CALCULUS

Newton's method

NEW
0530-1. We wish to solve \( x^5 - 5x^3 + 8 = 0 \). Starting with an initial guess of \( x_1 = 2 \), compute the next two guesses, \( x_2 \) and \( x_3 \), to at least four decimals, using Newton’s method.

**ANSWER:**

\[
x_2 = x_1 - \frac{x_1^5 - 5x_1^3 + 8}{5x_1^4 - 15x_1^2}
\]

\[
= 2 - \frac{32 - 40 + 8}{80 - 60} = 2
\]

\[
x_3 = x_2 - \frac{x_2^5 - 5x_2^3 + 8}{5x_2^4 - 15x_2^2}
\]

\[
= 2 - \frac{32 - 40 + 8}{80 - 60} = 2
\]
We wish to solve \( x^5 - 5x^3 - 12 = 0 \). Starting with an initial guess of \( x_1 = 2 \), compute the next two guesses, \( x_2 \) and \( x_3 \), to at least four decimals, using Newton’s method.

**ANSWER:**

\[
x_2 = x_1 - \frac{x_1^5 - 5x_1^3 - 12}{5x_1^4 - 15x_1^2} = 2 - \frac{32 - 40 - 12}{80 - 60} = 3
\]

\[
x_3 = x_2 - \frac{x_2^5 - 5x_2^3 - 12}{5x_2^4 - 15x_2^2} = 3 - \frac{243 - 135 - 12}{405 - 135} = 2.6444
\]
0530-3. We wish to solve \( x^5 - 4 = 0 \). Starting with an initial guess of \( x_1 = 1 \), compute the next two guesses, \( x_2 \) and \( x_3 \), to at least four decimals, using Newton’s method.

**ANSWER:**

\[
x_2 = x_1 - \frac{x_1^5 - 4}{5x_1^4} = 1 - \frac{1 - 4}{5} = 1.6
\]

\[
x_3 = x_2 - \frac{x_2^5 - 4}{5x_2^4} = 1.6 - \frac{(1.6^5) - 4}{5(1.6^4)} = 1.4021
\]
We wish to solve \( x^5 + 2x^3 + 3 = 0 \). Starting with an initial guess of \( x_1 = -1 \), compute the next two guesses, \( x_2 \) and \( x_3 \), to at least four decimals, using Newton’s method.

**ANSWER:**

\[
x_2 = x_1 - \frac{x_1^5 + 2x_1^3 + 3}{5x_1^4 + 6x_1^2} = -1 \frac{-1 - 2 + 3}{5 + 6} = -1
\]

\[
x_3 = x_2 - \frac{x_2^5 + 2x_2^3 + 3}{5x_2^4 + 6x_2^2} = -1 \frac{-1 - 2 + 3}{5 + 6} = -1
\]
We wish to solve \( x^3 - 8 = 0 \). Starting with an initial guess of \( x_1 = -1 \), compute the next two guesses, \( x_2 \) and \( x_3 \), to at least four decimals, using Newton’s method.

**ANSWER:**

\[
x_2 = x_1 - \frac{x_1^3 - 8}{3x_1^2} = -1 - \frac{-1 - 8}{3} = 2
\]

\[
x_3 = x_2 - \frac{x_2^3 - 8}{3x_2^2} = 2 - \frac{8 - 8}{12} = 2
\]
Using Newton’s method, calculate \( \sqrt[3]{6} \), to five decimal places.

**ANS:** Want: the unique root of \( f(x) = x^3 - 6 \).

\[
x_{n+1} = x_n - \frac{x_n^3 - 6}{3x_n^2}
\]

Let’s try \( x_1 := 1 \).

\[
x_2 = x_1 - \frac{x_1^3 - 6}{3x_1^2} = 1 - \frac{1^3 - 6}{3 \cdot 1^2} = \frac{8}{3}
\]

\[
x_3 = x_2 - \frac{x_2^3 - 6}{3x_2^2} = \frac{8}{3} - \frac{\left(\frac{8}{3}\right)^3 - 6}{3 \cdot \left(\frac{8}{3}\right)^2} \approx 2.05903
\]

\[
x_4 \approx 1.84443 \quad x_6 \approx 1.81712
\]

\[
x_5 \approx 1.81752 \quad x_7 \approx 1.81712
\]
0530-7. Find the unique solution to \(3x = \cos x\), to five decimal places.

\[x_{n+1} = x_n - \frac{3x_n - \cos x_n}{3 + \sin x_n}\]

**ANSWER:** We want the root of \(f(x) = 3x - \cos x\).

Let's try \(x_1 := 0\).

\[x_2 = 0 - \frac{3 \cdot 0 - \cos 0}{3 + \sin 0} = \frac{1}{3}\]

\[x_3 \approx 0.31679\]

\[x_4 \approx 0.31675\]

\[x_5 \approx 0.31675\]
0530-8. Find a solution to \(3x^{5/3} + x = -2\), to five decimal places, by applying Newton's method to \(f(x) = 3x^{5/3} + x + 2\), with \(x_1 = 1.5\).

**ANSWER:** Want: a root of \(3x^{5/3} + x + 2\).

\[
x_{n+1} = x_n - \frac{3x_n^{5/3} + x_n + 2}{5x_n^{2/3} + 1}
\]

Let's try \(x_1 := 1.5\).

\[
x_2 = 1.5 - \frac{3(1.5^{5/3}) + 1.5 + 2}{5(1.5^{2/3}) + 1} \approx 0.25571
\]

\[
x_3 \approx -0.59514
\]

\[
x_4 \approx -0.62635
\]

\[
x_5 \approx -0.62594
\]

\[
x_6 \approx -0.62594
\]
We wish to solve \( t + \sqrt[3]{t} = 0 \).

Let \( t_1 := \frac{1}{3\sqrt{3}} \). Starting with this initial guess \( t_1 \), compute the next six guesses, \( t_2, \ldots, t_7 \), using Newton’s method. Draw a picture, to illustrate what is happening.

**ANS:** Want: the root of \( f(t) = t + \sqrt[3]{t} \).

\[
t_{n+1} = t_n - \frac{t_n + t_n^{1/3}}{1 + (1/3)t_n^{-2/3}} = t_n - \frac{3t_n^{5/3} + 3t_n}{3t_n^{2/3} + 1}
\]

\[
= \frac{3t_n^{5/3} + t_n}{3t_n^{2/3} + 1} - \frac{3t_n^{5/3} + 3t_n}{3t_n^{2/3} + 1} = \frac{-2t_n}{3t_n^{2/3} + 1} = \frac{-2t_n}{3\sqrt[3]{t_n^2} + 1}
\]

\[
t_2 = -\frac{2\left(1/(3\sqrt{3})\right)}{3\sqrt[3]{(1/(3\sqrt{3}))^2} + 1} = -\frac{2/(3\sqrt{3})}{3\sqrt[3]{1/27} + 1} = -\frac{2/(3\sqrt{3})}{3(1/3) + 1} = -t_1
\]

\[
t_3 = t_1 \quad t_5 = t_1 \quad t_7 = t_1
\]

\[
t_4 = -t_1 \quad t_6 = -t_1
\]
0530-9. We wish to solve \( t + \frac{3}{\sqrt[3]{t}} = 0 \).

Let \( t_1 := \frac{1}{3\sqrt{3}} \). Starting with this initial guess \( t_1 \), compute the next six guesses, \( t_2, \ldots, t_7 \), using Newton’s method. Draw a picture, to illustrate what is happening.

**ANS:** Want: the root of \( f(t) = t + \frac{3}{\sqrt[3]{t}} \).

\[
\begin{align*}
t_2 &= -t_1 \\
t_3 &= t_1 \\
t_4 &= -t_1 \\
t_5 &= t_1 \\
t_6 &= -t_1 \\
t_7 &= t_1
\end{align*}
\]