CALCULUS
Optimization
OLD
Among all pairs of real numbers whose sum is equal to 9, we seek the pair whose product is maximized. We examine the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 9 - x )</th>
<th>( x(9 - x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
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<tr>
<td>2</td>
<td>7</td>
<td>14</td>
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<tr>
<td>3</td>
<td>6</td>
<td>18</td>
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<tr>
<td>4</td>
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<td>2</td>
<td>14</td>
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<tr>
<td>8</td>
<td>1</td>
<td>8</td>
</tr>
</tbody>
</table>

a. Based on this table, guess a solution.

b. Find the exact solution by finding the maximum of all the values of

\[
f(x) = x(9 - x)
\]
at critical numbers.
Among all pairs of real numbers whose sum is equal to 9, we seek the pair whose product is maximized. We examine the following table.

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a. Based on this table, **guess** a solution.

**ANSWER:** a. $x$ is a number between 4 and 5, possibly 4.5. In that case, $9 - x = 9 - 4.5 = 4.5$ and the product is $(4.5)(4.5) = 20.25$.  


Among all pairs of real numbers whose sum is equal to 9, we seek the pair whose product is maximized.

b. Find the exact solution by finding the maximum of all the values of

\[ f(x) = x(9 - x) \] at critical numbers.

**ANSWER: b.**

\[ f'(x) = [1][9 - x] + [x][−1] = 9 - 2x \]

is positive on \( x < 4.5 \)

and negative on \( 4.5 < x \).

\( f \) is increasing on \( (−∞, 4.5] \)

and decreasing on \( [4.5, ∞) \).

\( f \) has a global maximum at 4.5.

The maximum value is

\[ f(4.5) = (4.5)(9 - 4.5) = 20.25. \]
0510-2. Among all pairs of real numbers whose difference is equal to 7, find the pair whose product is minimized.

**ANSWER:** Call the two numbers $x$ and $x + 7$. We seek to minimize $f(x) = x(x + 7)$.

$$f'(x) = [1][x + 7] + [x][1] = 2x + 7$$

is negative on $x < -3.5$

and positive on $-3.5 < x$.

$f$ is decreasing on $(-\infty, -3.5]$ and increasing on $[-3.5, \infty)$.

$f(x)$ attains a minimum at $x = -3.5$.

The two numbers are $-3.5$ and $-3.5 + 7 = 3.5$.
Their product is $(-3.5)(3.5) = -12.25$. 


0510-3. Maximize \( uv \)

subject to \( 3u + 4v = 100 \).

**ANS:** \( v = (100 - 3u)/4 \), so \( uv = u(100 - 3u)/4 \)

We wish to maximize \( f(u) = u(100 - 3u)/4 \).

\[
f'(u) = \frac{([1][100 - 3u] + [u][-3])}{4} = \frac{(100 - 6u)}{4} = \frac{(50 - 3u)}{2}
\]

is positive on \( u < 50/3 \)
and negative on \( 50/3 < u \).

\( f \) is increasing on \( (-\infty, 50/3] \)
and decreasing on \( [50/3, \infty) \).

\( f(u) \) attains a global maximum at \( u = 50/3 \).

\[
[v]_{u:50/3} = \left[\frac{(100 - 3u)}{4}\right]_{u:50/3} = 25/2
\]

**Solution:**

\( u = 50/3, \ v = 25/2, \ uv = 625/3 \)
Maximize \( r + s \) subject to \( r^6 + s^6 = 10 \).

**Answer:** \( s = (10 - r^6)^{1/6} \)

We wish to maximize \( f(r) = r + (10 - r^6)^{1/6} \).

**Domain:** \([-10^{1/6}, 10^{1/6}]\)

\[
f'(r) = 1 + \frac{1}{6}(10 - r^6)^{-5/6}(-6r^5) = 1 - (10 - r^6)^{-5/6}(r^5)
\]

\[f'(r) = 0 \text{ iff } r^5 = (10 - r^6)^{5/6} \text{ iff } r^6 = 10 - r^6 \]

\[\text{iff } r^6 = 5 \text{ iff } r = \pm 5^{1/6}\]

Critical numbers: \(-10^{1/6}, -5^{1/6}, 5^{1/6}, 10^{1/6}\)
Maximize \( r + s \)
subject to \( r^6 + s^6 = 10 \).

**ANSWER:** \( s = (10 - r^6)^{1/6} \)

We wish to maximize \( f(r) = r + (10 - r^6)^{1/6} \).

**Domain:** \([-10^{1/6}, 10^{1/6}]\)

**Critical numbers:** \(-10^{1/6}, -5^{1/6}, 5^{1/6}, 10^{1/6}\)

\[ f(\pm 10^{1/6}) = \pm 10^{1/6} \quad f(-5^{1/6}) = 0 \]

\[ f(5^{1/6}) = (2)(5^{1/6}) = (2^6 \cdot 5)^{1/6} \]

\( f(r) \) attains a maximum at \( r = 5^{1/6} \).

\[ [s]_{r \rightarrow 5^{1/6}} = [(10 - r^6)^{1/6}]_{r \rightarrow 5^{1/6}} = 5^{1/6} \]

**Solution:** \( r = 5^{1/6}, s = 5^{1/6}, r + s = (2)(5^{1/6}) \)
0510-5. The top of the object in the diagram is a semicircle. The total perimeter of the object is 15 meters. Find the dimensions $x$ and $y$ that maximize the area enclosed.
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**Answer:** radius of the semicircle is $x/2$.

The perimeter of the object is

$$
\pi \left(\frac{x}{2}\right) + (y - \left(\frac{x}{2}\right)) + x + (y - \left(\frac{x}{2}\right)) = 15
$$

Area enclosed:

$$
\left[ \pi \left(\frac{x}{2}\right)^2 / 2 \right] + \left[ (x)(y - \left(\frac{x}{2}\right)) \right] = \frac{\pi}{8} x^2 + xy - \frac{1}{2} x^2
$$
0510-5. The top of the object in the diagram is a semicircle. The total perimeter of the object is 15 meters. Find the dimensions $x$ and $y$ that maximize the area enclosed.

**ANSWER:**

$$15 = \pi \left( \frac{x}{2} \right) + 2y$$

$$y = \frac{(15 - \pi \left( \frac{x}{2} \right))}{2}$$

$$= \frac{15}{2} - \frac{\pi}{4}x$$

Area enclosed:

$$\left[ \frac{\pi \left( \frac{x}{2} \right)^2}{2} \right] + [(x)(y - \left( \frac{x}{2} \right))]$$

$$= \frac{\pi}{8}x^2 + xy - \frac{1}{2}x^2$$
0510-5. The top of the object in the diagram is a semicircle. The total perimeter of the object is 15 meters. Find the dimensions $x$ and $y$ that maximize the area enclosed.

**ANSWER:** \[ y = \frac{15}{2} - \frac{\pi}{4}x \]

area enclosed: \[ \frac{\pi}{8}x^2 + xy - \frac{1}{2}x^2 \]

\[ = \left( \frac{\pi}{8} - \frac{1}{2} \right) x^2 + x \left( \frac{15}{2} - \frac{\pi}{4}x \right) \]

\[ = \left( \frac{\pi}{8} - \frac{1}{2} - \frac{\pi}{4} \right) x^2 + \frac{15}{2}x \]

\[ = - \left( \frac{\pi}{8} + \frac{1}{2} \right) x^2 + \frac{15}{2}x \]
0510-5. The top of the object in the diagram is a semicircle. The total perimeter of the object is 15 meters. Find the dimensions $x$ and $y$ that maximize the area enclosed.

**ANSWER:**

\[ y = \frac{15}{2} - \frac{\pi}{4} x \]

Area enclosed: \[- \left( \frac{\pi}{8} + \frac{1}{2} \right) x^2 + \frac{15}{2} x\]

**Goal:** Maximize

\[ f(x) = - \left( \frac{\pi}{8} + \frac{1}{2} \right) x^2 + \frac{15}{2} x \]

\[ f'(x) = - \left( \frac{\pi}{4} + 1 \right) x + \frac{15}{2} \]

\[ f'(x) = 0 \text{ iff } x = \left( \frac{15/2}{\pi/4 + 1} \right) = \frac{30}{\pi + 4} \]
0510-5. The top of the object in the diagram is a semicircle. The total perimeter of the object is 15 meters. Find the dimensions $x$ and $y$ that maximize the area enclosed.

**ANSWER:** \[ y = \frac{15}{2} - \frac{\pi}{4}x \]

**Goal:** Maximize \[ f(x) = -\left(\frac{\pi}{8} + \frac{1}{2}\right)x^2 + \frac{15}{2}x \]

\[ f'(x) = -\left(\frac{\pi}{4} + 1\right)x + \frac{15}{2} \]

is pos. on $x < \frac{30}{\pi+4}$ and neg. on $\frac{30}{\pi+4} < x$.

\[ f'(x) = 0 \iff x = \left(\frac{15}{2}\right) / \left(\frac{\pi}{4} + 1\right) = \frac{30}{\pi+4} \]
The top of the object in the diagram is a semicircle. The total perimeter of the object is 15 meters. Find the dimensions $x$ and $y$ that maximize the area enclosed.

**Answer:**

$$y = \frac{15}{2} - \frac{\pi}{4}x$$

**Goal:** Maximize $f(x) = -\left(\frac{\pi}{8} + \frac{1}{2}\right)x^2 + \frac{15}{2}x$

$$f'(x) = -\left(\frac{\pi}{4} + 1\right)x + \frac{15}{2}$$

is pos. on $x < \frac{30}{\pi + 4}$ and neg. on $\frac{30}{\pi + 4} < x$.

$f$ is incr. on $(-\infty, \frac{30}{\pi + 4}]$ and decr. on $\left[\frac{30}{\pi + 4}, \infty\right)$.

$f(x)$ has its global max at $x = \frac{30}{\pi + 4}$.
The top of the object in the diagram is a semicircle. The total perimeter of the object is 15 meters. Find the dimensions \( x \) and \( y \) that maximize the area enclosed.

**ANSWER:**

\[
y = \frac{15}{2} - \frac{\pi}{4} x
\]

\[
[y]_{x \rightarrow \frac{30}{\pi + 4}} = \left[ \frac{15}{2} - \frac{\pi}{4} x \right]_{x \rightarrow \frac{30}{\pi + 4}}
\]

\[
= \frac{15}{2} - \left[ \frac{\pi}{4} \right] \left[ \frac{30}{\pi + 4} \right]
\]

\[
= \left[ \frac{15}{2} \right] \left[ 1 - \frac{\pi}{\pi + 4} \right]
\]

\( f \) is incr. on \( (-\infty, \frac{30}{\pi + 4}] \) and decre. on \( [\frac{30}{\pi + 4}, \infty) \).

\( f(x) \) has its global max at \( x = \frac{30}{\pi + 4} \).
0510-5. The top of the object in the diagram is a semicircle. The total perimeter of the object is 15 meters. Find the dimensions $x$ and $y$ that maximize the area enclosed.

**ANSWER:**

$$y = \frac{15}{2} - \frac{\pi}{4} x$$

$$[y]_{x: \rightarrow 30/(\pi+4)} = \left[ \frac{15}{2} \right] \left[ 1 - \frac{\pi}{\pi+4} \right]$$

$$= \left[ \frac{15}{2} \right] \left[ \frac{\pi+4}{\pi+4} - \frac{\pi}{\pi+4} \right]$$

$$= \left[ \frac{15}{2} \right] \left[ \frac{4}{\pi+4} \right] = \frac{30}{\pi+4}$$

$f$ is incr. on $(-\infty, \frac{30}{\pi+4}]$ and decr. on $[\frac{30}{\pi+4}, \infty)$.

$f(x)$ has its global max at $x = \frac{30}{\pi+4}$. 

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0510-5. The top of the object in the diagram is a semicircle. The total perimeter of the object is 15 meters. Find the dimensions $x$ and $y$ that maximize the area enclosed.

**Answer:** $f(x)$ has its global max at $x = \frac{30}{\pi+4}$.

$$[y]_{x: \rightarrow 30/(\pi+4)} = \frac{30}{\pi+4}$$

Area is maximized when $x = y = \frac{30}{\pi+4}$. 
0510-6. We plan to build a fenced area with six pens for livestock, as shown. We wish for the total length of the fencing (including both the perimeter and the interior fencing) to be 100 yards. **Find** the maximum possible enclosed area.
0510-6. We plan to build a fenced area with six pens for livestock, as shown. We wish for the total length of the fencing (including both the perimeter and the interior fencing) to be 100 yards. Find the maximum possible enclosed area.

**ANSWER:**
Max $xy$ subj. to $3x + 4y = 100$.

$$y = 25 - \frac{3}{4}x$$

$$f(x) = xy = x \left(25 - \frac{3}{4}x\right)$$

$$f'(x) = \left(25 - \frac{3}{4}x\right) + x \left(-\frac{3}{4}\right) = 25 - \frac{3}{2}x$$
We plan to build a fenced area with six pens for livestock, as shown. We wish for the total length of the fencing (including both the perimeter and the interior fencing) to be 100 yards. Find the maximum possible enclosed area.

**ANSWER:**

\[ f'(x) = 25 - \frac{3}{2}x \]

pos on \( x < \frac{50}{3} \), neg on \( \frac{50}{3} < x \)

\[ f(x) = xy = x \left( 25 - \frac{3}{4}x \right) \]

incr on \( x \leq \frac{50}{3} \), decr on \( \frac{50}{3} \leq x \)

global max at \( x = \frac{50}{3} \)
We plan to build a fenced area with six pens for livestock, as shown. We wish for the total length of the fencing (including both the perimeter and the interior fencing) to be 100 yards. Find the maximum possible enclosed area.

**ANSWER:**

\[ f(x) = xy = x \left( 25 - \frac{3}{4}x \right) \]

Global max at \( x = \frac{50}{3} \)

\[ f \left( \frac{50}{3} \right) = \left( \frac{50}{3} \right) \left( 25 - \frac{50}{4} \right) \]

\[ = \frac{625}{3} \text{ sq. yards} \]
We plan to build a fenced area with six pens for livestock, as shown. We wish for the total area enclosed (including all six pens) to be 600 feet\(^2\). Each foot of fencing costs one dollar. Find the minimum possible cost of the fencing.
0510-7. We plan to build a fenced area with six pens for livestock, as shown. We wish for the total area enclosed (including all six pens) to be 600 feet$^2$. Each foot of fencing costs one dollar. Find the minimum possible cost of the fencing.

**ANS:**

Minimize $3x + 4y$

subject to $xy = 600$.

$y = 600x^{-1}$

$3x + 4y = 3x + 2400x^{-1}$

$f(x) = 3x + 2400x^{-1}$

is the cost of the fencing.
We plan to build a fenced area with six pens for livestock, as shown. We wish for the total area enclosed (including all six pens) to be 600 feet². Each foot of fencing costs one dollar. Find the minimum possible cost of the fencing.

\[ f(x) = 3x + 2400x^{-1} \]

\[ f'(x) = 3 - 2400x^{-2} \]

is neg. on \( 0 < x < 20\sqrt{2} \)
and pos. on \( 20\sqrt{2} < x \).
We plan to build a fenced area with six pens for livestock, as shown. We wish for the total area enclosed (including all six pens) to be 600 feet$^2$. Each foot of fencing costs one dollar. **Find** the minimum possible cost of the fencing.

**ANS:** Cost of the fencing:

\[ f(x) = 3x + 2400x^{-1} \]

is decr. on \( 0 < x \leq 20\sqrt{2} \)

and incr. on \( 20\sqrt{2} \leq x \), so

has its global min at \( x = 20\sqrt{2} \).

\[ f'(x) = 3 - 2400x^{-2} \]

is neg. on \( 0 < x < 20\sqrt{2} \)

and pos. on \( 20\sqrt{2} < x \).
We plan to build a fenced area with six pens for livestock, as shown. We wish for the total area enclosed (including all six pens) to be 600 feet$^2$. Each foot of fencing costs one dollar. \textbf{Find} the minimum possible cost of the fencing.

\textbf{ANS:} Cost of the fencing:

\[ f(x) = 3x + 2400x^{-1} \]

has its global min at \( x = 20\sqrt{2} \).

\[
\begin{align*}
    f(20\sqrt{2}) &= 60\sqrt{2} + \left( \frac{120}{\sqrt{2}} \right) \\
    &= 60\sqrt{2} + \left( \frac{120\sqrt{2}}{2} \right) \\
    &= 60\sqrt{2} + 60\sqrt{2} \\
    &= 120\sqrt{2} \text{ dollars}
\end{align*}
\]
0510-8. We have 950 meters\(^2\) of material from which to build an open-topped cylindrical container. **Find** the radius and height that maximizes the volume enclosed.

**ANSWER:**

\[ r := \text{radius}, \ h := \text{height} \]
\[ \text{area of side} = 2\pi rh \]
\[ \text{area of base} = \pi r^2 \]
\[ \text{volume enclosed} = \pi r^2 h \]

\[ 2\pi rh + \pi r^2 = 950 \]

\[ h = \frac{475}{\pi} r^{-1} - \frac{1}{2} r \]

\[ f(r) = \pi r^2 \left( \frac{475}{\pi} r^{-1} - \frac{1}{2} r \right) \] is the volume enclosed.

**Goal:** Maximize \( f(r) \) on \( r > 0 \).
0510-8. We have 950 meters$^2$ of material from which to build an open-topped cylindrical container. Find the radius and height that maximizes the volume enclosed.

**Answer:** \( h = \frac{475}{\pi}r^{-1} - \frac{1}{2}r \)

\[ f(r) = \pi r^2 \left( \frac{475}{\pi}r^{-1} - \frac{1}{2}r \right) \] is the volume enclosed.

**Goal:** Maximize \( f(r) \) on \( r > 0 \).

\[ f(r) = 475r - \frac{\pi}{2}r^3 \]

\[ f'(r) = 475 - \frac{3\pi}{2}r^2 \] is positive on \( 0 < r < \sqrt{\frac{950}{3\pi}} \) and negative on \( \sqrt{\frac{950}{3\pi}} < r \).

\( f(r) \) is increasing on \( 0 < r \leq \sqrt{\frac{950}{3\pi}} \) and decreasing on \( \sqrt{\frac{950}{3\pi}} \leq r \), so attains its global max on \( r > 0 \) at \( r = \sqrt{\frac{950}{3\pi}} \).
0510-8. We have 950 meters$^2$ of material from which to build an open-topped cylindrical container. Find the radius and height that maximizes the volume enclosed.

**ANSWER:** \[ h = \frac{475}{\pi} r^{-1} - \frac{1}{2} r \]

\[ f(r) = \pi r^2 \left( \frac{475}{\pi} r^{-1} - \frac{1}{2} r \right) \] is the volume enclosed.

**Goal:** Maximize \( f(r) \) on \( r > 0 \).

\( f(r) \) attains its global max on \( r > 0 \) at \( r = \sqrt{\frac{950}{3\pi}} \).

\[
\begin{align*}
\left[ h \right]_{r \to \sqrt{\frac{950}{3\pi}}} &= \left[ \frac{475}{\pi} r^{-1} - \frac{1}{2} r \right]_{r \to \sqrt{\frac{950}{3\pi}}} \\
&= \frac{950}{2\pi} \sqrt{\frac{3\pi}{950}} - \frac{1}{2} \sqrt{\frac{950}{3\pi}} \\
&= \frac{\sqrt{3}}{2} \sqrt{\frac{950}{\pi}} - \frac{1}{2\sqrt{3}} \sqrt{\frac{950}{\pi}} \\
&= \left[ \frac{3}{2\sqrt{3}} - \frac{1}{2\sqrt{3}} \right] \sqrt{\frac{950}{\pi}} = \sqrt{\frac{950}{3\pi}}
\end{align*}
\]
0510-9. Let $L$ be the line $3x + 2y = 5$.

a. Find the point $P$ on $L$ closest to $(4,3)$,
   by minimizing $(x - 4)^2 + (y - 3)^2,$
   subject to $3x + 2y = 5$.

b. Find an equation of the line $N$ that is
   perpendicular to $L$ and passes through
   the point $(4,3)$.

c. Find the point $Q$ that is on the intersection
   of $L$ and $N$.

d. Sketch the graph of $L$, and then
   add in $(4,3)$, $P$, $Q$ and $N$. 
0510-9. Let \( L \) be the line \( 3x + 2y = 5 \). 

a. Find the point \( P \) on \( L \) closest to \((4, 3)\), by minimizing \((x - 4)^2 + (y - 3)^2\), subject to \(3x + 2y = 5\).

**ANSWER:** a.

\[
y = \frac{5}{2} - \frac{3}{2}x
\]

\[
f(x) := (x - 4)^2 + \left(\frac{5}{2} - \frac{3}{2}x - 3\right)^2
\]

\[
= (x - 4)^2 + \left(-\frac{1}{2} - \frac{3}{2}x\right)^2
\]

\[
= (x - 4)^2 + (-1)^2 \left(\frac{1}{2} + \frac{3}{2}x\right)^2
\]

\[
= (x - 4)^2 + \left(\frac{3}{2}x + \frac{1}{2}\right)^2
\]

**Goal:** Minimize \( f \).
0510-9. Let $L$ be the line $3x + 2y = 5$.

a. Find the point $P$ on $L$ closest to $(4, 3)$, by minimizing $(x - 4)^2 + (y - 3)^2$, subject to $3x + 2y = 5$.

**ANSWER:** a. Goal: Minimize $f$. 

$$y = \frac{5}{2} - \frac{3}{2}x$$

$$f(x) = (x - 4)^2 + \left(\frac{3}{2}x + \frac{1}{2}\right)^2$$

$$f'(x) = 2(x - 4)(1) + 2\left(\frac{3}{2}x + \frac{1}{2}\right)\left(\frac{3}{2}\right)$$

$$= 2(x - 4) + 3\left(\frac{3}{2}x + \frac{1}{2}\right)$$

$$= (2x - 8) + \left(\frac{9}{2}x + \frac{3}{2}\right)$$

$$= \left(\frac{4}{2}x - \frac{16}{2}\right) + \left(\frac{9}{2}x + \frac{3}{2}\right)$$

$$= \frac{13}{2}x - \frac{13}{2}$$
Let $L$ be the line $3x + 2y = 5$.

a. Find the point $P$ on $L$ closest to $(4, 3)$, by minimizing $(x - 4)^2 + (y - 3)^2$, subject to $3x + 2y = 5$.

**Answer:** a. Goal: Minimize $f$.

$$y = \frac{5}{2} - \frac{3}{2}x$$

$$f(x) = (x - 4)^2 + \left(\frac{3}{2}x + \frac{1}{2}\right)^2$$

$$f'(x) = \frac{13}{2}x - \frac{13}{2}$$

is negative on $x < 1$

and positive on $1 < x$.

$f(x)$ is decreasing on $x \leq 1$

and increasing on $1 \leq x$,

so $f(x)$ attains its global minimum at $x = 1$. 

Let $L$ be the line $3x + 2y = 5$.

a. Find the point $P$ on $L$ closest to $(4, 3)$, by minimizing $(x - 4)^2 + (y - 3)^2$, subject to $3x + 2y = 5$.

**ANSWER:** a. Goal: Minimize $f$. 

$$y = \frac{5}{2} - \frac{3}{2}x$$

$$f(x) = (x - 4)^2 + \left(\frac{3}{2}x + \frac{1}{2}\right)^2$$

$f(x)$ attains its global minimum at $x = 1$.

$$[y]_{x\rightarrow 1} = \left[\frac{5}{2} - \frac{3}{2}x\right]_{x\rightarrow 1} = \frac{5}{2} - \frac{3}{2} = 1$$

$P$ is the point $(1, 1)$. 

0510-9. Let $L$ be the line $3x + 2y = 5$.

b. Find an equation of the line $N$ that is perpendicular to $L$ and passes through the point $(4, 3)$.

c. Find the point $Q$ that is on the intersection of $L$ and $N$.

**ANSWER:**

b. $3x + 2y = 5 \iff y = -\frac{3}{2}x + \frac{5}{2}$, so $L$ has slope $-\frac{3}{2}$, so $N$ has slope $\frac{2}{3}$.

An eq’n for $N$: $y - 3 = \frac{2}{3}(x - 4)$

c. Goal: Solve $y - 3 = \frac{2}{3}(x - 4)$, $y = -\frac{3}{2}x + \frac{5}{2}$.

\[
\begin{align*}
6 \times & \quad \left( -\frac{3}{2}x + \frac{5}{2} - 3 = \frac{2}{3}(x - 4) \right) \\
-9x + 15 - 18 &= 4(x - 4) \\
-9x - 3 &= 4x - 16 \\
-13x &= -13 \\
x &= 1
\end{align*}
\]
0510-9. Let $L$ be the line $3x + 2y = 5$.

b. Find an equation of the line $N$ that is perpendicular to $L$ and passes through the point $(4, 3)$.

c. Find the point $Q$ that is on the intersection of $L$ and $N$.

**ANSWER:**

b. $3x + 2y = 5 \iff y = -\frac{3}{2}x + \frac{5}{2}$, so $L$ has slope $-\frac{3}{2}$, so $N$ has slope $\frac{2}{3}$.

An eq’n for $N$: $y - 3 = \frac{2}{3}(x - 4)$

c. Goal: Solve $y - 3 = \frac{2}{3}(x - 4)$, $y = -\frac{3}{2}x + \frac{5}{2}$.

$$[y]_{x: \rightarrow 1} = \left[-\frac{3}{2}x + \frac{5}{2}\right]_{x: \rightarrow 1} = -\frac{3}{2} + \frac{5}{2} = 1$$

$Q$ is the point $(1, 1)$.

$$x = 1$$
0510-9. Let $L$ be the line $3x + 2y = 5$.

d. Sketch the graph of $L$, and then add in $(4,3), \quad P, \quad Q$ and $N$.

**ANSWER:**
d. $P$ is the point $(1, 1)$.
$Q$ is the point $(1, 1)$.

an equation for $N$: $y - 3 = \frac{2}{3}(x - 4)$

The moral: The shortest route from a point to a line is on a perpendicular line.
0510-10. Let $E$ be the ellipse $9x^2 + 16y^2 = 25$. Find the dimensions of a rectangle inscribed ... whose area is maximal.

**ANSWER:** For any point $(u, v)$ on the ellipse, if $u \geq 0$ and $v \geq 0$, then we can form an inscribed rectangle with corners $(u, v), (u, -v), (-u, -v)$ and $(-u, v)$. The area of that rectangle is $(2u)(2v)$.

**Goal:** Maximize $(2u)(2v)$ over $u, v \geq 0$, subject to the constraint that $9u^2 + 16v^2 = 25$. 

![Diagram of the ellipse and inscribed rectangle]
Let $E$ be the ellipse $9x^2 + 16y^2 = 25$. Find the dimensions of a rectangle inscribed ... whose area is maximal.

**ANSWER:** Goal: Maximize $(2u)(2v)$ over $u, v \geq 0$, subject to $9u^2 + 16v^2 = 25$.

\[ v = \frac{1}{4}\sqrt{25 - 9u^2} \]

\[ (2u)(2v) = 4uv = u\sqrt{25 - 9u^2} \]

Let $f(u) = u\sqrt{25 - 9u^2}$.

Goal: Maximize $f(u)$ over $0 \leq u \leq \frac{5}{3}$. 
Let $E$ be the ellipse $9x^2 + 16y^2 = 25$. Find the dimensions of a rectangle inscribed . . . whose area is maximal.

**ANSWER:** Let $f(u) = u\sqrt{25 - 9u^2}$.

**Goal:** Maximize $f(u)$ over $0 \leq u \leq 5/3$.

$$f(u) = u\left[25 - 9u^2\right]^{1/2}$$

$$f'(u) = \left[25 - 9u^2\right]^{1/2} + u(1/2)\left[25 - 9u^2\right]^{-1/2}(-18u)$$

$$= \frac{25 - 9u^2 + u(1/2)(-18u)}{\left[25 - 9u^2\right]^{1/2}}$$

$$= \frac{25 - 9u^2 - 9u^2}{\sqrt{25 - 9u^2}} = \frac{25 - 18u^2}{\sqrt{25 - 9u^2}}$$
0510-10. Let \( E \) be the ellipse \( 9x^2 + 16y^2 = 25 \).
Find the dimensions of a rectangle inscribed . . . whose area is maximal.

**ANSWER:** Let \( f(u) = u\sqrt{25 - 9u^2} \).

**Goal:** Maximize \( f(u) \) over \( 0 \leq u \leq 5/3 \).

\[
f'(u) = \frac{25 - 18u^2}{\sqrt{25 - 9u^2}}
\]

is positive on \( 0 \leq u < \sqrt{25/18} \)
and negative on \( \sqrt{25/18} < u < 5/3 \).

\( f(u) \) is increasing on \( 0 \leq u \leq \sqrt{25/18} \)
and decreasing on \( \sqrt{25/18} \leq u \leq 5/3 \).
0510-10. Let \( E \) be the ellipse \( 9x^2 + 16y^2 = 25 \). Find the dimensions of a rectangle inscribed ... whose area is maximal.

**ANSWER:** Let \( f(u) = u\sqrt{25 - 9u^2} \).

**Goal:** Maximize \( f(u) \) over \( 0 \leq u \leq 5/3 \).

\( f(u) \) is increasing on \( 0 \leq u \leq \sqrt{25/18} \) and decreasing on \( \sqrt{25/18} \leq u \leq 5/3 \).

Then \( f(u) \) attains its global max at \( u = \sqrt{25/18} \).

\[
[v]_{u: \rightarrow \sqrt{25/18}} = \left[ \frac{1}{4} \sqrt{25 - 9u^2} \right]_{u: \rightarrow \sqrt{25/18}} = \frac{1}{4} \sqrt{25 - 9 \left( \frac{25}{18} \right)} = \frac{1}{4} \sqrt{25 - \frac{25}{2}} = \frac{1}{4} \sqrt{\frac{25}{2}} = \frac{5}{4\sqrt{2}}
\]
Let $E$ be the ellipse $9x^2 + 16y^2 = 25$. Find the dimensions of a rectangle inscribed...whose area is maximal.

**ANSWER:** Let $f(u) = u \sqrt{25 - 9u^2}$.

**Goal:** Maximize $f(u)$ over $0 \leq u \leq 5/3$.

Then $f(u)$ attains its global max at $u = \sqrt{25/18}$.

$$[v]_{u \to \sqrt{25/18}} = \frac{5}{4\sqrt{2}}.$$ 

Height: $\frac{5\sqrt{2}}{4}$, Width: $\frac{5\sqrt{2}}{3}$.
Let $S$ be a sphere of radius 10.
Find the radius and height of a right circular cylinder inscribed in $S$ whose volume is maximal.

**ANSWER:** Let $h$ be the height of the cylinder.
Let $r$ be its radius.

$$\text{volume of cylinder} = \pi r^2 h$$
Let $S$ be a sphere of radius 10. Find the radius and height of a right circular cylinder inscribed in $S$ whose volume is maximal.

**Answer:** Let $h$ be the height of the cylinder. Let $r$ be its radius.

Volume of cylinder $= \pi r^2 h$

By Pythagorean Theorem on the green triangle,

$$r^2 + \left(\frac{h}{2}\right)^2 = 10^2.$$

**Goal:** $h, r \geq 0$ that maximize $\pi r^2 h$, subject to

$$r^2 + \left(\frac{h}{2}\right)^2 = 10^2.$$
Let $S$ be a sphere of radius 10. Find the radius and height of a right circular cylinder inscribed in $S$ whose volume is maximal.

**ANSWER:** Goal: $h, r \geq 0$ that maximize $\pi r^2 h$, subject to $r^2 + (h/2)^2 = 10^2$.

For an easier approach, see the “2nd ANS” below.

\[
h = 2\sqrt{100 - r^2}
\]

\[
\pi r^2 h = 2\pi r^2 \sqrt{100 - r^2}
\]

Let $f(r) = 2\pi r^2 \sqrt{100 - r^2}$.

**Subgoal:** $r \in [0, 10]$ that maximizes $f(r)$. 

0510-11. Let $S$ be a sphere of radius 10. Find the radius and height of a right circular cylinder inscribed in $S$ whose volume is maximal.

**ANSWER:** Let \( f(r) = 2\pi r^2 \sqrt{100 - r^2} \).

**Subgoal:** \( r \in [0, 10] \) that maximizes \( f(r) \).

\[
f(r) = 2\pi r^2 \left[ 100 - r^2 \right]^{1/2}
\]

\[
f'(r) = 4\pi r \left[ 100 - r^2 \right]^{1/2} + 2\pi r^2 \left( \frac{1}{2} \right) \left[ 100 - r^2 \right]^{-1/2} [-2r]
\]

\[
= 4\pi r \left[ 100 - r^2 \right]^{1/2} + 2\pi r^2 \left( \frac{1}{2} \right) [-2r]
\]

\[
= 2\pi \left[ 200r - 2r^3 \right] + 2\pi \left[ -r^3 \right]
\]

\[
= \frac{2\pi \left[ 200r - 2r^3 \right] + 2\pi \left[ -r^3 \right]}{\sqrt{100 - r^2}}
\]
Let $S$ be a sphere of radius 10. Find the radius and height of a right circular cylinder inscribed in $S$ whose volume is maximal.

**ANSWER:** Let $f(r) = 2\pi r^2 \sqrt{100 - r^2}$.

**Subgoal:** $r \in [0, 10]$ that maximizes $f(r)$.

$$f'(r) = \frac{2\pi \left[ 200r - 2r^3 \right] + 2\pi \left[ -r^3 \right]}{\sqrt{100 - r^2}}$$

$$= \frac{2\pi \left[ 200r - 3r^3 \right]}{\sqrt{100 - r^2}} = \frac{2\pi r \left[ 200 - 3r^2 \right]}{\sqrt{100 - r^2}}$$
Let $S$ be a sphere of radius 10. Find the radius and height of a right circular cylinder inscribed in $S$ whose volume is maximal.

**Answer:** Let $f(r) = 2\pi r^2 \sqrt{100 - r^2}$.

**Subgoal:** $r \in [0, 10]$ that maximizes $f(r)$.

$$f'(r) = \frac{2\pi r \left[ 200 - 3r^2 \right]}{\sqrt{100 - r^2}}$$

is positive on $0 < r < \sqrt{200/3}$ and negative on $\sqrt{200/3} < r < 10$.

$f(r)$ is increasing on $0 \leq r \leq \sqrt{200/3}$ and decreasing on $\sqrt{200/3} \leq r \leq 10$.

On $r \in [0, 10]$, $f(r)$ attains its global maximum at $r = \sqrt{200/3}$. 
Let $S$ be a sphere of radius 10. Find the radius and height of a right circular cylinder inscribed in $S$ whose volume is maximal.

**ANSWER:** Let $f(r) = 2\pi r^2 \sqrt{100 - r^2}$. On $r \in [0, 10]$, $f(r)$ attains its global maximum at $r = \sqrt{200/3}$.

**Goal:** $h, r \geq 0$ that maximize $\pi r^2 h$, subject to $r^2 + (h/2)^2 = 10^2$.

\[
\left[h\right]_{r \to \sqrt{200/3}} = \left[2\sqrt{100 - r^2}\right]_{r \to \sqrt{200/3}} = 2\sqrt{(300/3) - (200/3)} = 2\sqrt{100/3} = \sqrt{400/3}
\]
Let $S$ be a sphere of radius 10. Find the radius and height of a right circular cylinder inscribed in $S$ whose volume is maximal.

2nd ANS: Goal: $h, r \geq 0$ that maximize $\pi r^2 h$, subject to $r^2 + (h/2)^2 = 10^2$.

\[ r^2 = 100 - (h^2/4) \]

\[ \pi r^2 h = \pi (100 - (h^2/4))h \]
\[ = (\pi/4)(400h - h^3) \]

Let $\phi(h) = (\pi/4)(400h - h^3)$.

Subgoal: $h \in [0, 20]$ that maximizes $\phi(h)$. 
Let $S$ be a sphere of radius 10. Find the radius and height of a right circular cylinder inscribed in $S$ whose volume is maximal.

2nd ANS: Goal: $h, r \geq 0$ that maximize $\pi r^2 h$, subject to $r^2 + (h/2)^2 = 10^2$.

Let $\phi(h) = (\pi/4)(400h - h^3)$.

Subgoal: $h \in [0, 20]$ that maximizes $\phi(h)$.

$\phi'(h) = (\pi/4)(400 - 3h^2)$ is positive on $0 \leq h < \sqrt{400/3}$ and negative on $\sqrt{400/3} < h \leq 20$.

Then $\phi(h)$ is increasing on $0 \leq h \leq \sqrt{400/3}$ and decreasing on $\sqrt{400/3} \leq h \leq 20$, so $\phi(h)$ attains its global max on $0 \leq h \leq 20$ at $h = \sqrt{400/3}$. 
Let $S$ be a sphere of radius 10. Find the radius and height of a right circular cylinder inscribed in $S$ whose volume is maximal.

2nd ANS: Goal: $h, r \geq 0$ that maximize $\pi r^2 h$, subject to $r^2 + (h/2)^2 = 10^2$.

$$h = \sqrt{400/3}$$

$$r = \sqrt{100 - (h^2/4)}$$

$$[r]_{h: \rightarrow \sqrt{400/3}} = \sqrt{100 - ((400/3)/4)}$$

$$= \sqrt{100 - (100/3)}$$

$$= \sqrt{200/3}$$

Let $\phi(h) = (\pi/4)(400h - h^3)$. Then $\phi(h)$ attains its global max on $0 \leq h \leq 20$ at $h = \sqrt{400/3}$. 64
We build two holding pens with 90 feet of fencing. One is in the shape of a circle. The shape of the other is an isosceles right triangle. Find the maximum total area that can be enclosed in the two pens.

**ANS:**

$r := \text{radius of circle}$  
$s := \text{length of leg of triangle}$

**Goal:** Max value of $A := \pi r^2 + \frac{1}{2} s^2$, over $r, s \geq 0$, subject to $2\pi r + s + s + s\sqrt{2} = 90$. 
We build two holding pens with 90 feet of fencing. One is in the shape of a circle. The shape of the other is an isosceles right triangle. Find the maximum total area that can be enclosed in the two pens.

**ANS: Goal:** Max value of $A := \pi r^2 + (1/2)s^2$, over $r, s \geq 0$, subject to $2\pi r + s + s + s\sqrt{2} = 90$.

$$2\pi r + (2 + \sqrt{2})s = 90$$

$r \geq 0$, so $(2 + \sqrt{2})s \leq 90$, so $s \leq \frac{90}{2+\sqrt{2}}$

$$r = \frac{1}{2\pi} \left[ 90 - (2 + \sqrt{2})s \right]$$

$$A = \pi \left( \frac{1}{2\pi} \left[ 90 - (2 + \sqrt{2})s \right] \right)^2 + \frac{1}{2}s^2$$

**Goal:** Max value of $A$ over $0 \leq s \leq \frac{90}{2+\sqrt{2}}$. 
We build two holding pens with 90 feet of fencing. One is in the shape of a circle. The shape of the other is an isosceles right triangle. Find the maximum total area that can be enclosed in the two pens.

**ANS:** \[ A = \pi \left( \frac{1}{2\pi} \left[ 90 - (2 + \sqrt{2})s \right] \right)^2 + \frac{1}{2} s^2 \]

**Goal:** Max value of \( A \) over \( 0 \leq s \leq \frac{90}{2+\sqrt{2}} \).

\( A \) is quadratic in \( s \), with positive leading coefficient, and is therefore concave up,

so its max value over \( 0 \leq s \leq \frac{90}{2+\sqrt{2}} \) occurs either at \( s = 0 \) or at \( s = \frac{90}{2+\sqrt{2}} \).
We build two holding pens with 90 feet of fencing. One is in the shape of a circle. The shape of the other is an isosceles right triangle. Find the maximum total area that can be enclosed in the two pens.

**ANS:** 
\[ A = \pi \left( \frac{1}{2\pi} \left[ 90 - \left( 2 + \sqrt{2} \right) s \right] \right)^2 + \frac{1}{2} s^2 \]

**Goal:** Max value of \( A \) over \( 0 \leq s \leq \frac{90}{2 + \sqrt{2}} \).

Occurs either at \( s = 0 \) or at \( s = \frac{90}{2 + \sqrt{2}} \).

\[
[A]_{s \to 0} = \pi \left( \frac{1}{2\pi} [90] \right)^2 = \pi \left( \frac{8100}{4\pi^2} \right) = \frac{2025}{\pi} \approx 645
\]

\[
[A]_{s \to \frac{90}{2 + \sqrt{2}}} = \pi \left( \frac{1}{2\pi} [0] \right)^2 + \frac{1}{2} \left( \frac{90}{2 + \sqrt{2}} \right)^2 \approx 347
\]

Maximum value is \( \frac{2025}{\pi} \approx 645 \) at \( s = 0 \).
We build two holding pens with 90 feet of fencing. One is in the shape of a circle. The shape of the other is an isosceles right triangle. Find the maximum total area that can be enclosed in the two pens.

**ANS:** \[ A = \pi \left( \frac{1}{2\pi} \left[ 90 - \left( 2 + \sqrt{2} \right) s \right] \right)^2 + \frac{1}{2} s^2 \]

**Goal:** Max value of \( A \) over \( 0 \leq s \leq \frac{90}{2+\sqrt{2}} \).

Maximum value is \( \frac{2025}{\pi} \approx 645 \) at \( s = 0 \).

**Note:** The solution is to put all of the fencing toward building a circular pen, and not to bother with a triangular pen at all. In fact, it turns out that a circle captures more area per length than any other shape. A result like that belongs to a branch of mathematics called the calculus of variations.
On an (8.5 in) \times (11 in) sheet of paper, we mark a green-dashed square in each corner, each one of side length $s$. After cutting out these squares, we fold along the red-dashed lines as shown, creating an open-topped box. What is the maximum volume of such a box?
0510-13. What is the max vol. of such a box?

**OLD**

**ANSWER:** Goal: Max value of 

\[ V(s) := (8.5 - 2s)(11 - 2s)s \]

over \( 0 \leq s \leq 8.5/2 \).

\[
V'(s) = (-2)(11 - 2s)s + (8.5 - 2s)(-2)s + (8.5 - 2s)(11 - 2s)(1)
\]

\[
= (-22s + 4s^2) + (-17s + 4s^2) + (93.5 - 39s + 4s^2)
\]

\[
= 93.5 - 78s + 12s^2
\]

\[
= 12s^2 - 78s + 93.5
\]
What is the max vol. of such a box?

**ANSWER:** Goal: Max value of

\[ V(s) := (8.5 - 2s)(11 - 2s)s \]

over \( 0 \leq s \leq 8.5/2 \).

\[ V'(s) = 12s^2 - 78s + 93.5 \]

\[ \alpha := \frac{78 - \sqrt{78^2 - 4(12)(93.5)}}{2(12)} \]

\[ = \frac{78 - \sqrt{1596}}{24} \approx 1.5854 \]

\[ \beta := \frac{78 + \sqrt{78^2 - 4(12)(93.5)}}{2(12)} \]

\[ = \frac{78 + \sqrt{1596}}{24} \approx 4.9146 \]

\[ 0 < \alpha < 8.5/2 < \beta \]

\( V'(s) \) is pos. on \( 0 \leq s < \alpha \)

and neg. on \( \alpha < s \leq 8.5/2 \).
0510-13. What is the max vol. of such a box?

**Answer:** 

**Goal:** Max value of $V(s) := (8.5 - 2s)(11 - 2s)s$

over $0 \leq s \leq 8.5/2$.

$$V'(s) = 12s^2 - 78s + 93.5$$

$$\alpha = \frac{78 - \sqrt{1596}}{24} \approx 1.5854$$

$V'(s)$ is pos. on $0 \leq s < \alpha$

and neg. on $\alpha < s \leq 8.5/2$.

$V(s)$ is incr. on $0 \leq s \leq \alpha$

and decre. on $\alpha \leq s \leq 8.5/2$.

Over $0 \leq s \leq 8.5/2$,

$V(s)$ attains its global max at $s = \alpha$.

Max value $= V(\alpha) = (8.5 - 2\alpha)(11 - 2\alpha)\alpha$

$\approx (5.3292)(7.8292)(1.5854)$

$\approx 66.1482$ cubic inches.
0510-14. I must travel from point $P$ on land to point $R$ in the water. My land speed is 5 mph. My water speed is 3 mph. I choose the point $Q_0$ so as to minimize travel time. Define $\theta_0$ and $\phi_0$ as shown in the picture and show that \[
\frac{\sin \theta_0}{\sin \phi_0} = \frac{5}{3}.
\]

Note: This is called Snell’s Law.

Note: You may assume that there exists a unique $Q_0$ that minimizes the travel time from $P$ to $R$. 
Goal:
Prove that \( \frac{\sin \theta_0}{\sin \phi_0} = \frac{5}{3} \).

Replace \( Q_0 \) by a variable point \( Q \), replace \( \theta_0, \phi_0 \) by variable angles \( \theta, \phi \), add coordinates, and minimize travel time...
Define $u, v, w, x$ by

\[ P = (0, u), \quad Q = (x, 0), \quad R = (v, -w). \]

Let $f(x) :=$ travel time from $P$ to $R$ via $Q$.

Suppose $f(x)$ attains its global min at $x = x_0$, whence $Q_0 = (x_0, 0)$.

\[
\sin \theta = \frac{x}{\sqrt{u^2 + x^2}} \quad \text{and} \quad \sin \phi = \frac{v - x}{\sqrt{w^2 + (v - x)^2}}
\]
\[ f(x) := \frac{\text{travel time}}{5} = \frac{\sqrt{x^2 + u^2} + \sqrt{(v - x)^2 + w^2}}{3} \]

attains its global minimum at \( x = x_0 \).

\[ \sin \theta = \frac{x}{\sqrt{u^2 + x^2}} \]
\[ \sin \phi = \frac{v - x}{\sqrt{w^2 + (v - x)^2}} \]

Goal: Prove that \( \frac{\sin \theta_0}{\sin \phi_0} = \frac{5}{3} \),

i.e., that
\[ \left[ \frac{x_0}{\sqrt{u^2 + x_0^2}} \right] \left[ \frac{\sqrt{w^2 + (v - x_0)^2}}{v - x_0} \right] = \frac{5}{3} \].

\[ P = (0, u) \]
\[ Q = (x, 0) \]
\[ Q_0 = (x_0, 0) \]
\[ R = (v, -w) \]
\[ f(x) = \frac{\sqrt{x^2 + u^2}}{5} + \frac{\sqrt{(v - x)^2 + w^2}}{3} \]

attains its global minimum at \( x = x_0 \).

**Goal:** Prove that

\[
\begin{bmatrix}
\frac{x_0}{\sqrt{u^2 + x_0^2}} \\
\frac{\sqrt{w^2 + (v - x_0)^2}}{v - x_0}
\end{bmatrix}
= \frac{5}{3}.
\]

\[ f(x) = \frac{\left[ x^2 + u^2 \right]^{1/2}}{5} + \frac{\left[ (v - x)^2 + w^2 \right]^{1/2}}{3} \]

\[ f'(x) = \frac{1/2}{5} \left[ x^2 + u^2 \right]^{-1/2} \left[ 2x \right] \\
+ \frac{1/2}{3} \left[ (v - x)^2 + w^2 \right]^{-1/2} \left[ 2(v - x)(-1) \right] \]
\[ f'(x) = \frac{\left[\frac{1}{2}\right] \left[ x^2 + u^2 \right]^{-1/2} [2x]}{5} + \frac{\left[\frac{1}{2}\right] \left[ (v - x)^2 + w^2 \right]^{-1/2} [2(v - x)(-1)]}{3} \]

\[ = \frac{x}{5 \left[ x^2 + u^2 \right]^{1/2}} - \frac{v - x}{3 \left[ (v - x)^2 + w^2 \right]^{1/2}} \]
\[ f'(x) = \frac{x}{5 \left[ x^2 + u^2 \right]^{1/2}} - \frac{v - x}{3 \left[ (v - x)^2 + w^2 \right]^{1/2}} \]

Note: The problem does not require us to prove that \( f \) has a unique global minimum, but we will do so, at the end of this writeup. For now, we assume that \( f(x) \) has a unique global minimum, and that this global minimum occurs at \( x = x_0 \).

Then \( f'(x_0) = 0 \), and we wish to prove that
\[
\left[ \frac{x_0}{\sqrt{u^2 + x_0^2}} \right] \left[ \frac{\sqrt{w^2 + (v - x_0)^2}}{v - x_0} \right] = \frac{5}{3}.
\]
0510-14. ANSWER:

\[ f'(x) = \frac{x}{5 \left( x^2 + u^2 \right)^{1/2}} - \frac{v - x}{3 \left( (v - x)^2 + w^2 \right)^{1/2}} \]

\[ f'(x_0) = 0, \text{ and we wish to prove that} \]

\[ \left[ \frac{x_0}{\sqrt{u^2 + x_0^2}} \right] \left[ \frac{\sqrt{w^2 + (v - x_0)^2}}{v - x_0} \right] = \frac{5}{3}. \]

\[ 0 = f'(x_0) = \frac{x_0}{5 \left( x_0^2 + u^2 \right)^{1/2}} - \frac{v - x_0}{3 \left( (v - x_0)^2 + w^2 \right)^{1/2}} \]

\[ \frac{x_0}{5 \left( x_0^2 + u^2 \right)^{1/2}} = \frac{v - x_0}{3 \left( (v - x_0)^2 + w^2 \right)^{1/2}} \]
\[ f'(x) = \frac{x}{5 \left[ x^2 + u^2 \right]^{1/2}} - \frac{v - x}{3 \left[ (v - x)^2 + w^2 \right]^{1/2}} \]

\[ f'(x_0) = 0, \text{ and we wish to prove that} \]
\[ \left[ \frac{x_0}{\sqrt{u^2 + x_0^2}} \right] \left[ \frac{\sqrt{w^2 + (v - x_0)^2}}{v - x_0} \right] = \frac{5}{3}. \]

\[ \frac{x_0}{5 \left[ x_0^2 + u^2 \right]^{1/2}} = \frac{v - x_0}{3 \left[ (v - x_0)^2 + w^2 \right]^{1/2}} \]
\[ \left[ \frac{x_0}{\left[ x_0^2 + u^2 \right]^{1/2}} \right] \left[ \frac{\left[ (v - x_0)^2 + w^2 \right]^{1/2}}{v - x_0} \right] = \frac{5}{3} \]

\[ \text{QED} \]
0510-14. ANSWER:

\[ f'(x) = \frac{x}{5 \left[ x^2 + u^2 \right]^{1/2}} - \frac{v - x}{3 \left[ (v - x)^2 + w^2 \right]^{1/2}} \]

**Proof that \( f \) has a unique global minimum:**

\[ f''(x) = \frac{\left[ x^2 + u^2 \right]^{1/2} \left[ 1 - [x][\frac{1}{2}(x^2 + u^2)^{-1/2}(2x)] \right]}{5 \left[ x^2 + u^2 \right]} \]

\[ - \frac{\left[ (v - x)^2 + w^2 \right]^{1/2} \left[ -1 - [v - x][(1/2)((v - x)^2 + w^2)^{-1/2}(2(v - x)(-1))] \right]}{3 \left[ (v - x)^2 + w^2 \right]} \]

\[ = \frac{\left[ x^2 + u^2 \right] - [x][x]}{5 \left[ x^2 + u^2 \right]^{3/2}} \]

\[ - \frac{\left[ (v - x)^2 + w^2 \right] \left[ -1 - [v - x][(v - x)(-1)] \right]}{3 \left[ (v - x)^2 + w^2 \right]^{3/2}} \]
Proof that $f$ has a unique global minimum:

$$f''(x) = \frac{[x^2 + u^2] - [x][x]}{5 \left[ x^2 + u^2 \right]^{3/2}}$$

$$+ \frac{[u - x]^2 + w^2} {3 \left[ (u - x)^2 + w^2 \right]^{3/2}} \left[ +1 \right] - \frac{[v - x][(v - x)(+1)]} {3 \left[ (v - x)^2 + w^2 \right]^{3/2}}$$

$$= \frac{[x^2 + u^2] - [x^2]}{5 \left[ x^2 + u^2 \right]^{3/2}} + \frac{[u - x]^2 + w^2} {3 \left[ (u - x)^2 + w^2 \right]^{3/2}} - \frac{[(v - x)]^2}{3 \left[ (v - x)^2 + w^2 \right]^{3/2}}$$

$$= \frac{u^2}{5 \left[ x^2 + u^2 \right]^{3/2}} + \frac{w^2}{3 \left[ (u - x)^2 + w^2 \right]^{3/2}} > 0 + 0 = 0,$$

so $f$ is concave up.
Proof that $f$ has a unique global minimum:

$f$ is concave up.

\[ f(x) = \frac{\sqrt{x^2 + u^2}}{5} + \frac{\sqrt{(v - x)^2 + w^2}}{3} \]

\[ \lim_{x \to -\infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty, \]

so, by the Extreme Value Theorem, $f$ attains a global minimum.

Because $f$ is concave up, $f$ has at most one global minimum. QED