CALCULUS
The Mean Value Theorem
NEW
Let \( f(x) = x^2 - 2x + 3 \).

a. Check that \( f \) satisfies the conditions of Rolle’s Theorem on the interval \([0, 2]\). That is, check

   (i) that \( f \) is continuous on \([0, 2]\),
   (ii) that \( f \) is differentiable on \((0, 2)\) and
   (iii) that \( f(0) = f(2) \).

b. Find all solutions to the equation in the conclusion of Rolle’s Th’m for \( f \) on \([0, 2]\). That is, find all \( c \in (0, 2) \) s.t. \( f'(c) = 0 \).
0460-1. Let \( f(x) = x^2 - 2x + 3 \).

a. Check that \( f \) satisfies the conditions of Rolle’s Theorem on the interval \([0, 2]\). That is, check

(i) that \( f \) is continuous on \([0, 2]\),
(ii) that \( f \) is differentiable on \((0, 2)\) and
(iii) that \( f(0) = f(2) \).

**ANSWER:**

(i) \( f \) is a poly., so \( f \) is continuous on \( \mathbb{R} \), so \( f \) is continuous on \([0, 2]\).

(ii) \( f \) is a poly., so \( f \) is differentiable on \( \mathbb{R} \), so \( f \) is differentiable on \((0, 2)\).

(iii) \( f(0) = 0 - 0 + 3 = 3 \)

\( f(2) = 4 - 4 + 3 = 3 \)
0460-1. Let \( f(x) = x^2 - 2x + 3 \).

b. Find all solutions to the equation in the conclusion of Rolle’s Th’m for \( f \) on \([0, 2]\). That is, find all \( c \in (0, 2) \) s.t. \( f'(c) = 0 \).

\[
\text{ANSWER: } f'(x) = 2x - 2
\]

\[
f'(c) = 0 \iff 2c - 2 = 0
\]

\[
\iff c = 1
\]
Let \( f(x) = x^2 + x + 3 \).

a. Check that \( f \) satisfies the conditions of the MVT on the interval \([0, 2]\). That is, check

(i) that \( f \) is continuous on \([0, 2]\) and (ii) that \( f \) is differentiable on \((0, 2)\).

b. Find all solutions to the equation in the conclusion of the MVT for \( f \) on \([0, 2]\). That is, find all \( c \in (0, 2) \) s.t.

\[
\frac{f'(c)}{2 - 0} = \frac{f(2) - f(0)}{2 - 0}.
\]
Let \( f(x) = x^2 + x + 3 \).

a. Check that \( f \) satisfies the conditions of the MVT on the interval \([0, 2]\).
   That is, check
   
   (i) that \( f \) is continuous on \([0, 2]\) and (ii) that \( f \) is differentiable on \((0, 2)\).

**ANSWER:**

(i) \( f \) is a poly., so \( f \) is continuous on \( \mathbb{R} \), so \( f \) is continuous on \([0, 2]\).

(ii) \( f \) is a poly., so \( f \) is differentiable on \( \mathbb{R} \), so \( f \) is differentiable on \((0, 2)\).
b. Find all solutions to the equation in the conclusion of the MVT for \( f \) on \([0, 2]\). That is, find all \( c \in (0, 2) \) s.t.

\[
f'(c) = \frac{[f(2)] - [f(0)]}{2 - 0}.
\]

**ANSWER:** \( f'(x) = 2x + 1 \)

\[
\frac{[f(2)] - [f(0)]}{2 - 0} = \frac{[4 + 2 + 3] - [0 + 0 + 3]}{2} = \frac{9 - 3}{2} = 3
\]

\[
f'(c) = \frac{[f(2)] - [f(0)]}{2 - 0} \iff 2c + 1 = 3 \iff c = 1\]
Let \( f(x) = 3 + 2|x - 5| \).

a. Show that \( f \) is continuous on \([1, 9]\).

b. Show that \( f(1) = f(9) \).

c. Show that the conclusion of Rolle’s Th’m, for \( f \) on \([1, 9]\), fails. That is, show that there is no \( c \in (1, 9) \) s.t. \( f'(c) = 0 \).

d. Explain why this does not contradict Rolle’s Theorem.
0460-3. Let $f(x) = 3 + 2|x - 5|$.

a. Show that $f$ is continuous on $[1, 9]$.

**ANSWER:** a. $x - 5$ is a polynomial in $x$, so $x - 5$ is continuous in $x$.

$|\cdot|$ is continuous.

A composition of a continuous function and a continuous expression is continuous. Then $|x - 5|$ is continuous in $x$.

1 is a constant, so 1 is continuous in $x$.
A lin. comb. of contin. expressions is contin. Then $3 \cdot 1 + 2 \cdot |x - 5|$ is continuous in $x$.
That is, $f$ is continuous. The domain of $f$ is $\mathbb{R}$.
So $f$ is continuous on $\mathbb{R}$, hence on $[1, 9]$. 
Let \( f(x) = 3 + 2|x - 5| \).

b. Show that \( f(1) = f(9) \).

**ANSWER:** b.

\[
\begin{align*}
  f(1) &= 3 + 2|1 - 5| = 3 + 2|-4| = 3 + 2 \cdot 4 = 11 \\
  f(9) &= 3 + 2|9 - 5| = 3 + 2|4| = 3 + 2 \cdot 4 = 11
\end{align*}
\]
Let \( f(x) = 3 + 2|x - 5| \).

c. Show that the conclusion of Rolle’s Th’m, for \( f \) on \([1, 9]\), fails. That is, show that there is no \( c \in (1, 9) \) s.t. \( f'(c) = 0 \).

**ANSWER:**

c.

\[
f(x) = \begin{cases} 
3 - 2(x - 5), & \text{if } x \leq 5 \\
3 + 2(x - 5), & \text{if } x \geq 5 
\end{cases}
\]

\[
f'(x) = \begin{cases} 
-2, & \text{if } x < 5 \\
2, & \text{if } x > 5 
\end{cases}
\]

Therefore there is no \( c \in (3, 5) \) s.t. \( f'(c) = 0 \).
0460-3. Let \( f(x) = 3 + 2|x - 5| \).

d. Explain why this does not contradict Rolle’s Theorem.

**ANSWER:**

\[
f'(x) = \begin{cases} 
-2, & \text{if } x < 5 \\
2, & \text{if } x > 5 
\end{cases}
\]

\( f \) is not differentiable at 5, so \( f \) is not differentiable on \((1, 9)\).
0460-4. Let $f(x) = -x + 2|x - 5|$.

a. Show that $f$ is continuous on $[1, 9]$.

b. Show that the conclusion of the MVT, for $f$ on $[1, 9]$, fails. That is, show that there is no $c \in (1, 9)$ s.t.

$$f'(c) = \frac{[f(9)] - [f(1)]}{9 - 1}.$$

c. Explain why this does not contradict the MVT.
0460-4. Let \( f(x) = -x + 2|x - 5| \).

a. Show that \( f \) is continuous on \([1, 9]\).

**ANSWER:** a. \( x - 5 \) is a polynomial in \( x \), so \( x - 5 \) is continuous in \( x \).

| \( \bullet \) | is continuous.

A composition of a continuous function and a continuous expression is continuous.

Then \( |x - 5| \) is continuous in \( x \).

\( x \) is a polynomial, so \( x \) is continuous in \( x \).

A lin. comb. of contin. expressions is contin.

Then \((-1) \cdot x + 2 \cdot |x - 5| \) is continuous in \( x \).

That is, \( f \) is continuous. The domain of \( f \) is \( \mathbb{R} \).

So \( f \) is continuous on \( \mathbb{R} \), hence on \([1, 9]\).
 Let \( f(x) = -x + 2|x - 5| \).

b. Show that the conclusion of the MVT, for \( f \) on \([1, 9]\), fails. That is, show that there is no \( c \in (1, 9) \) s.t.

\[
f'(c) = \frac{[f(9)] - [f(1)]}{9 - 1}.
\]

**ANSWER:** b.

\[
\frac{[f(9)] - [f(1)]}{9 - 1} = \frac{[-9 + 2|4|] - [-1 + 2| - 4|]}{8} = \frac{-1 - 7}{8} = \frac{-8}{8} = -1
\]
0460-4. Let \( f(x) = -x + 2|x - 5| \).

b. Show that the conclusion of the MVT, for \( f \) on \([1, 9]\), fails. That is, show that there is no \( c \in (1, 9) \) s.t.

\[
f'(c) = \frac{[f(9)] - [f(1)]}{9-1} = -1.
\]

\[
f(x) = \begin{cases} 
-x - 2(x - 5), & \text{if } x \leq 5 \\
-x + 2(x - 5), & \text{if } x > 5
\end{cases}
\]

\[
f'(x) = \begin{cases} 
-1 - 2, & \text{if } x < 5 \\
-1 + 2, & \text{if } x > 5
\end{cases}
\]

Therefore there is no \( c \in (1, 9) \) s.t. \( f'(c) = -1 \).
Let $f(x) = -x + 2|x - 5|$.

c. Explain why this does not contradict the MVT.

**ANSWER:** c.

$$f'(x) = \begin{cases} -1 - 2, & \text{if } x < 5 \\ -1 + 2, & \text{if } x > 5 \end{cases}$$

$f$ is not differentiable at 5, so $f$ is not differentiable on $(1, 9)$. ☐
Let \( f(x) = \begin{cases} 
398, \text{ if } x = 1 \\
3x - 5, \text{ if } 1 < x \leq 9 
\end{cases} \)

a. Show that \( f \) is differentiable on \((1, 9)\).

b. Show that the conclusion of the MVT, for \( f \) on \([1, 9]\), fails. That is, show that there is no \( c \in (1, 9) \) s.t.

\[
f'(c) = \frac{[f(9)] - [f(1)]}{9 - 1}.
\]

c. Explain why this does not contradict the MVT.
Let \( f(x) = \begin{cases} 
398, & \text{if } x = 1 \\
3x - 5, & \text{if } 1 < x \leq 9 
\end{cases} \)

\[ a. \text{ Show that } f \text{ is differentiable on } (1, 9). \]

\text{ANSWER:}

\( \forall x \in (1, 9), \quad f(x) = 3x - 5, \)

so, \( \forall x \in (1, 9), \quad f'(x) = 3, \)

so \( f \) is differentiable at \( x \).

Then \( f \) is differentiable on \( (1, 9) \).
0460-5. Let $f(x) = \begin{cases} 398, & \text{if } x = 1 \\ 3x - 5, & \text{if } 1 < x \leq 9 \end{cases}$

b. Show that the conclusion of the MVT, for $f$ on $[1,9]$, fails. That is, show that there is no $c \in (1,9)$ s.t.

$$f'(c) = \frac{[f(9)] - [f(1)]}{9 - 1}.$$  

**ANSWER:**

$$\frac{[f(9)] - [f(1)]}{9 - 1} = \frac{22 - 398}{8} = -47$$  

$\forall x \in (1,9), \quad f(x) = 3x - 5,$  

so, $\forall x \in (1,9), \quad f'(x) = 3,$  

so, $\forall c \in (1,9), \quad f'(c) = 3 > -47.$  

Then there is no $c \in (1,9)$ s.t. $f'(c) = -47.$
0460-5. Let \( f(x) = \begin{cases} 
398, & \text{if } x = 1 \\
3x - 5, & \text{if } 1 < x \leq 9 
\end{cases} \)

c. Explain why this does not contradict the MVT.

**ANSWER:**
\[
\lim_{x \to 1^+} [f(x)] = \lim_{x \to 1^+} [3x - 5] = -2
\]

\[
f(1) = 398
\]

\( f \) is not continuous from the right at 1
so \( f \) is not continuous on \([1, 9]\).
Show that $xe^{x^2} = -5$ has exactly one real solution.

**ANS:** Let $f(x) = xe^{x^2}$.

\[ f(-100) = (-100) \left(e^{10000}\right) < -5 \]

\[ f(100) = (100) \left(e^{10000}\right) > -5 \]

Therefore, by the Intermediate Value Thm, $f(x) = -5$ has at least one real sol’n.

Want: $f(x) = -5$ does not have two real sol’ns.
Show that $xe^{x^2} = -5$ has exactly one real solution.

**ANS:** Let $f(x) = xe^{x^2}$.

$f(x) = -5$ has at least one real sol’n.

**Want:** $f(x) = -5$ does not have two real sol’n.

Suppose $s < u$ and $f(s) = -5 = f(u)$.

**Want:** Contradiction.

By Rolle’s Th’m, fix $t \in (s, u)$ s.t. $f'(t) = 0$.

$f'(x) = e^{x^2} + xe^{x^2}(2x)$

$= (1 + 2x^2)e^{x^2}$

Then $0 = f'(t) = (1 + 2t^2)e^{t^2} \geq e^{t^2} > 0$.

Contradiction.
Let $c$ be any constant.

Show that $e^{x^3} + x + c = 0$ has at most one real solution on $\mathbb{R}$.

**ANS:** Let $f(x) = e^{x^3} + x + c$.

Suppose $s < u$ and

$$f(s) = 0 = f(u).$$

Want: Contradiction.

By Rolle’s Th’m, fix $t \in (s, u)$ s.t. $f'(t) = 0$.

$$f'(x) = e^{x^3} (3x^2) + 1$$

Then

$$e^{t^3} (3t^2) + 1 = 0.$$  

However, $e^{t^3} (3t^2) + 1 \geq 1 > 0$.

Contradiction.
At noon on some day, a certain car is at the 250 mile marker on some road. The speed limit on the road is 50 mph. A driver drives the car for nine hours, obeying the speed limit.

Let $f(t)$ denote the position of the car $t$ hours after noon; then

$$f(0) = 250$$

and

$$\forall t \in [0, 9], \quad f'(t) \leq 50.$$

With these constraints, what is the largest possible value for $f(9)$?
With these constraints, what is the largest possible value for $f(9)$?

**ANSWER:**
In the special case where $f(t) = 250 + 50t$,
we have $f(0) = 250$,
we have $\forall t \in [0, 9], f'(t) \leq 50$
and we have $f(9) = 700$.

Then 700 is a possible value for $f(9)$.
We will show that 700 is the largest possible value for $f(9)$. 

$f(0) = 250$ and $\forall t \in [0, 9], f'(t) \leq 50$. 

$f(0) = 250$

and

$\forall t \in [0, 9], \quad f'(t) \leq 50.$

With these constraints, what is the largest possible value for $f(9)$?

**ANSWER:** 700 is a possible value for $f(9)$.

We will show that 700 is the largest possible value for $f(9)$.

Suppose $f$ satisfies the two conditions above, i.e., suppose

$\forall t \in [0, 9], \quad f'(t) \leq 50.$

Want: $f(9) \leq 700$
\[ f(0) = 250 \]
and
\[ \forall t \in [0, 9], \quad f'(t) \leq 50. \]

With these constraints, what is the largest possible value for \( f(9) \)?

**ANSWER:** Want: \( f(9) \leq 700 \)

By the MVT, fix \( c \in (0, 9) \) s.t.
\[
\frac{[f(9)] - [f(0)]}{9 - 0} = f'(c).
\]

Then \[
\frac{[f(9)] - [250]}{9} = f'(c) \leq 50.
\]

Then \[
[f(9)] - [250] \leq 9(50) = 450.
\]

Then \[
f(9) \leq 450 + 250 = 700. \]