CALCULUS
Derivatives of trigonometric functions

§4.3, p. 68, l.-7: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

$\theta :\to h$

§4.3, p. 69, l.-10: $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0$

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

$$(-1) \quad \times \quad \left(\lim_{h\to 0} \frac{1-\cos h}{h} = 0 \right)$$

$$\lim_{h\to 0} \frac{(\cos h) - 1}{h} = 0$$

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

$$\lim_{h \to 0} \frac{(\cos h) - 1}{h} = 0$$

Goal: $\sin'$

$$\lim_{h \to 0} \frac{(\cos h) - 1}{h} = 0$$

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

$$\lim_{h\to 0} \frac{(\cos h) - 1}{h} = 0$$

Goal: $\sin'$

$$\frac{[\sin(x+h)] - [\sin x]}{h}$$

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

$$\lim_{h\to 0} \frac{(\cos h) - 1}{h} = 0$$

Goal: $\sin'$

$$\frac{[\sin(x+h)] - [\sin x]}{h} = \frac{[\sin x][\cos h] + [\cos x][\sin h] - [\sin x]}{h}$$

$$\sin(\alpha + \beta) = [\sin \alpha][\cos \beta] + [\cos \alpha][\sin \beta]$$

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

$$\lim_{h\to 0} \frac{(\cos h) - 1}{h} = 0$$

Goal: $\sin'$

$$\frac{[\sin(x+h)] - [\sin x]}{h} = \frac{[\sin x][\cos h] + [\cos x][\sin h] - [\sin x]}{h}$$

$$= \frac{[\sin x][(\cos h) - 1] + [\cos x][\sin h]}{h}$$

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

$$\lim_{h\to 0} \frac{(\cos h) - 1}{h} = 0$$

Goal: $\sin'$

$$\frac{[\sin(x+h)] - [\sin x]}{h} = \frac{[\sin x][(\cos h) - 1] + [\cos x][\sin h]}{h}$$

$$= \frac{[\sin x]\left[\frac{(\cos h) - 1}{h}\right] + [\cos x]\left[\frac{\sin h}{h}\right]}{h}$$

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

$$\lim_{h\to 0} \frac{(\cos h) - 1}{h} = 0$$

Goal: $\sin' = \cos$ ☺

§4.4, p. 70
l.+3 to l.+5

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\underbrace{\frac{[\sin(x+h)] - [\sin x]}{h}}_{\downarrow\ \frac{d}{dx}(\sin x)} = \frac{[\sin x][(\cos h) - 1] + [\cos x][\sin h]}{h}$$

$$= [\sin x]\left[\underbrace{\frac{(\cos h) - 1}{h}}_{\downarrow\ 0}\right] + [\cos x]\left[\underbrace{\frac{\sin h}{h}}_{\downarrow\ 1}\right]$$

$h \to 0$

$$\frac{d}{dx}(\sin x) = [\sin x][0] + [\cos x][1] = \cos x$$

§4.4

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

$$\lim_{h\to 0} \frac{(\cos h) - 1}{h} = 0$$

Goal: $\cos'$

§4.4, p. 70
l.+3 to l.+5

$$\frac{d}{dx}(\sin x) = \cos x$$

INTERCHANGE

sin	$\leftrightarrow$	cos
tan	$\leftrightarrow$	cot
sec	$\leftrightarrow$	csc

$$\frac{d}{dx}(\cos x) \stackrel{?}{=} \sin x$$

close,
but not quite,
as we'll see. . .

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

$$\lim_{h\to 0} \frac{(\cos h) - 1}{h} = 0$$

Goal: $\cos'$

§4.4, p. 70
l.+3 to l.+5

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{[\cos(x+h)] - [\cos x]}{h}$$

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

$$\lim_{h\to 0} \frac{(\cos h) - 1}{h} = 0$$

Goal: $\cos'$

§4.4, p. 70
l.+3 to l.+5

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{[\cos(x+h)] - [\cos x]}{h} = \frac{[\cos x][\cos h] - [\sin x][\sin h] - [\cos x]}{h}$$

$$\cos(\alpha + \beta) = [\cos \alpha][\cos \beta] - [\sin \alpha][\sin \beta]$$

§4.5

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

$$\lim_{h\to 0} \frac{(\cos h) - 1}{h} = 0$$

Goal: $\cos'$

§4.4, p. 70
l.+3 to l.+5

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{[\cos(x+h)] - [\cos x]}{h} = \frac{[\cos x][\cos h] - [\sin x][\sin h] - [\cos x]}{h}$$

$$= \frac{[\cos x][(\cos h) - 1] - [\sin x][\sin h]}{h}$$

$$\lim_{h\to 0}\frac{\sin h}{h} = 1$$

$$\lim_{h\to 0}\frac{(\cos h) - 1}{h} = 0$$

Goal: $\cos'$

§4.4, p. 70
l.+3 to l.+5

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{[\cos(x+h)] - [\cos x]}{h} = \frac{[\cos x][(\cos h) - 1] - [\sin x][\sin h]}{h}$$

$$= [\cos x]\left[\frac{(\cos h) - 1}{h}\right] - [\sin x]\left[\frac{\sin h}{h}\right]$$

$$= \frac{}{h}$$

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

$$\lim_{h\to 0} \frac{(\cos h) - 1}{h} = 0$$

Goal: $\cos'$

§4.4, p. 70
l.+3 to l.+5

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\underbrace{\frac{[\cos(x+h)] - [\cos x]}{h}}_{\frac{d}{dx}(\cos x)} = [\cos x]\left[\underbrace{\frac{(\cos h) - 1}{h}}\right] - [\sin x]\left[\underbrace{\frac{\sin h}{h}}\right]$$

$$\overset{h\to 0}{=} [\cos x]\left[\frac{(\cos h) - 1}{h}\right] - [\sin x]\left[\frac{\sin h}{h}\right]$$

0 1

§4.5

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

$$\lim_{h\to 0} \frac{(\cos h) - 1}{h} = 0$$

Goal: $\cos' = -\sin$ ☺

§4.4, p. 70
l.+3 to l.+5

$$\frac{d}{dx}(\sin x) = \cos x$$

§4.5, p. 71
l.+10

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\underbrace{\frac{[\cos(x+h)] - [\cos x]}{h}}_{\downarrow\ \frac{d}{dx}(\cos x)} = [\cos x]\left[\underbrace{\frac{(\cos h) - 1}{h}}_{\downarrow\ 0}\right] - [\sin x]\left[\underbrace{\frac{\sin h}{h}}_{\downarrow\ 1}\right]$$

$h \to 0$

$$\frac{d}{dx}(\cos x) = [\cos x][0] - [\sin x][1] = -\sin x$$

§4.5

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sin x) = \cos x$$

Here's a way to remember these formulas, using geometry...

$$\frac{d}{dx}(\sin x) = \cos x$$

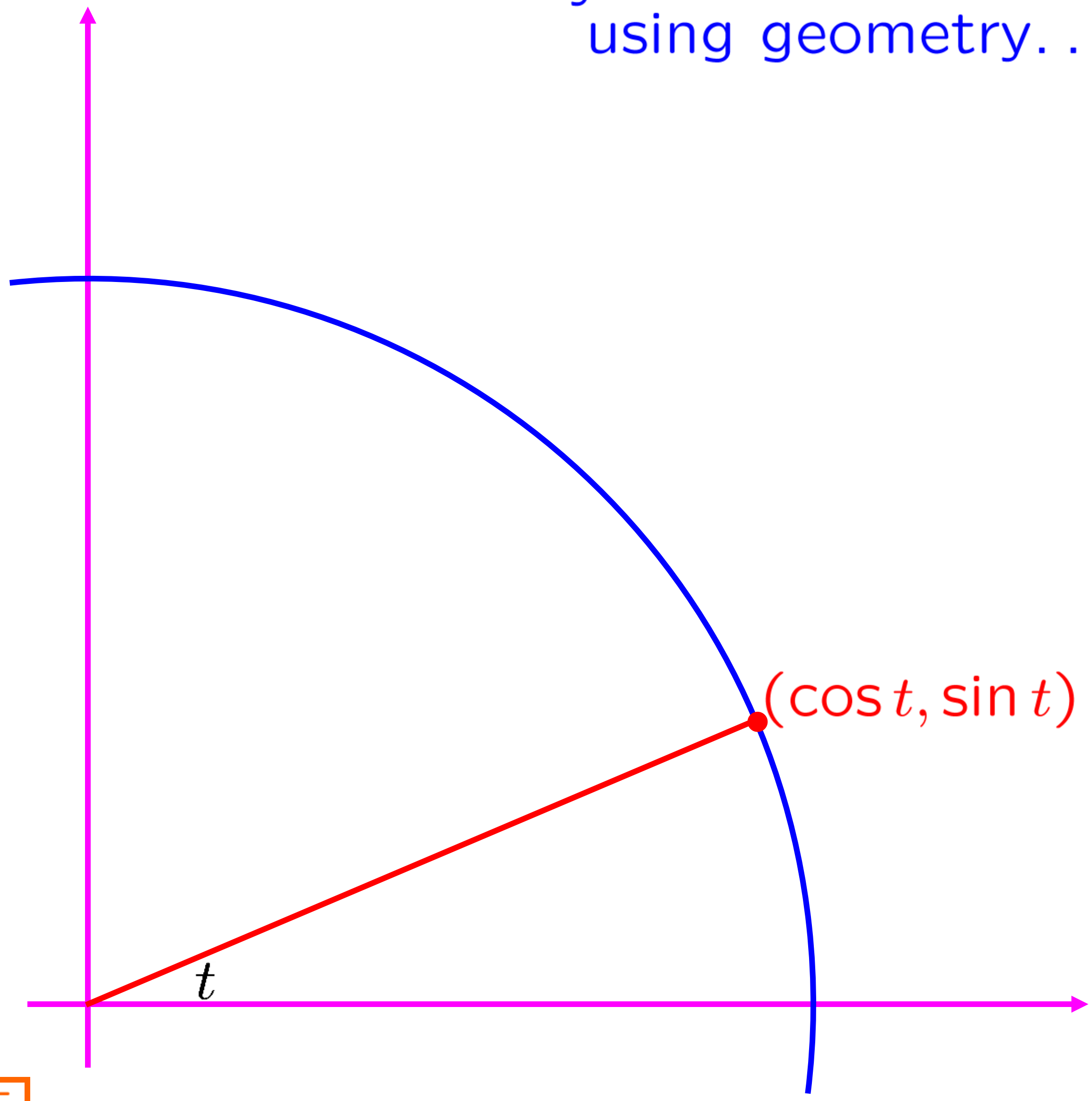
$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad x :\to t \qquad \frac{d}{dx}(\sin x) = \cos x$$

Here's a way to remember these formulas, using geometry. . .

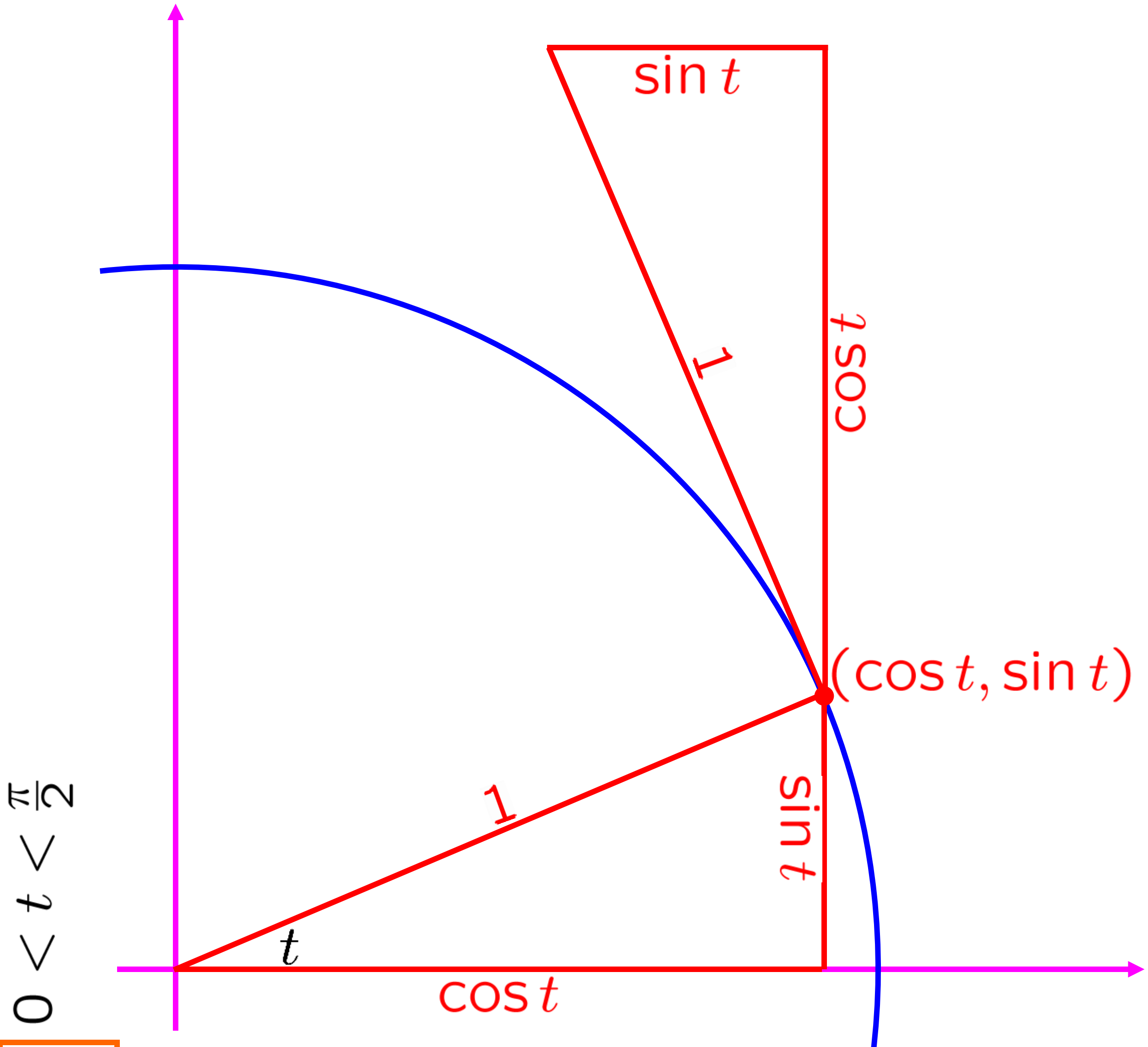
$$\frac{d}{dt}(\cos t) = -\sin t \qquad \frac{d}{dt}(\sin t) = \cos t$$

Here's a way to remember these formulas, using geometry. . .

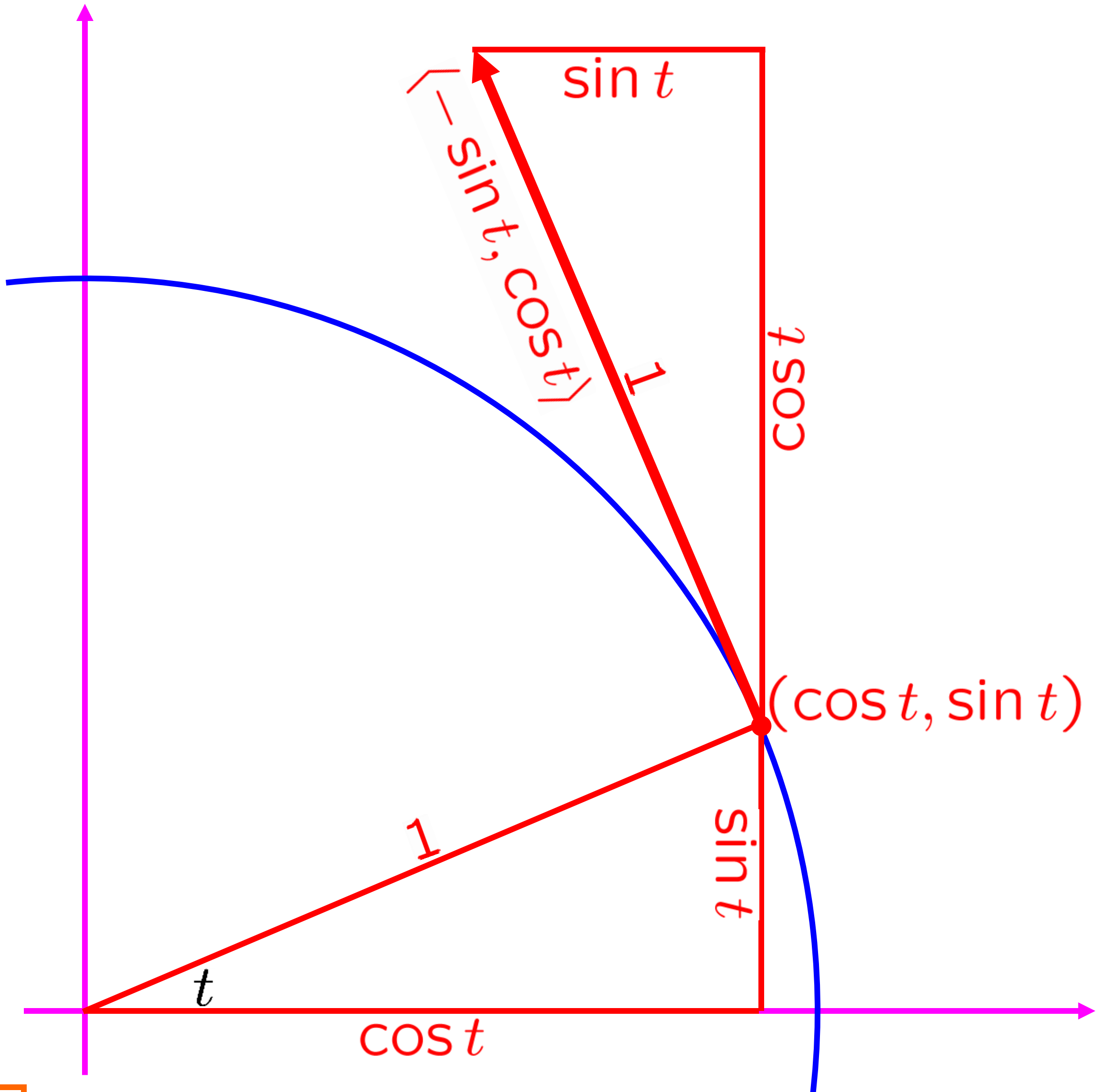


$0 < t < \frac{\pi}{2}$

$$\frac{d}{dt}(\cos t) = -\sin t \qquad \frac{d}{dt}(\sin t) = \cos t$$

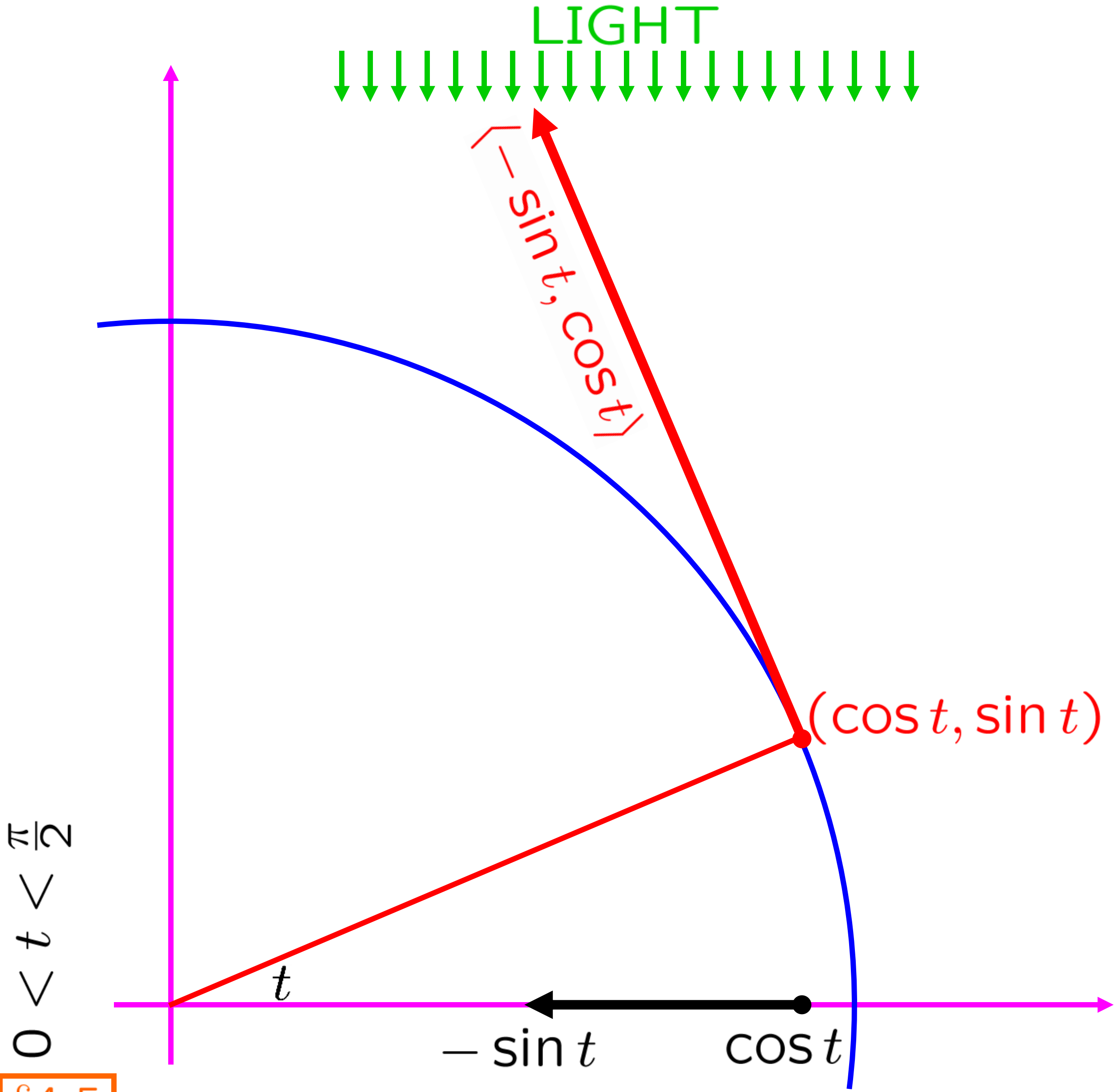


$$\frac{d}{dt}(\cos t) = -\sin t \qquad \frac{d}{dt}(\sin t) = \cos t$$



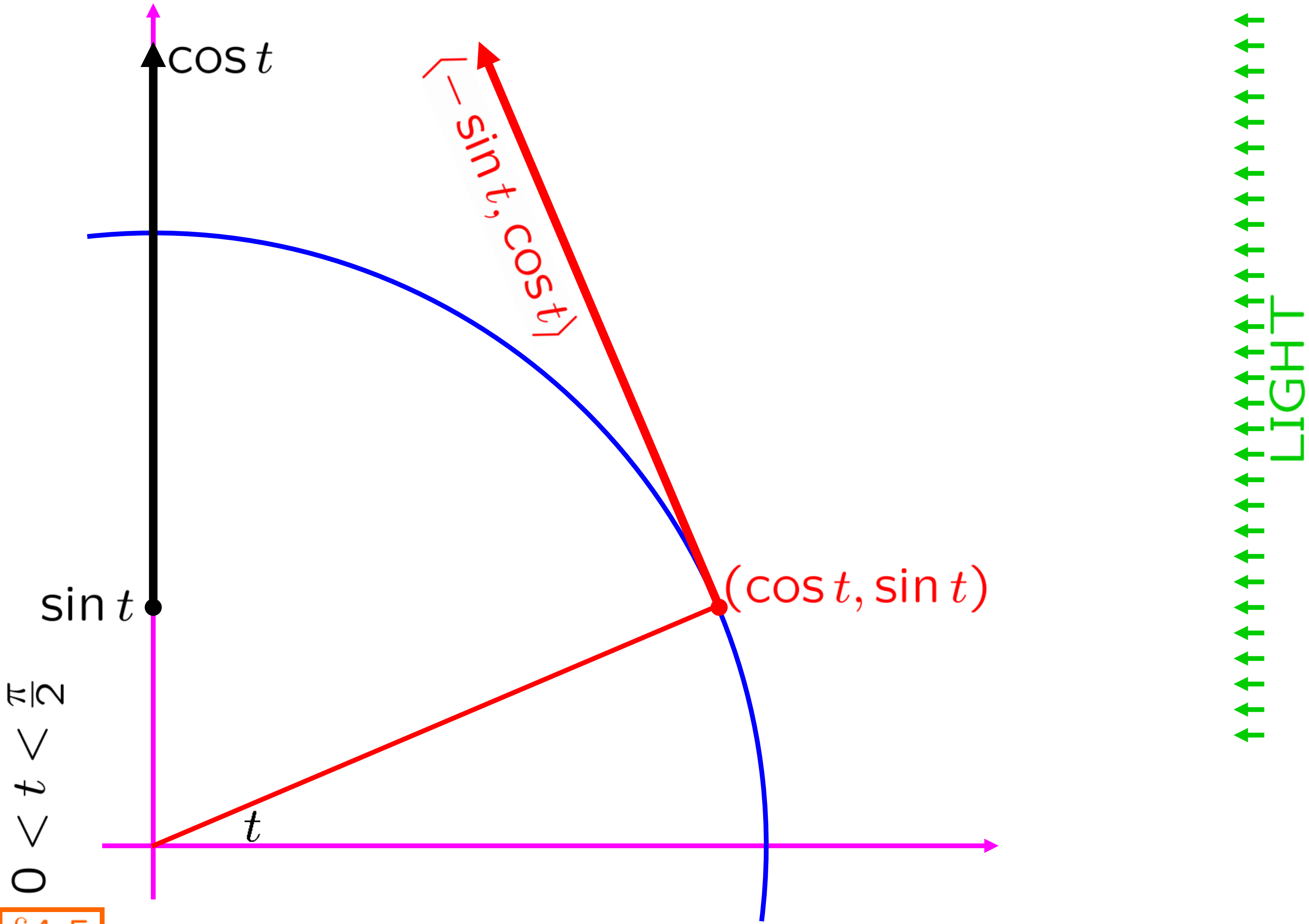
$0 < t < \frac{\pi}{2}$

$$\frac{d}{dt}(\cos t) = -\sin t \qquad \frac{d}{dt}(\sin t) = \cos t$$

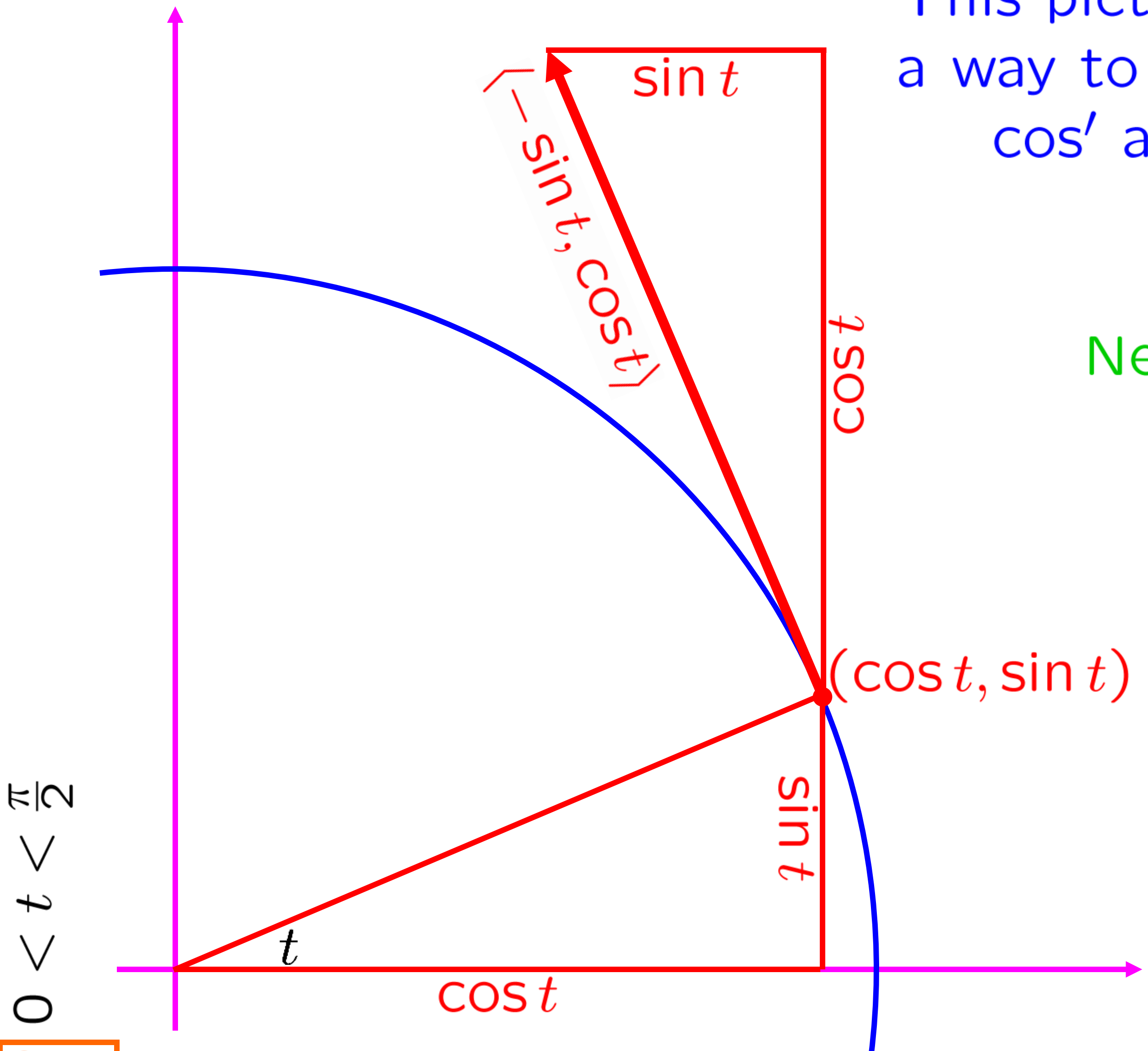


$$\frac{d}{dt}(\cos t) = -\sin t$$

$$\boxed{\frac{d}{dt}(\sin t) = \cos t}$$



$$\frac{d}{dt}(\cos t) = -\sin t \qquad \frac{d}{dt}(\sin t) = \cos t$$



This picture gives a way to remember $\cos'$ and $\sin'$.

Next goal: $\tan'$

$$\frac{d}{dt}(\tan t) = \frac{d}{dt}\left(\frac{\sin t}{\cos t}\right)$$

Quotient rule. . .

Quotient Rule

HI

$$\frac{d}{dt}\left(\frac{\sin t}{\cos t}\right)$$

LO

Next goal: $\tan'$

$$\frac{d}{dt}\left(\frac{\sin t}{\cos t}\right)$$

Quotient rule

Next goal: $\tan'$

Quotient Rule

LO DEE HI less HI DEE LO,

HI

$$\frac{d}{dt}\left(\frac{\sin t}{\cos t}\right) = \frac{[\cos t]\,[\cos t] - [\sin t]\,[-\sin t]}{\cos^2 t}$$

LO

and underneath,
LO SQUARED'LL GO

Next goal: tan′

$$\frac{d}{dt}\left(\frac{\sin t}{\cos t}\right) = \frac{[\cos t]\,[\cos t] \;+\; [\sin t]\,[+\sin t]}{\cos^2 t}$$

$$\frac{d}{dt}\,(\tan t) = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} = \sec^2 t$$

Next goal: tan′

$$t :\to x$$

$$\frac{d}{dt}(\tan t) = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} = \sec^2 t$$

Next goal: tan′

$$\frac{d}{dx}(\tan x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

§4.5 Next goal: $\tan' = \sec^2$ ☺

§4.4, p. 70 l.+3 to l.+5	$\frac{d}{dx}(\sin x) = \cos x$
§4.5, p. 71 l.+10	$\frac{d}{dx}(\cos x) = -\sin x$
§4.5, p. 71 l.+11	$\frac{d}{dx}(\tan x) = \sec^2 x$

$$\frac{d}{dx}(\tan x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Next goal: $\tan' = \sec^2$ ☺

§4.5, p. 71, l.+12: Prove that

$$\frac{d}{dx}(\sec x) = [\sec x][\tan x].$$

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left[\frac{1}{\cos x}\right]$$

$$= \frac{[\cos x][\ \ 0\ \ ] + [1][\ +\sin x\ \ ]}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \left[\frac{1}{\cos x}\right]\left[\frac{\sin x}{\cos x}\right] = [\sec x][\tan x]$$

§4.4, p. 70
l.+3 to l.+5

$$\frac{d}{dx}(\sin x) = \cos x$$

§4.5, p. 71
l.+10

$$\frac{d}{dx}(\cos x) = -\sin x$$

§4.5, p. 71
l.+11

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x)$$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS:

$$\frac{d}{dx}(\sec x) = [\sec x][\tan x]$$

$$= [\sec x][\tan x]$$

§4.4, p. 70
l.+3 to l.+5

$$\frac{d}{dx}(\sin x) = \cos x$$

§4.5, p. 71
l.+10

$$\frac{d}{dx}(\cos x) = -\sin x$$

§4.5, p. 71
l.+11

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS:

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = [\sec x][\tan x] \qquad \frac{d}{dx}(\csc x) = -[\csc x][\cot x]$$

HOMEWORK §4.5, p. 71: #5, 6.

FORMULA		COMPLEMENTARY FORMULA
$\frac{d}{dx}(\sin x) = \cos x$	$\longleftrightarrow$	$\frac{d}{dx}(\cos x) = -\sin x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\longleftrightarrow$	$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\sin x) = \cos x$ $x][\tan x]$ $\longleftrightarrow$ $\frac{d}{dx}(\cos x) = -\sin x$ $x][\cot x]$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = [\sec x][\tan x] \qquad \frac{d}{dx}(\csc x) = -[\csc x][\cot x]$$

FORMULA		COMPLEMENTARY FORMULA
$\frac{d}{dx}(\sin x) = \cos x$	$\longleftrightarrow$	$\frac{d}{dx}(\cos x) = -\sin x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\longleftrightarrow$	$\frac{d}{dx}(\cot x) = -\csc^2 x$
$\frac{d}{dx}(\sec x) = [\sec x][\tan x]$	$\longleftrightarrow$	$\frac{d}{dx}(\csc x) = -[\csc x][\cot x]$

MEMORIZE THESE

INTERCHANGE

sin	$\leftrightarrow$	cos
tan	$\leftrightarrow$	cot
sec	$\leftrightarrow$	csc

GET THESE FOR FREE

NOTE THE MINUS SIGNS

TO BE EXPLAINED IN §3.5 THE CHAIN RULE

EXAMPLE: Differentiate: $y = [\sqrt[5]{x}][\cot x]$

$x^{1/5}$

$$\frac{dy}{dx} = [\,(1/5)x^{-4/5}\,][\cot x] + [x^{1/5}][\,-\csc^2 x\,]\ \blacksquare$$

EXAMPLE: Differentiate: $y = e^{u+c}(ku + \tan u)$

$$y = e^c\left[e^u(ku + \tan u)\right]$$

$$\frac{dy}{du} = e^c\left[(e^u)(k + \sec^2 u) + (e^u)(ku + \tan u)\right] \blacksquare$$

EXAMPLE: Differentiate: $y = \dfrac{1 - \sin x}{x^2 + x \tan x}$

$$\frac{dy}{dx} = \frac{(x^2 + x\tan x)(-\cos x) - (1 - \sin x)(2x + (x)(\sec^2 x) + (1)(\tan x))}{(x^2 + x\tan x)^2}$$

■

EXAMPLE: Differentiate: $y = x^3[\sec x][\cos x]$

$$\frac{dy}{dx} = [3x^2][\sec x][\cos x] + x^3[(\sec x)(\tan x)][\cos x] + x^3[\sec x][-\sin x] \quad \blacksquare$$

EXAMPLE: Find an equation of the tangent line to the curve $y = \dfrac{1}{\cos x + 3\sin x}$ at $(0, 1)$.

$$\frac{dy}{dx} = \frac{(\cos x + 3\sin x)(0) - (1)(-\sin x + 3\cos x)}{(\cos x + 3\sin x)^2}$$

$$= \frac{\sin x - 3\cos x}{(\cos x + 3\sin x)^2}$$

$$\left[\frac{dy}{dx}\right]_{x:\to 0} = \frac{0 - 3}{(1+0)^2} = -3$$

$$y - 1 = -3(x - 0)$$ ■

EXAMPLE: If $f(x) = \sec x - \csc x$, find $f''(\pi/4)$.

$$f'(x) = [\sec x][\tan x] + [\csc x][\cot x]$$

$$f''(x) = [(\sec x)(\tan x)][\tan x] + [\sec x][\sec^2 x] + [-(\csc x)(\cot x)][\cot x] + [\csc x][-\csc^2 x]$$

$$= (\sec x)(\tan^2 x) + \sec^3 x - (\csc x)(\cot^2 x) - \csc^3 x$$

$$= \left(\frac{1}{\cos x}\right)\left(\frac{\sin^2 x}{\cos^2 x}\right) + \frac{1}{\cos^3 x} - \left(\frac{1}{\sin x}\right)\left(\frac{\cos^2 x}{\sin^2 x}\right) - \frac{1}{\sin^3 x}$$

EXAMPLE: If $f(x) = \sec x - \csc x$, find $f''(\pi/4)$.

$$f''(x) = \left(\frac{1}{\cos x}\right)\left(\frac{\sin^2 x}{\cos^2 x}\right) + \frac{1}{\cos^3 x}$$

$$f''(x) \qquad - \left(\frac{1}{\sin x}\right)\left(\frac{\cos^2 x}{\sin^2 x}\right) - \frac{1}{\sin^3 x}$$

$$\sin(\pi/4) = 1/\sqrt{2} = \cos(\pi/4)$$

$$= \left(\frac{1}{\cos x}\right)\left(\frac{\sin^2 x}{\cos^2 x}\right) + \frac{1}{\cos^3 x}$$

$$- \left(\frac{1}{\sin x}\right)\left(\frac{\cos^2 x}{\sin^2 x}\right) - \frac{1}{\sin^3 x}$$

EXAMPLE: If $f(x) = \sec x - \csc x$, find $f''(\pi/4)$.

$$f''(x) = \left(\frac{1}{\cos x}\right)\left(\frac{\sin^2 x}{\cos^2 x}\right) + \frac{1}{\cos^3 x} - \left(\frac{1}{\sin x}\right)\left(\frac{\cos^2 x}{\sin^2 x}\right) - \frac{1}{\sin^3 x}$$

$$\sin(\pi/4) = 1/\sqrt{2} = \cos(\pi/4)$$

$$f''(\pi/4) = \left(\frac{1}{1/\sqrt{2}}\right)\left(\frac{1/2}{1/2}\right) + \frac{1}{1/(2\sqrt{2})} - \left(\frac{1}{1/\sqrt{2}}\right)\left(\frac{1/2}{1/2}\right) - \frac{1}{1/(2\sqrt{2})}$$

$$= 0 \blacksquare$$

EXAMPLE: Find the 35th derivative of $\cos x$.

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d^2}{dx^2}[\cos x] = -\cos x$$

$$\frac{d^3}{dx^3}[\cos x] = \sin x$$

$$\frac{d^4}{dx^4}[\cos x] = \cos x$$

$$\vdots$$

$$\frac{d^{32}}{dx^{32}}[\cos x] = \cos x$$

$$\frac{d^{33}}{dx^{33}}[\cos x] = -\sin x$$

$$\frac{d^{34}}{dx^{34}}[\cos x] = -\cos x$$

$$\frac{d^{35}}{dx^{35}}[\cos x] = \sin x \blacksquare$$

EXAMPLE: Differentiate $f(x) = \dfrac{\csc x}{\sqrt{3} + \cot x}$.

For what values of x does the graph of f have a horizontal tangent?

$$0 = f'(x) = \frac{(\sqrt{3} + \cot x)(-(\csc x)(\cot x)) - (\csc x)(-\csc^2 x)}{(\sqrt{3} + \cot x)^2}$$

$$0 = (\sqrt{3} + \cot x)(-(\csc x)(\cot x)) - (\csc x)(-\csc^2 x)$$

$$0 = (\sqrt{3} + \cot x)(-\cot x) + \csc^2 x$$

csc x is NEVER 0

$$0 = -\sqrt{3}\cot x - \cot^2 x + \csc^2 x$$

$$1 = \frac{1 - \cos^2 x}{\sin^2 x} = -\frac{\cos^2 x}{\sin^2 x} + \frac{1}{\sin^2 x}$$

$$\sqrt{3}\cot x = -\cot^2 x + \csc^2 x$$

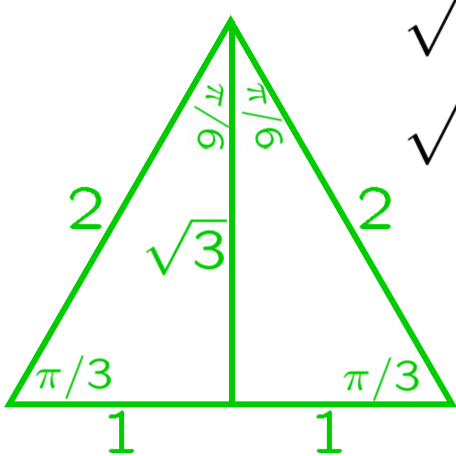
$$\sqrt{3}\cot x = 1$$

$$\cot x = 1/\sqrt{3}$$

$$\Updownarrow$$

$$x = \pi/3 + n\pi,\ n \in \mathbb{Z}$$

cot is π-periodic and 1-1 on $(n\pi, \pi + n\pi)$



SKILL
trig limits
Whitman problems
§4.3, p. 69, #1-5

SKILL
squeeze thm
Whitman problems
§4.3, p. 69, #6-8

SKILL
trig derivs
Whitman problems
§4.4, p. 70–71, #1-5

SKILL
trig derivs
Whitman problems
§4.5, p. 71, #1-12

SKILL
horizontal tan line
Whitman problems
§4.5, p. 71–72, #13-14,18

SKILL
eqn tan line
Whitman problems
§4.5, p. 72, #15-17

SKILL
lim trig with geometry
Whitman problems
§4.5, p. 72, #19

STOP