CALCULUS
Newton’s method
OLD
We wish to solve $x^3 + 3x - 4 = 0$. Starting with an initial guess of $x_1 = 1$, compute the next two guesses, $x_2$ and $x_3$, to at least four decimals, using Newton’s method.

**ANSWER:**

$$x_2 = x_1 - \frac{x_1^3 + 3x_1 - 4}{3x_1^2 + 3}$$

$$= 1 - \frac{1 + 3 - 4}{3 + 3} = 1$$

$$x_3 = x_2 - \frac{x_2^3 + 3x_2 - 4}{3x_2^2 + 3}$$

$$= 1 - \frac{1 + 3 - 4}{3 + 3} = 1$$
We wish to solve \( x^3 + 3x - 5 = 0 \). Starting with an initial guess of \( x_1 = 1 \), compute the next two guesses, \( x_2 \) and \( x_3 \), to at least four decimals, using Newton’s method.

**ANSWER:**

\[
x_2 = x_1 - \frac{x_1^3 + 3x_1 - 5}{3x_1^2 + 3} = 1 - \frac{1 + 3 - 5}{3 + 3} = \frac{7}{6}
\]

\[
x_3 = x_2 - \frac{x_2^3 + 3x_2 - 4}{3x_2^2 + 3} = \frac{7}{6} - \frac{(\frac{7}{6})^3 + 3(\frac{7}{6}) - 4}{3 (\frac{7}{6})^2 + 3} \approx 1.0131
\]
We wish to solve \( x^3 - 5 = 0 \).

Starting with an initial guess of \( x_1 = 1 \), compute the next two guesses, \( x_2 \) and \( x_3 \), to at least four decimals, using Newton’s method.

**ANSWER:**

\[
x_2 = x_1 - \frac{x_1^3 - 5}{3x_1^2} = 1 - \frac{1 - 5}{3} = \frac{7}{3}
\]

\[
x_3 = x_2 - \frac{x_2^3 - 5}{3x_2^2} = \frac{7}{3} - \frac{\left(\frac{7}{3}\right)^3 - 5}{3 \left(\frac{7}{3}\right)^2} \approx 1.8617
\]
0530-4. We wish to solve \( x^5 + 3x - 4 = 0 \).
Starting with an initial guess of \( x_1 = 1 \),
compute the next two guesses, \( x_2 \) and \( x_3 \), to
at least four decimals, using Newton’s method.

**ANSWER:**

\[
x_2 = x_1 - \frac{x_1^5 + 3x_1 - 4}{5x_1^4 + 3} = 1 - \frac{1 + 3 - 4}{5 + 3} = 1
\]

\[
x_3 = x_2 - \frac{x_2^5 + 3x_2 - 4}{5x_2^4 + 3} = 1 - \frac{1 + 3 - 4}{5 + 3} = 1
\]
We wish to solve \( x^2 - 4 = 0 \). Starting with an initial guess of \( x_1 = 1 \), compute the next two guesses, \( x_2 \) and \( x_3 \), to at least four decimals, using Newton’s method.

**ANSWER:**

\[
x_2 = x_1 - \frac{x_1^2 - 4}{2x_1} = 1 - \frac{1 - 4}{2} = \frac{5}{2}
\]

\[
x_3 = x_2 - \frac{x_2^2 - 4}{2x_2} = \frac{5}{2} - \frac{\left(\frac{5}{2}\right)^2 - 4}{2 \left(\frac{5}{2}\right)} = 2.05
\]
0530-6. Using Newton’s method, calculate $\sqrt[3]{7}$, to five decimal places.

ANS: Want: the unique root of $f(x) = x^3 - 7$.

$$x_{n+1} = x_n - \frac{x_n^3 - 7}{3x_n^2}$$

Let’s try $x_1 := 1$.

$$x_2 = x_1 - \frac{x_1^3 - 7}{3x_1^2} = 1 - \frac{1^3 - 7}{3.12} = 3$$

$$x_3 = x_2 - \frac{x_2^3 - 7}{3x_2^2} = 3 - \frac{3^3 - 7}{3.32} \approx 2.25926$$

$$x_4 \approx 1.96331$$  \hspace{1cm}  $x_6 \approx 1.91293$

$$x_5 \approx 1.91421$$  \hspace{1cm}  $x_7 \approx 1.91293$
0530-7. Find the unique solution to $2x = \cos x$, to five decimal places.

**ANSWER:** Want: the root of $f(x) = 2x - \cos x$.

$$x_{n+1} = x_n - \frac{2x_n - \cos x_n}{2 + \sin x_n}$$

Let's try $x_1 := 0$.

$$x_2 = 0 - \frac{2 \cdot 0 - \cos 0}{2 + \sin 0} = -\frac{-1}{2} = 0.5$$

$x_3 \doteq 0.45063$

$x_4 \doteq 0.45018$

$x_5 \doteq 0.45018$
0530-8. Find a solution to \( \tan x = 2x \),
to five decimal places,
by applying Newton’s method to
\[ f(x) = 2x - (\tan x) \], with \( x_1 = 1 \).

**ANSWER:** Want: a root of \( f(x) = 2x - \tan x \).

\[
x_{n+1} = x_n - \frac{2x_n - \tan x_n}{2 - \sec^2 x_n}
\]

Let’s try \( x_1 := 1 \).

\[
x_2 = 1 - \frac{2 \cdot 1 - \tan 1}{2 - \sec^2 1} \approx 1.31048
\]

\[
x_3 \approx 1.22393
\]

\[
x_4 \approx 1.17605
\]

\[
x_5 \approx 1.16593
\]

\[
x_6 \approx 1.16556
\]

\[
x_7 \approx 1.16556
\]
0530-9. We wish to solve \( \sin t = 0 \).

Let \( t_1 > 0 \) satisfy \( \tan t_1 = 2t_1 \). In 0530-8, \( t_1 \) is found, to five decimals. Starting with this initial guess \( t_1 \), compute the next six guesses, \( t_2, \ldots, t_7 \), using Newton’s method. 

**Draw** a picture, to illustrate what is happening.

**ANSWER:** Want: the root of \( f(t) = \sin t \).

\[
\begin{align*}
\tan t_1 &= 2t_1, \text{ so } \tan(-t_1) = -2t_1. \\
t_{n+1} &= t_n - \frac{\sin t_n}{\cos t_n} = t_n - (\tan t_n)
\end{align*}
\]

\[
\begin{align*}
t_2 &= t_1 - (\tan t_1) = t_1 - (2t_1) = -t_1 \\
t_3 &= t_2 - (\tan t_2) = -t_1 - (-2t_1) = t_1 \\
t_4 &= t_3 - (\tan t_3) = t_1 - (2t_1) = -t_1 \\
t_5 &= t_4 - (\tan t_4) = -t_1 - (-2t_1) = t_1 \\
t_6 &= t_5 - (\tan t_5) = t_1 - (2t_1) = -t_1 \\
t_7 &= t_6 - (\tan t_6) = -t_1 - (-2t_1) = t_1
\end{align*}
\]
We wish to solve \( \sin t = 0 \).

Let \( t_1 > 0 \) satisfy \( \tan t_1 = 2t_1 \). In 0530-8, \( t_1 \) is found, to five decimals. Starting with this initial guess \( t_1 \), compute the next six guesses, \( t_2, \ldots, t_7 \), using Newton’s method.

Draw a picture, to illustrate what is happening.

**ANSWER:**

Want: the root of \( f(t) = \sin t \).

\[
\begin{align*}
t_2 &= t_1 - (\tan t_1) = t_1 - (2t_1) = -t_1 \\
t_3 &= t_2 - (\tan t_2) = -t_1 - (2(-t_1)) = t_1
\end{align*}
\]

etc., etc., etc., \ldots