,1)$, which lies both on $G_0$ and on $H_e$.

Show that the tangent lines to $G_0$ and $H_e$ at $p$ are perpendicular.

ANSWER: a. For $G_0$: $e^{-x} - e^{-y} = 0$

$$-e^{-x} + e^{-y}y' = 0$$

$$e^{-y}y' = e^{-x}$$

$$y' = e^{-x}/e^{-y} = e^{y-x}$$

$$[\text{slope at } p] = [y']_{x: \to 1, y=1} = e^{1-1} = e^0 = 1$$
Ch. prb: $G_a: e^{-x} - e^{-y} = a; \quad H_b: e^x + e^y = 2b$

a. Let $p$ be the point $(1,1)$, which lies both on $G_0$ and on $H_e$.

Show that the tangent lines to $G_0$ and $H_e$ at $p$ are perpendicular.

**ANSWER:**

a. For $G_0$: [slope at $p$] = 1

For $H_e$: $e^x + e^y = 2e$

$$e^x + e^y y' = 0$$
$$e^y y' = -e^x$$
$$y' = -e^x / e^y$$
$$= -e^{x-y}$$

[slope at $p$] = $[y']_{x \to 1, y=1} = -e^{1-1} = -e^0 = -1$
Ch. prb: \( G_a: e^{-x} - e^{-y} = a; \quad H_b: e^x + e^y = 2b \)

a. Let \( p \) be the point \((1,1)\), which lies both on \( G_0 \) and on \( H_e \).

Show that the tangent lines to \( G_0 \) and \( H_e \) at \( p \) are perpendicular.

\[ \text{ANSWER:} \quad a. \text{ For } G_0: \text{[slope at } p]\} = 1 \]
\[ \text{For } H_e: \text{[slope at } p]\} = -1 \]
\[ (1)(-1) = -1, \]
so the lines are perpendicular.
Ch. prb: \( G_a: e^{-x} - e^{-y} = a; \quad H_b: e^x + e^y = 2b \)

b. \( q \) on \( G_a \) and on \( H_b \).

Show that the tangent lines to \( G_a \) and \( H_b \) at \( q \) are perpendicular.

**ANSWER:** b. Write \( q = (s, t) \).

For \( G_a: e^{-x} - e^{-y} = a \)
\[-e^{-x} + e^{-y} y' = 0 \]
\[e^{-y} y' = e^{-x} \]
\[y' = e^{-x}/e^{-y} = e^{y-x} \]

[slope at \( q \)] \( \left[ y' \right]_{x:s \rightarrow s, y:t} = e^{t-s} \)
Ch. prb: \( G_a: \ e^{-x} - e^{-y} = a; \quad H_b: \ e^x + e^y = 2b \)

b. \( q \) on \( G_a \) and on \( H_b \).

Show that the tangent lines to \( G_a \) and \( H_b \) at \( q \) are perpendicular.

**ANSWER:** b. Write \( q = (s, t) \).

For \( G_a: \) [slope at \( q \)] = \( e^{t-s} \)

For \( H_b: \ e^x + e^y = 2b \)

\[ e^x + e^y y' = 0 \]

\[ e^y y' = -e^x \]

\[ y' = -e^x / e^y = -e^{x-y} \]

[slope at \( q \)] = \( [y'] \) \( x: \rightarrow s, y = t \) = \( -e^{s-t} \)
Ch. prb: \(G_a: e^{-x} - e^{-y} = a; \quad H_b: e^x + e^y = 2b\)

b. \(q\) on \(G_a\) and on \(H_b\).

Show that the tangent lines to \(G_a\) and \(H_b\) at \(q\) are perpendicular.

\[\text{ANSWER: b. Write } q = (s, t).\]

For \(G_a\): [slope at \(q\)] = \(e^{t-s}\)

For \(H_b\): [slope at \(q\)] = \(-e^{s-t}\)

\[
\begin{bmatrix} e^{t-s} \\ -e^{s-t} \end{bmatrix} \begin{bmatrix} e^{t-s} \\ -e^{s-t} \end{bmatrix} = -e^{t-s+s-t} = -e^0 = -1,
\]

so the lines are perpendicular.