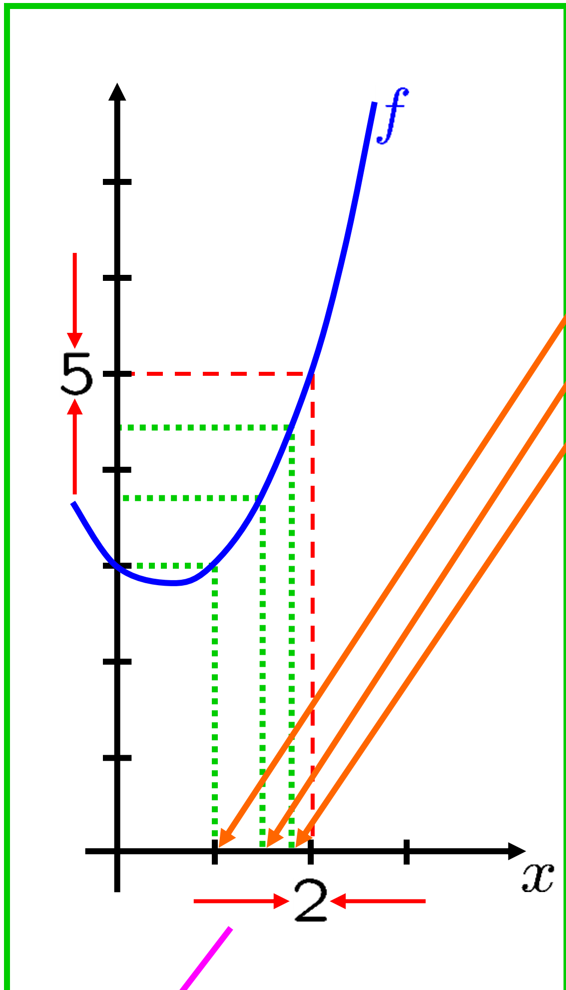
# CALCULUS
Limits

As $x \to 2^-$, $f(x) \to 5^-$
As $x \to 2^+$, $f(x) \to 5^+$

$f(x) = x^2 - x + 3$

Understood: $f : \mathbb{R} \to \mathbb{R}$




| $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|
| 1.0 | 3.000000 | 3.0 | 9.000000 |
| 1.5 | 3.750000 | 2.5 | 6.750000 |
| 1.8 | 4.440000 | 2.2 | 5.640000 |
| 1.9 | 4.710000 | 2.1 | 5.310000 |
| 1.95 | 4.852500 | 2.05 | 5.152500 |
| 1.99 | 4.970100 | 2.01 | 5.030100 |
| 1.995 | 4.985025 | 2.005 | 5.015025 |
| 1.999 | 4.997001 | 2.001 | 5.003001 |

$2^-$ $5^-$ $2^+$ $5^+$

INTUITIVE DEFINITION:

$$\lim_{x \to 2} f(x) = 5$$

because $f(2) = 5$

means:

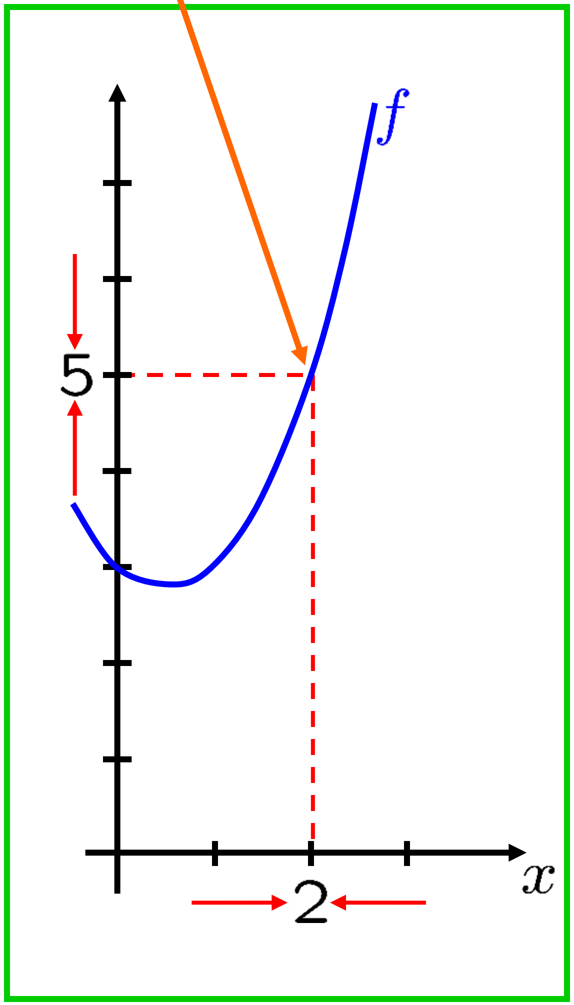
if $x$ is close to 2 (on left or right),
but not actually *equal* to 2,
then $f(x)$ is close to 5.

irrelevent
in *this* case,
but. . .

$x \to 2^=$

$$g(x) = \frac{(x^2 - x + 3)(x-2)}{x-2} \underset{x \neq 2}{=} x^2 - x + 3 = f(x)$$

Understood: $g : \mathbb{R}\backslash\{2\} \to \mathbb{R}$




| $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|
| 1.0 | 3.000000 | 3.0 | 9.000000 |
| 1.5 | 3.750000 | 2.5 | 6.750000 |
| 1.8 | 4.440000 | 2.2 | 5.640000 |
| 1.9 | 4.710000 | 2.1 | 5.310000 |
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INTUITIVE DEFINITION:

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means:

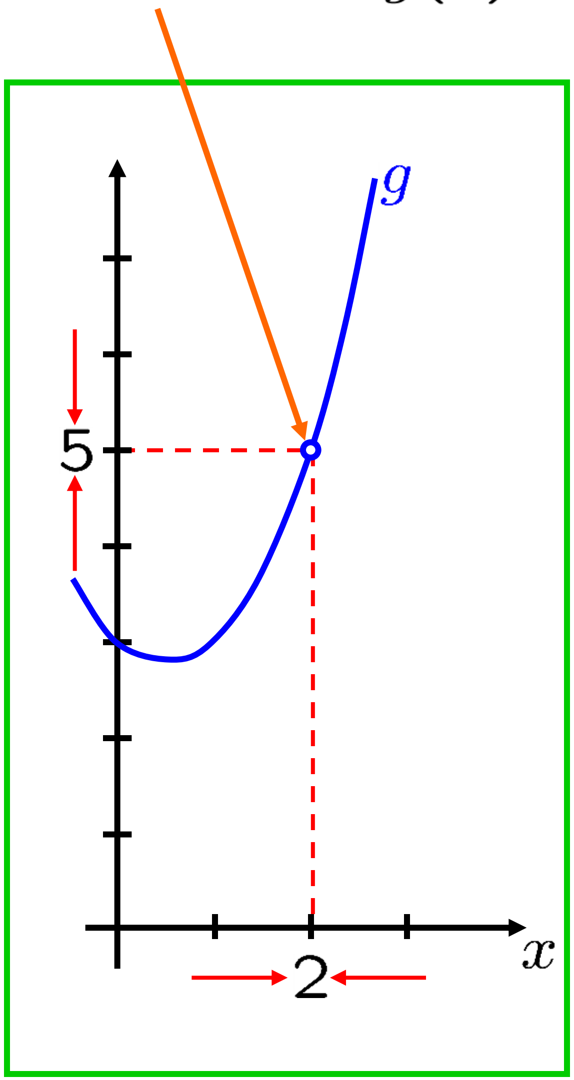
if $x$ is close to 2 (on left or right),

but not actually *equal* to 2, irrelevent in *this* case, but. . .

then $f(x)$ is close to 5.

$$g(x) = \frac{(x^2 - x + 3)(x-2)}{x-2} \underset{x \neq 2}{=} x^2 - x + 3 = f(x)$$

Understood: $g : \mathbb{R}\backslash\{2\} \to \mathbb{R}$




| $x$ | $g(x)$ | $x$ | $g(x)$ |
|---|---|---|---|
| 1.0 | 3.000000 | 3.0 | 9.000000 |
| 1.5 | 3.750000 | 2.5 | 6.750000 |
| 1.8 | 4.440000 | 2.2 | 5.640000 |
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INTUITIVE DEFINITION:

$$\lim_{\substack{x \to 2 \\ \neq}} g(x) = 5$$

means:

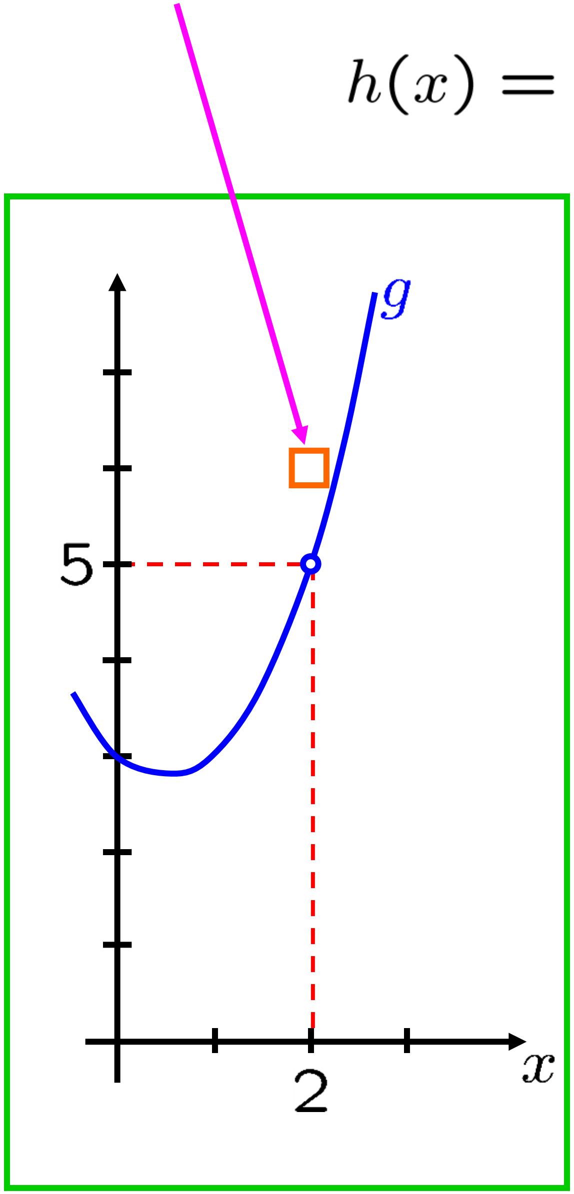
if $x$ is close to 2 (on left or right),

CRUCIAL → but not actually *equal* to 2,

then $g(x)$ is close to 5.

$$h(x) = \begin{cases} x^2 - x + 3, & \text{if } x \neq 2 \\ 6, & \text{if } x = 2 \end{cases}$$

Understood: $h : \mathbb{R} \to \mathbb{R}$




| $x$ | $g(x)$ |
|---|---|
| 1.0 | 3.000000 |
| 1.5 | 3.750000 |
| 1.8 | 4.440000 |
| 1.9 | 4.710000 |
| 1.95 | 4.852500 |
| 1.99 | 4.970100 |
| 1.995 | 4.985025 |
| 1.999 | 4.997001 |

| $x$ | $g(x)$ |
|---|---|
| 3.0 | 9.000000 |
| 2.5 | 6.750000 |
| 2.2 | 5.640000 |
| 2.1 | 5.310000 |
| 2.05 | 5.152500 |
| 2.01 | 5.030100 |
| 2.005 | 5.015025 |
| 2.001 | 5.003001 |

INTUITIVE DEFINITION:

$$\lim_{\substack{x \to 2 \\ \neq}} g(x) = 5$$

means:

if $x$ is close to 2 (on left or right),

but not actually *equal* to 2,

then $g(x)$ is close to 5.

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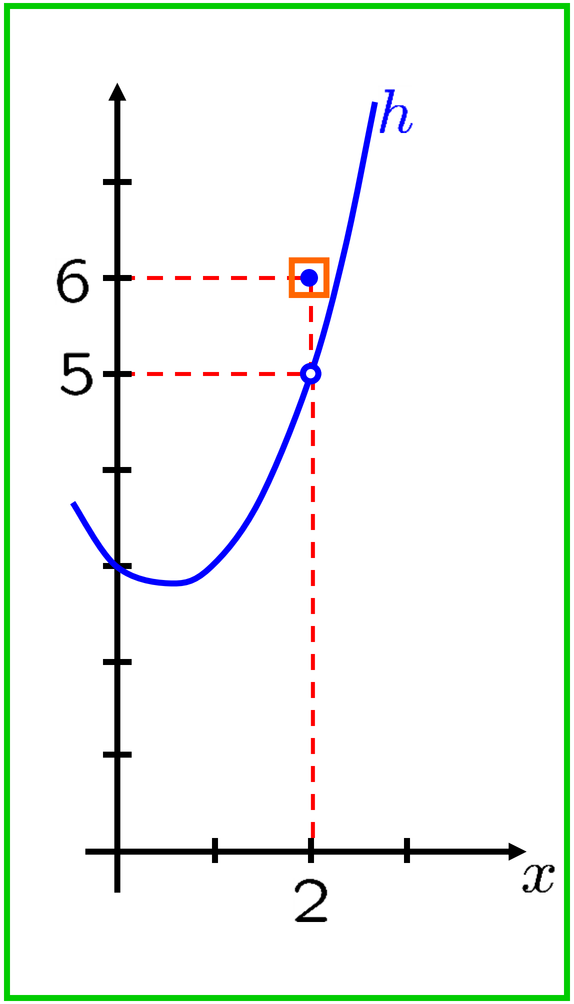
Understood: $h : \mathbb{R} \to \mathbb{R}$




| $x$ | $h(x)$ | $x$ | $h(x)$ |
|---|---|---|---|
| 1.0 | 3.000000 | 3.0 | 9.000000 |
| 1.5 | 3.750000 | 2.5 | 6.750000 |
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| 1.99 | 4.970100 | 2.01 | 5.030100 |
| 1.995 | 4.985025 | 2.005 | 5.015025 |
| 1.999 | 4.997001 | 2.001 | 5.003001 |

INTUITIVE DEFINITION:

$$\lim_{x \to 2} h(x) = 5$$

$\neq$ NONSTANDARD

To be discussed later. . .

means:

if $x$ is close to 2 (on left or right),

CRUCIAL → but not actually *equal* to 2,

irrelevent in *this* case, but. . .

then $h(x)$ is close to 5

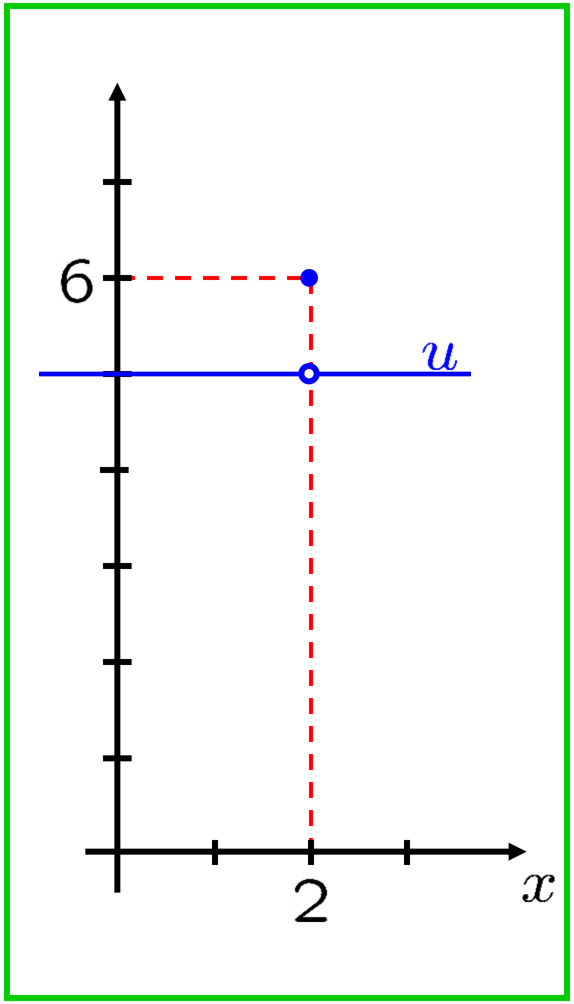
How close is close?

and possibly equal to 5.

$$u(x) = \begin{cases} 5, \text{ if } x \neq 2 \\ 6, \text{ if } x = 2 \end{cases}$$

As $x \underset{\neq}{\to} 2$, $u(x) \to 5$

Understood: $u : \mathbb{R} \to \mathbb{R}$




| $x$ | $u(x)$ | $x$ | $u(x)$ |
|---|---|---|---|
| 1.0 | 5.000000 | 3.0 | 5.000000 |
| 1.5 | 5.000000 | 2.5 | 5.000000 |
| 1.8 | 5.000000 | 2.2 | 5.000000 |
| 1.9 | 5.000000 | 2.1 | 5.000000 |
| 1.95 | 5.000000 | 2.05 | 5.000000 |
| 1.99 | 5.000000 | 2.01 | 5.000000 |
| 1.995 | 5.000000 | 2.005 | 5.000000 |
| 1.999 | 5.000000 | 2.001 | 5.000000 |

INTUITIVE DEFINITION:

$$\lim_{x \to 2} u(x) = 5$$

means:

If $x$ is close to 2 (on left or right),

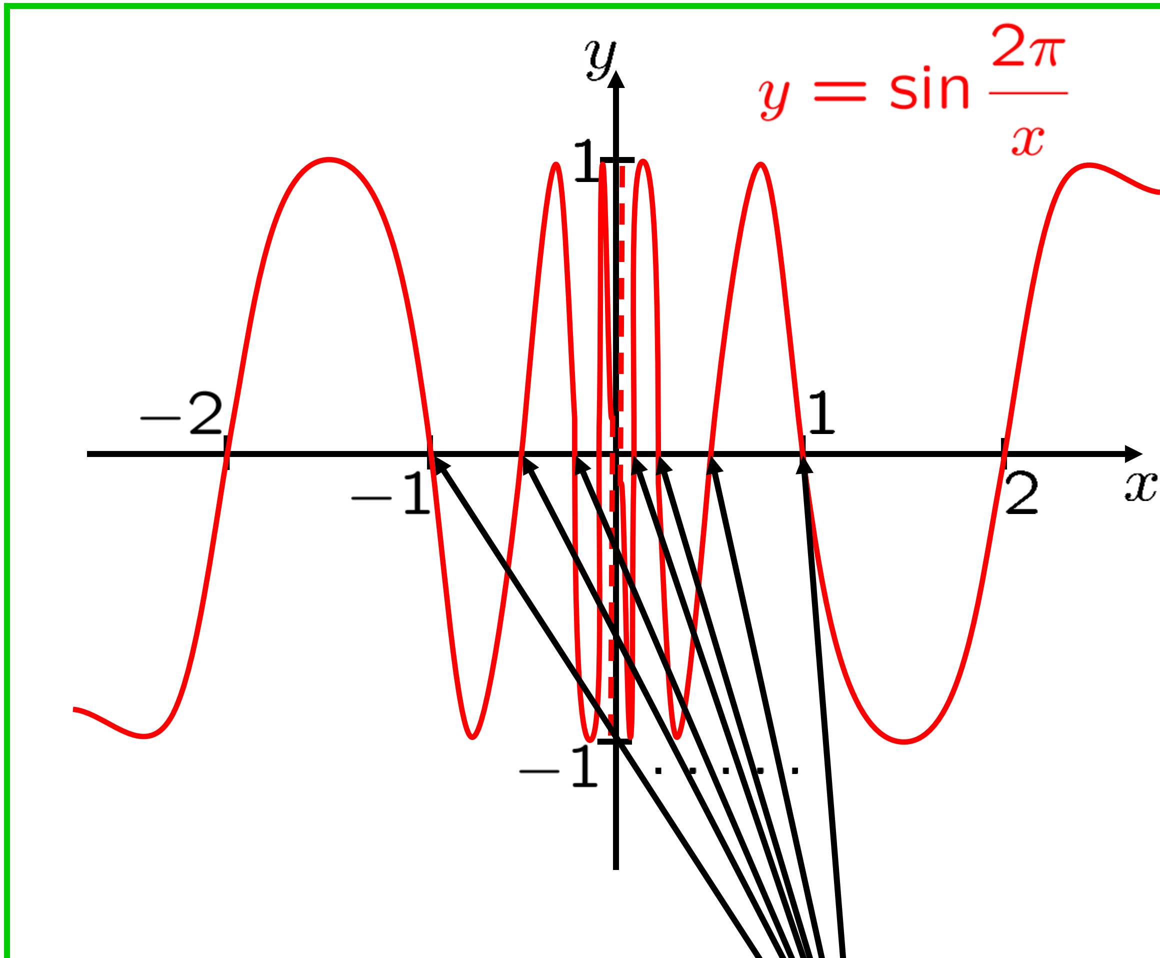
CRUCIAL → but not actually *equal* to 2,

CRUCIAL → then $u(x)$ is close to 5

and possibly equal to 5.

EXAMPLE: Guess $\lim_{x\to 0} \sin\frac{2\pi}{x}$.

| $x$ | $\sin\frac{2\pi}{x}$ |
|---|---|
| $\pm 0.1$ | 0 |
| $\pm 0.01$ | 0 |
| $\pm 0.001$ | 0 |
| $\pm 0.0001$ | 0 |
| $\pm 0.00001$ | 0 |
| $\pm 0.000001$ | 0 |
| $\pm 0.0000001$ | 0 |
| $\pm 0.00000001$ | 0 |
| $\pm 0.000000001$ | 0 |
| $\pm 0.0000000001$ | 0 |



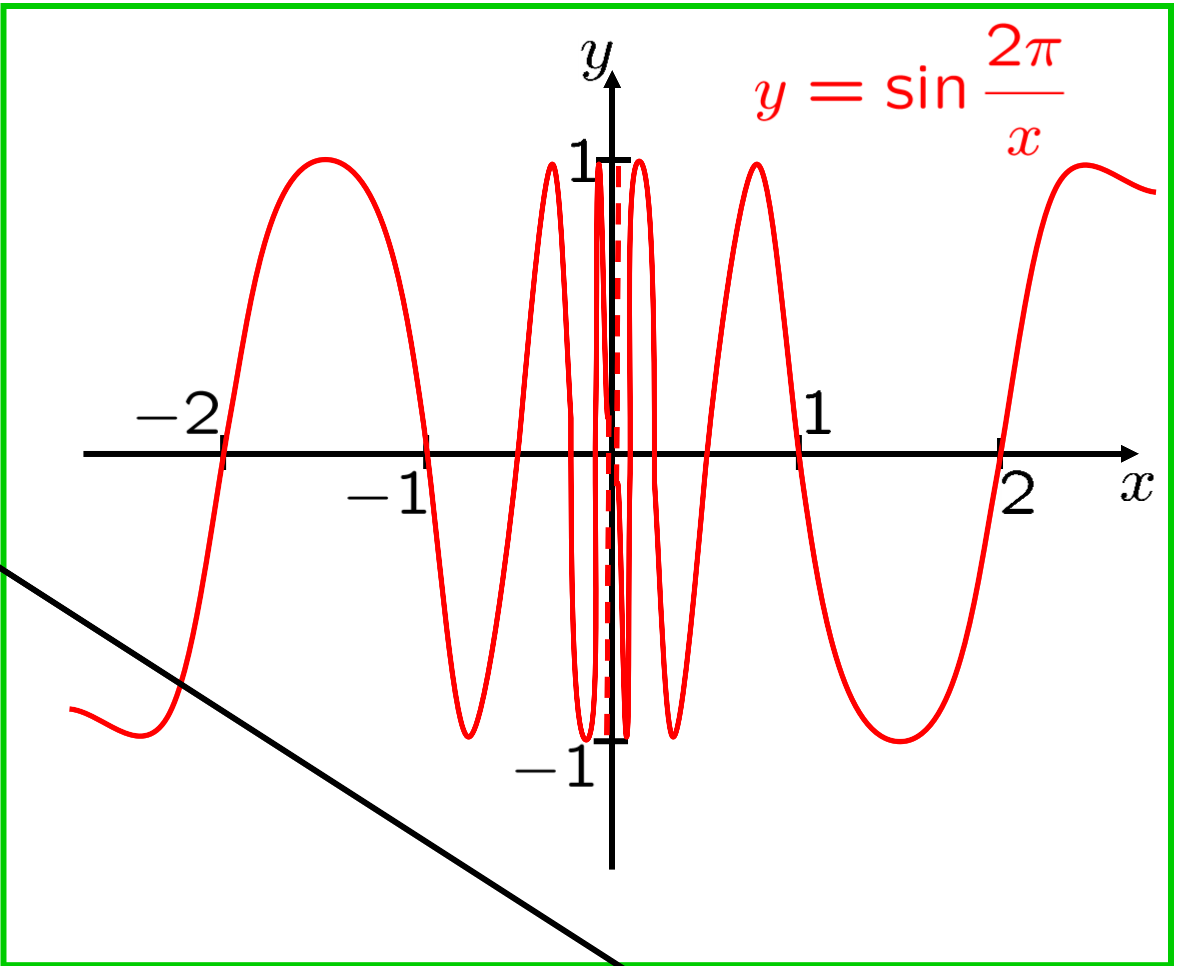

$$\lim_{x\to 0} \sin\frac{2\pi}{x} \overset{?}{=} 0$$

$\forall$integers $n \neq 0$,

$$\left[\sin\frac{2\pi}{x}\right]_{x:\to 1/n} = 0.$$

EXAMPLE: Guess $\lim_{x\to 0} \sin \frac{2\pi}{x}$.

| $x$ | $\sin \frac{2\pi}{x}$ |
|---|---|
| $\pm 0.1$ | 0 |
| $\pm 0.01$ | 0 |
| $\pm 0.001$ | 0 |
| $\pm 0.0001$ | 0 |
| $\pm 0.00001$ | 0 |
| $\pm 0.000001$ | 0 |
| $\pm 0.0000001$ | 0 |
| $\pm 0.00000001$ | 0 |
| $\pm 0.000000001$ | 0 |
| $\pm 0.0000000001$ | 0 |




$\lim_{x\to 0} \sin \frac{2\pi}{x} \overset{?}{=} 0$ NO

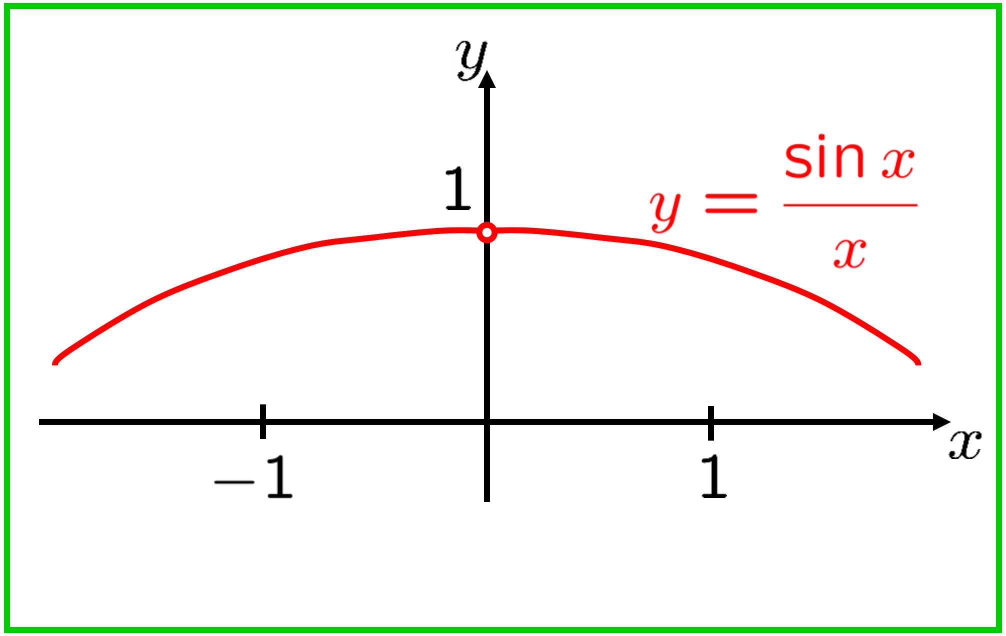
$\lim_{x\to 0} \sin \frac{2\pi}{x}$ does not exist.

$\forall$integers $n \neq 0$,

$$\left[\sin \frac{2\pi}{x}\right]_{x:\to 1/n} = 0.$$

EXAMPLE: Guess $\lim_{x \to 0} \frac{\sin x}{x}$.

| $x$ | $\frac{\sin x}{x}$ |
|---|---|
| $\pm 1.0$ | 0.841470984 |
| $\pm 0.4$ | 0.973545855 |
| $\pm 0.3$ | 0.985067355 |
| $\pm 0.2$ | 0.993346654 |
| $\pm 0.1$ | 0.998334166 |
| $\pm 0.05$ | 0.999583385 |
| $\pm 0.01$ | 0.999983333 |
| $\pm 0.005$ | 0.999995833 |
| $\pm 0.001$ | 0.999999833 |
| $\pm 0.0001$ | 0.999999998 |




How do we know *FOR SURE* that the graph doesn't exhibit *ANY* surprising behvior for $x$ very close to 0?

A proof requires
a definition of "limit". This leads back to:

How close is close?

To be discussed later. . .

This limit is important . . .

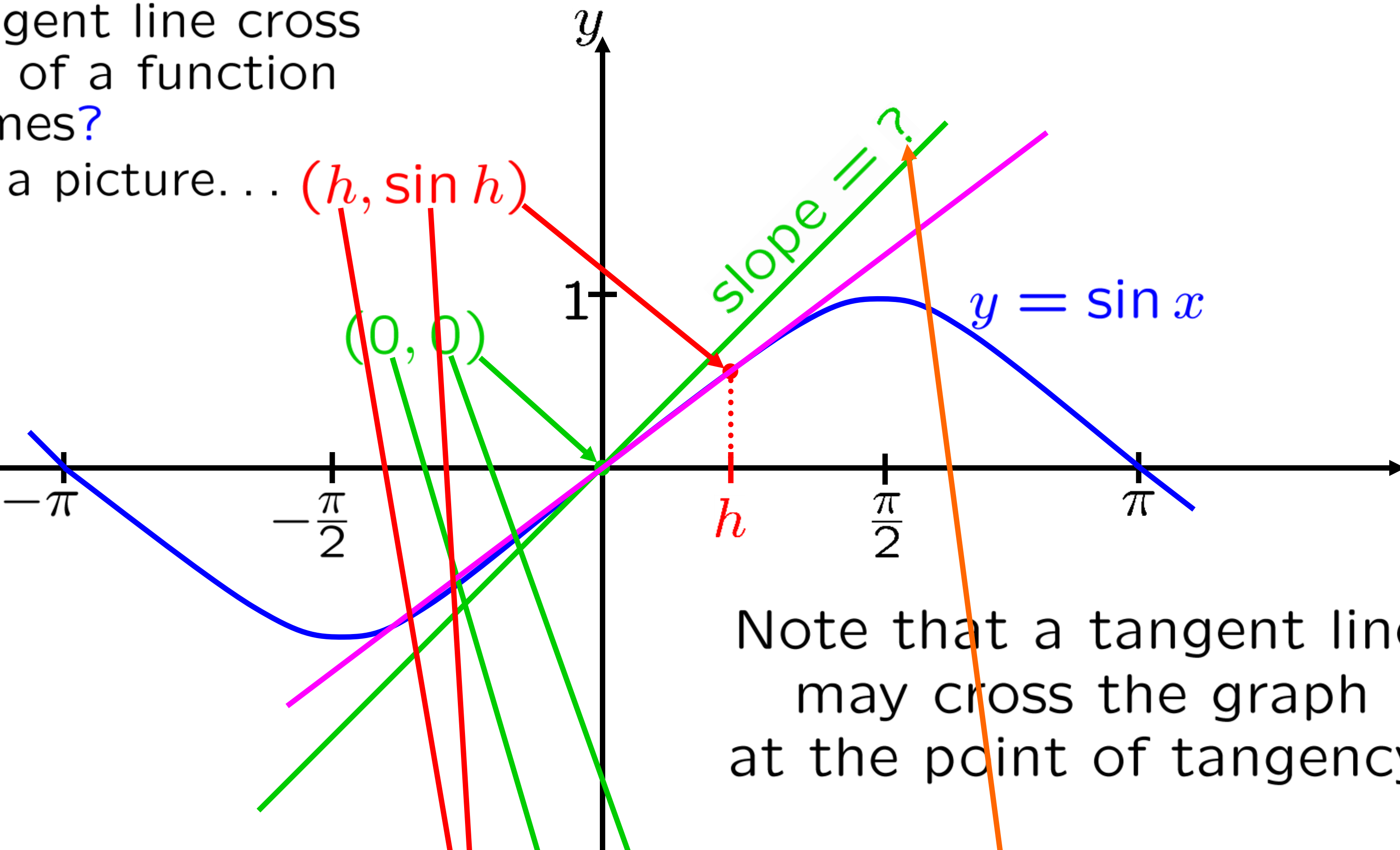
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

IOU a proof
§4.3, p. 68, l.-7

§2.3

Can a tangent line cross the graph of a function several times?

Answer in a picture. . .




Note that a tangent line may cross the graph at the point of tangency.

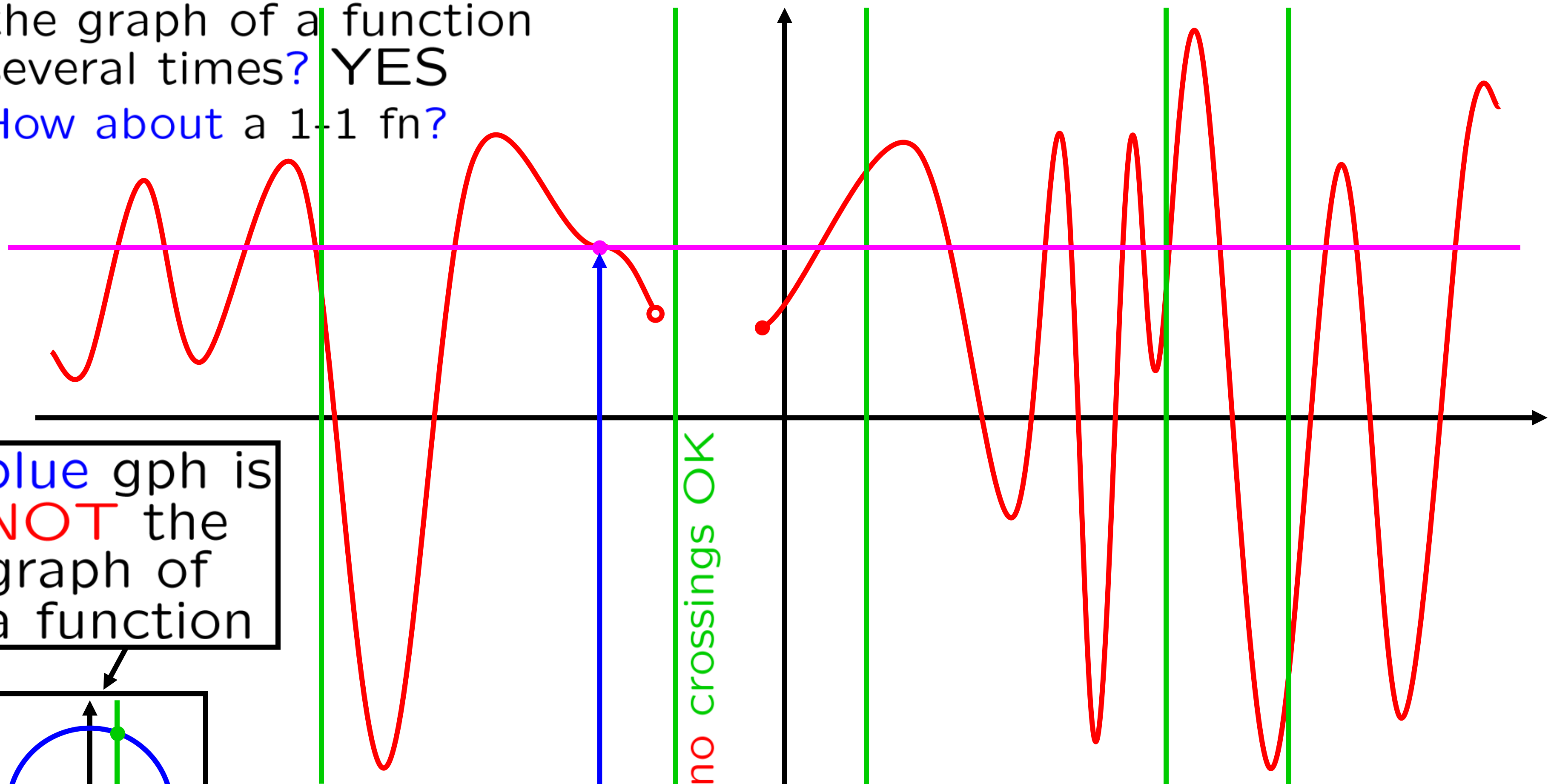
slope of secant line $= \dfrac{(\sin h) - 0}{h - 0} = \dfrac{\sin h}{h} \to 1$, as $h \to 0$.

$x :\to h$

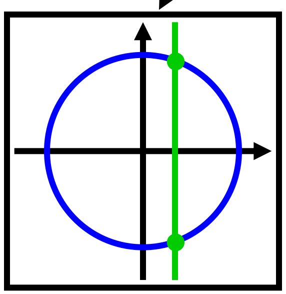
$x$ is a dummy variable: $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1 \Leftrightarrow \lim_{h\to 0} \frac{\sin h}{h} = 1$

Can a tangent line cross the graph of a function several times? YES

How about a 1-1 fn?




blue gph is NOT the graph of a function



No vertical line crosses the graph more than once.

SKILL recognize gph of fn

NOTE: A graph satisfies the vertical line test iff it is the graph of a function.

The red graph above is the graph of some function.

STOP