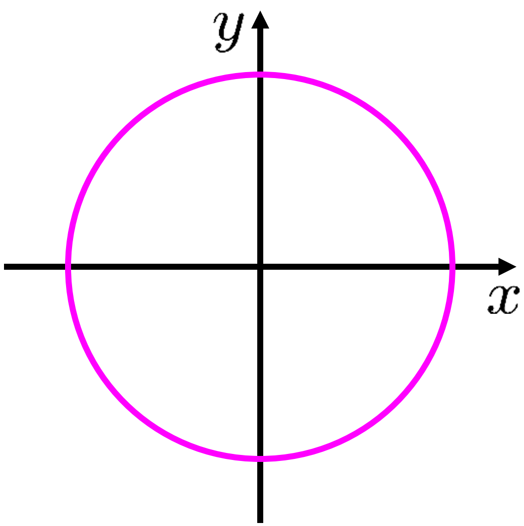
CALCULUS
Implicit differentiation

cf. §4.9, p. 85 EXAMPLE 4.16

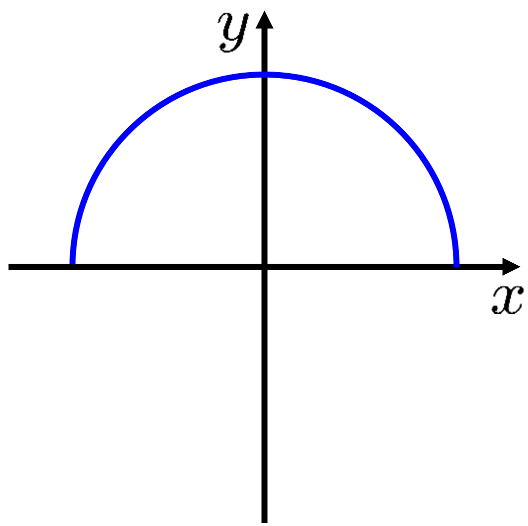
If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

$y = f(x)$, where either $f(x) = \sqrt{25 - x^2}$ or $f(x) = -\sqrt{25 - x^2}$

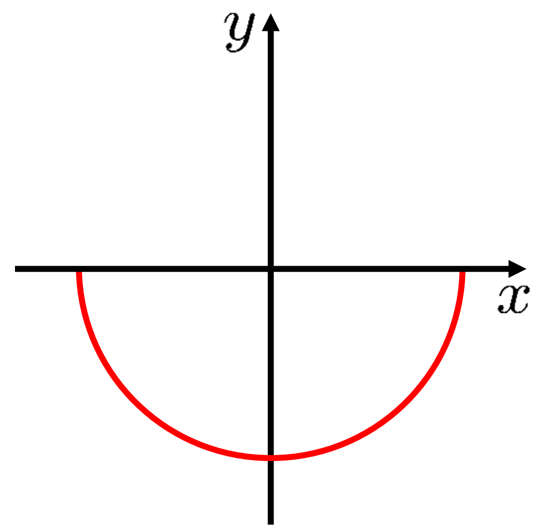
$x^2 + y^2 = 25$



$y = \sqrt{25 - x^2}$



$y = -\sqrt{25 - x^2}$



NOT THE GRAPH OF A FUNCTION

GRAPHS OF FUNCTIONS

cf. §4.9, p. 85 EXAMPLE 4.16

(a) If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

$y = f(x)$, where either $\boxed{f(x) = \sqrt{25 - x^2}}$ or $f(x) = -\sqrt{25 - x^2}$

but we don't know which.

LET'S PEEK AHEAD...

(b) Find an equation to the tangent line to

$x^2 + y^2 = 25$ at the point $(3, 4)$.

$$4 = \sqrt{25 - 3^2}$$

EVEN IF WE DON'T PEEK AHEAD,
WE CAN STILL DO SOME OF THE WORK...

cf. §4.9, p. 85 EXAMPLE 4.16

If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{x}{y}$$

THIS FORMULA IS TRUE IN EITHER CASE!

$y = f(x)$, where $\boxed{\text{either } f(x) = \sqrt{25 - x^2} \text{ or } f(x) = -\sqrt{25 - x^2}}$

but we don't know which.

$$\frac{d}{dx}\left[x^2 + \underline{[f(x)]^2}\right] = \frac{d}{dx}\left[25\right]$$

$$\| \qquad\qquad\qquad \|$$

$$2x + 2[f(x)][f'(x)] \qquad\qquad 0$$

$$\cancel{2}[f(x)][f'(x)] = -\cancel{2}x$$

$$f'(x) = \frac{-x}{f(x)} = -\frac{x}{y}$$

EVEN IF WE DON'T PEEK AHEAD,
WE CAN STILL DO SOME OF THE WORK. . .

cf. §4.9, p. 85 EXAMPLE 4.16

(a) If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$. $\qquad \dfrac{dy}{dx} = -\dfrac{x}{y}$

(b) Find an equation to the tangent line to $x^2 + y^2 = 25$ at the point $(3, 4)$.

$$\text{slope} = \left[\frac{dy}{dx}\right]_{x:\to 3} = \left[-\frac{x}{y}\right]_{\substack{x:\to 3\\ y=4}} = -\frac{3}{4}$$

EQUATION: $y - 4 = -\dfrac{3}{4}(x - 3)$ ■

cf. §4.9, p. 85 EXAMPLE 4.16

If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

THIS IS AN (UNKNOWN) EXPRESSION, NAMELY $f(x)$,
INSIDE A FUNCTION, NAMELY SQUARING.

$$\frac{d}{dx}\left[x^2 + \boxed{[f(x)]^2}\right] = \frac{d}{dx}\left[25\right]$$

$$\| \qquad\qquad\qquad \|$$

$$2x + \boxed{2[f(x)][f'(x)]} \qquad\qquad 0$$

CHAIN RULE

$$2[f(x)][f'(x)] = -2x$$

$$f'(x) = \frac{-x}{f(x)} = -\frac{x}{y}$$

TO SAVE WRITING, WRITE
y INSTEAD OF $f(x)$
§4.9 AND $\boxed{y'}$ INSTEAD OF $f'(x)$

NOT QUITE KOSHER. . .
OH, WELL.

cf. §4.9, p. 85 EXAMPLE 4.16

If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

THIS IS AN (UNKNOWN) EXPRESSION, NAMELY y,
INSIDE A FUNCTION, NAMELY SQUARING.

$$\frac{d}{dx}\left[\ x^2 + \boxed{y^2}\ \right] = \frac{d}{dx}[25]$$

$$\begin{array}{ccc} \| & & \| \\ 2x + \boxed{2[\ y\ ]\ [y']} & & 0 \end{array}$$

CHAIN RULE

$$2[y][y'] = -2x$$

WHEN YOU DIFFERENTIATE
y^2 WITH RESPECT TO $\boxed{y}$
YOU GET $2y$.

$$y' = \frac{-x}{y} = -\frac{x}{y}$$

TO SAVE WRITING, WRITE
y INSTEAD OF $f(x)$
$\boxed{§4.9}$ AND y' INSTEAD OF $f'(x)$

cf. §4.9, p. 85 EXAMPLE 4.16

If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

THIS IS AN (UNKNOWN) EXPRESSION, NAMELY y,
INSIDE A FUNCTION, NAMELY SQUARING.

$$\frac{d}{dx}\left[\; x^2 + \boxed{y^2} \;\right] = \frac{d}{dx}\left[25\right]$$

$$\begin{array}{ccc} \parallel & & \parallel \\ 2x + \boxed{2[\;y\;]\;[y']} & & 0 \end{array}$$

CHAIN RULE

$$2[y][y'] = -2x$$

WHEN YOU DIFFERENTIATE
y^2 WITH RESPECT TO $\boxed{x}$
YOU DO NOT GET $2y$,
BUT RATHER $2yy'$.

$$y' = \frac{-x}{y} = -\frac{x}{y}$$

TO SAVE WRITING, WRITE
y INSTEAD OF $f(x)$
AND y' INSTEAD OF $f'(x)$

cf. §4.9, p. 85 EXAMPLE 4.16

If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

THIS IS AN (UNKNOWN) EXPRESSION, NAMELY y,
INSIDE A FUNCTION, NAMELY SQUARING.

$$=$$

$$\boxed{2x + 2[\ y\ ]\ [y'] \ = \ 0}$$ LINEAR IN y'

$$2[y][y'] = -2x$$

WHEN YOU DIFFERENTIATE
y^2 WITH RESPECT TO x
YOU DO NOT GET $2y$,
BUT RATHER $2yy'$.

$$y' = \frac{-x}{y} = -\frac{x}{y}$$

TO SAVE WRITING, WRITE
y INSTEAD OF $f(x)$
AND y' INSTEAD OF $f'(x)$

cf. §4.9, p. 85 EXAMPLE 4.16

If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

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$$\boxed{2x + 2[\ y\ ]\ [y'] = 0}$$ LINEAR IN y'

$$2[y][y'] = -2x$$

WHEN YOU DIFFERENTIATE
y^2 WITH RESPECT TO x
YOU DO NOT GET $2y$,
BUT RATHER $2yy'$.

$$\boxed{y' = \frac{-x}{y} = -\frac{x}{y}}$$ SOLUTION

Next: Review how to
solve linear equations.

TO SAVE WRITING, WRITE
y INSTEAD OF $f(x)$
AND y' INSTEAD OF $f'(x)$

$\longleftarrow x$ terms no x terms$\longrightarrow$

BASIC ALGEBRA PROBLEM:

Solve $7(4x+8) = \dfrac{2x-6}{7}$ for x.

$$(7)(4)x + (7)(8) = \tfrac{2}{7}x - \tfrac{6}{7}$$

$$\left((7)(4) - \tfrac{2}{7}\right)x = (7)(4)x - \tfrac{2}{7}x = -(7)(8) - \tfrac{6}{7}$$

$$x = \frac{-(7)(8) - \frac{6}{7}}{(7)(4) - \frac{2}{7}} \quad \blacksquare$$

EXAMPLE: (a) Find y' if $x^3 + 2y^3 = 15xy$.
(b) Find the slope of the tangent line to $x^3 + 2y^3 = 15xy$ at the point $(6, 3)$.
(c) At what points is the tangent line to $x^3 + 2y^3 = 15xy$ horizontal?

---

d/dx

LINEAR IN y'

$\longleftarrow y'$

no $y' \longrightarrow$

(a) $3x^2 + 6y^2y' = 15y + 15xy'$

$$(6y^2 - 15x)y' = 6y^2y' - 15xy' = -3x^2 + 15y$$

$$y' = \frac{-3x^2 + 15y}{6y^2 - 15x}$$

(b) $$[y']_{\substack{x:\to 6\\ y=3}} = \left[\frac{-3x^2 + 15y}{6y^2 - 15x}\right]_{\substack{x:\to 6\\ y=3}} = \frac{-3 \cdot 6^2 + 15 \cdot 3}{6 \cdot 3^2 - 15 \cdot 6} = \frac{7}{4}$$

EXAMPLE: (a) Find y' if $x^3 + 2y^3 = 15xy$.

(b) Find the slope of the tangent line to

$x^3 + 2y^3 = 15xy$ at the point $(6, 3)$.

(c) At what points is the tangent line to

$x^3 + 2y^3 = 15xy$ horizontal?

(c) $$0 = y' = \frac{-3x^2 + 15y}{6y^2 - 15x}$$

$$y' = \frac{-3x^2 + 15y}{6y^2 - 15x}$$

EXAMPLE: (a) Find y' if $x^3 + 2y^3 = 15xy$.

(b) Find the slope of the tangent line to $x^3 + 2y^3 = 15xy$ at the point $(6, 3)$.

(c) At what points is the tangent line to $x^3 + 2y^3 = 15xy$ horizontal?

(c) $$0 = y' = \frac{-3x^2 + 15y}{6y^2 - 15x}$$

$$0 \neq 6y^2 - 15x$$

$$0 = \boxed{-3x^2} + 15y$$

$$3x^2 = 15y$$

$\frac{1}{15}\times$ $$15y = 3x^2$$

$$y = \frac{1}{5}x^2$$

$$x^3 + 2\left(\tfrac{1}{5}x^2\right)^3 = 15x\left(\tfrac{1}{5}x^2\right)$$

$$\boxed{x^3} + \frac{2}{5^3}x^6 = 3x^3$$

$\frac{5^3}{2}\times$ $$\frac{2}{5^3}x^6 = 2x^3$$

$\sqrt[3]{}$ $$x^6 = 5^3x^3$$

$$x^2 = \boxed{5x}$$

$$x(x-5) = x^2 - 5x = 0$$

$$x = 0 \quad \text{or} \quad x = 5$$

$$y = \frac{1}{5}x^2$$

$$y = 0$$

$$6y^2 - 15x = 0 \neq 6y^2 - 15x$$

EXAMPLE: (a) Find y' if $x^3 + 2y^3 = 15xy$.
(b) Find the slope of the tangent line to $x^3 + 2y^3 = 15xy$ at the point $(6, 3)$.
(c) At what points is the tangent line to $x^3 + 2y^3 = 15xy$ horizontal?

(c) $$0 = y' = \frac{-3x^2 + 15y}{6y^2 - 15x}$$

$$0 \neq 6y^2 - 15x$$

$$0 = -3x^2 + 15y$$

$$y = \tfrac{1}{5}x^2$$

$$x = 5$$

$$y = \tfrac{1}{5}(5^2)$$

$$x = 5$$

EXAMPLE: (a) Find y' if $x^3 + 2y^3 = 15xy$.
(b) Find the slope of the tangent line to $x^3 + 2y^3 = 15xy$ at the point $(6, 3)$.
(c) At what points is the tangent line to $x^3 + 2y^3 = 15xy$ horizontal?

(c) $$0 = y' = \frac{-3x^2 + 15y}{6y^2 - 15x}$$

$$0 \neq 6y^2 - 15x$$

$$0 = -3x^2 + 15y$$

$$y = \tfrac{1}{5}x^2$$

$$x = 5$$

$$y = \tfrac{1}{5}(5^2) = 5$$

$$(x, y) = (5, 5) \blacksquare$$

EXAMPLE: Find y' if $x^7 + y^7 = 3$.

LINEAR IN y'

$\frac{1}{7}\times \longrightarrow 7x^6 + 7y^6y' = 0$

$$\boxed{x^6} + y^6y' = 0$$

$\longleftarrow y'$

no $y' \longrightarrow$

$$\boxed{y^6}y' = -x^6$$

$$y' = -\frac{x^6}{y^6} \quad \blacksquare$$

EXAMPLE: $\cos x + \sqrt{y} = 5$

(a) Find y' by implicit differentiation.

(b) Solve the equation explicitly for y and differentiate to get y' in terms of x.

(c) Check that your solutions to parts (a) and (b) are consistent by substituting the expression for y into your solution for part (a).

---

d/dx

(a) $-\sin x + (1/2)y^{-1/2}y' = 0$ LINEAR IN y'

$$y' = \frac{\sin x}{(1/2)y^{-1/2}} = 2y^{1/2}\sin x$$

(b) $\cos x + \sqrt{y} = 5$

$$y = (5 - \cos x)^2$$

$$y' = 2(5 - \cos x)\sin x$$

EQUAL

(c) $y' = 2\sqrt{y}\sin x = 2\sqrt{(5-\cos x)^2}\sin x = 2(5 - \cos x)\sin x$ ■

EXAMPLE: $6x^2 + 5y^2 = 2$

(a) Find y' by implicit differentiation.

(b) Solve the equation explicitly for y and differentiate to get y' in terms of x.

(c) Check that your solutions to parts (a) and (b) are consistent by substituting the expression for y into your solution for part (a).

d/dx

(a) $12x + 10yy' = 0$

LINEAR IN y'

$$y' = \frac{-12x}{10y} = \frac{-6x}{5y}$$

(b) $5y^2 = 2 - 6x^2$ $\qquad d/dx \longrightarrow y = \pm\left(\frac{2}{5} - \frac{6}{5}x^2\right)^{1/2}$

$$y' = \boxed{\pm\frac{1}{2}}\left(\frac{2}{5} - \frac{6}{5}x^2\right)^{-1/2}\left(\boxed{-\frac{12}{5}x}\right) = \boxed{\mp\frac{6}{5}x}\left(\frac{2}{5} - \frac{6}{5}x^2\right)^{-1/2}$$

EQUAL

(c) $$\frac{-6x}{5y} = \frac{-6x}{5\left[\pm\left(\frac{2}{5} - \frac{6}{5}x^2\right)^{1/2}\right]} = \mp\frac{6}{5}x\left(\frac{2}{5} - \frac{6}{5}x^2\right)^{-1/2}$$

EXAMPLE: $\tan(x+2y) = \dfrac{y}{1+x^2}$

Find dy/dx by implicit differentiation.

$$[\sec^2(x+2y)][1+2y'] = \frac{(1+x^2)(\quad y' \quad) - y(\quad 2x \quad)}{(1+x^2)^2}$$

LINEAR IN y'

$$[\sec^2(x+2y)]+2[\sec^2(x+2y)]y' = \left[\frac{(1+x^2)}{(1+x^2)^2}\right] y' - \frac{2xy}{(1+x^2)^2}$$

$$-\left[\frac{(1+x^2)}{(1+x^2)^2}\right] y' + 2[\sec^2(x+2y)]y' = -[\sec^2(x+2y)] - \frac{2xy}{(1+x^2)^2}$$

$$\|$$

$$\left[-\frac{(1+x^2)}{(1+x^2)^2} + 2[\sec^2(x+2y)]\right] y'$$

$$y' = \frac{-[\sec^2(x+2y)] - \frac{2xy}{(1+x^2)^2}}{-\frac{(1+x^2)}{(1+x^2)^2} + 2[\sec^2(x+2y)]}$$ ■

EXAMPLE: $\tan(x + 2y) = \dfrac{y}{1 + x^2}$

Find dy/dx by implicit differentiation.

ALTERNATE BOOKKEEPING SYTEM...

$\longleftarrow x$ terms $\downarrow$ $\qquad\qquad$ $\uparrow$ no x terms $\longrightarrow$

BASIC ALGEBRA PROBLEM:

Solve $7(4x + 8) = \dfrac{2x - 6}{7}$ for x.

$$(7)(4)x + (7)(8) = \tfrac{2}{7}x - \tfrac{6}{7}$$

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$$x = \frac{-(7)(8) - \frac{6}{7}}{(7)(4) - \frac{2}{7}} \quad \blacksquare$$

EXAMPLE: $\tan(x+2y) = \dfrac{y}{1+x^2}$

Find dy/dx by implicit differentiation.

ALTERNATE BOOKKEEPING SYTEM...

$\longleftarrow x$ terms $\downarrow$ $\qquad\qquad$ $\uparrow$ no x terms$\longrightarrow$

BASIC ALGEBRA PROBLEM:

$$\downarrow \quad \uparrow \quad \downarrow \quad \uparrow$$
$$+ \quad - \quad - \quad +$$

Solve $7(\boxed{4}x+\boxed{8}) = \dfrac{\boxed{2}x\boxed{-6}}{7}$ for x.

$$x = \frac{-(7)(8)+\frac{-6}{7}}{(7)(4)-\frac{2}{7}} \blacksquare$$

$$x = \frac{-(7)(8)-\frac{6}{7}}{(7)(4)-\frac{2}{7}}$$

EXAMPLE: $\tan(x+2y) = \dfrac{y}{1+x^2}$

Find dy/dx by implicit differentiation.

ALTERNATE BOOKKEEPING SYTEM:

$\longleftarrow y'$ terms

no y' terms $\longrightarrow$

$$[\sec^2(x+2y)][\boxed{1}+\boxed{2}y'] = \frac{\boxed{(1+x^2)}(\quad y' \quad) \boxed{-\,y(\ 2x \ )}}{(1+x^2)^2}$$

$-$ $+$ $-$ $+$

EXAMPLE: $\tan(x+2y) = \dfrac{y}{1+x^2}$

Find dy/dx by implicit differentiation.

ALTERNATE BOOKKEEPING SYTEM:

$\longleftarrow y'$ terms $\qquad\qquad$ no y' terms$\longrightarrow$

$$[\sec^2(x+2y)][\boxed{1}+\boxed{2}y'] = \frac{(\boxed{1+x^2})(\quad y' \quad)\boxed{-y(\quad 2x \quad)}}{(1+x^2)^2}$$

$$y' = \frac{-[\sec^2(x+2y)] + \frac{-2xy}{(1+x^2)^2}}{2[\sec^2(x+2y)] - \frac{(1+x^2)}{(1+x^2)^2}} \qquad \blacksquare$$

EXAMPLE: If $[g(x)] + x^3[\sin(g(x))] = x^5$, find $g'(0)$.

$$\frac{d}{dx} \qquad\qquad \frac{d}{dx}$$

$$[g'(x)] \; + \; [3x^2]\,\boxed{[\sin(g(x))]} + [\boxed{x^3}][[\cos(g(x))][g'(x)]] = 5x^4$$

$x :\to 0$

$$[g'(0)] \; + \underbrace{\cancel{[3(0^2)]}}_{0}\cancel{[\sin(g(0))]} + \underbrace{\cancel{[0^3]}}_{0}\cancel{[[\cos(g(0))][g'(0)]]} = \underbrace{5(0^4)}_{0}$$

$g'(0) = 0$ ■

EXAMPLE: Use implicit differentiation to find an equation of the tangent line to the hyperbola

$$d/dx \longrightarrow x^2 + 4xy + 3y^2 + 2x + 4y = -1$$

at the point $(2, -3)$.

$$2x + 4y + 4xy' + 6yy' + 2 + 4y' = 0$$

$$y' = \frac{-2x - 4y - 2}{4x + 6y + 4}$$

$$\left[y'\right]_{x:\to 2, y=-3} = \frac{-2(2) - 4(-3) - 2}{4(2) + 6(-3) + 4}$$

$$= \frac{-4 + 12 - 2}{8 - 18 + 4} = \frac{6}{-6} = -1$$

$$y - (-3) = (-1)(x - 2) \blacksquare$$

EXAMPLE: Use implicit differentiation to find an equation of the tangent line to the hyperbola

$$d/dx \quad x^2 + 4xy + 3y^2 + 2x + 4y = -1$$

at the point $(2, -3)$.

Alternate solution: $y - (-3) \overset{?}{=} m(x - 2)$

$$\underset{2}{2x} + \underset{-3}{4y} + 4\underset{2}{x}\overset{m}{y'} + 6\underset{-3}{y}\overset{m}{y'} + 2 + 4\overset{m}{y'} = 0$$

$$4 - 12 + 8m - 18m + 2 + 4m = 0$$

$$-6 - 6m = 0$$

$$m = -1$$

$$y - (-3) = (-1)(x - 2) \blacksquare$$

$$y - (-3) = (-1)(x - 2)$$

EXAMPLE: Use implicit differentiation to find an equation of the tangent line to the "devil's curve"

$$\xrightarrow{d/dx} y^2(y^2-9) = x^2(x^2-3)$$

at the point $(0,-3)$.

$$y-(-3) \overset{?}{=} m(x-0)$$

$$y^2(\quad 2yy' \quad) + \quad 2yy'(y^2-9) = \quad x^2(\quad 2x \quad) + \quad 2x(x^2-3)$$

$$\frac{d}{dx} \qquad\qquad \frac{d}{dx}$$

$$2yy' \qquad 2yy' \qquad\qquad 2x \qquad 2x$$

EXAMPLE: Use implicit differentiation to find an equation of the tangent line to the "devil's curve"

$$y^2(y^2-9)=x^2(x^2-3)$$

at the point $(0,-3)$.

$$y-(-3)=\overset{?}{m}(x-0)$$

$$\underset{-3}{y}^2(\,2\underset{-3}{y}\overset{m}{y'}\,)+2\underset{-3}{y}\overset{m}{y'}(\underset{-3}{y}^2-9)=\underset{0}{x}^2(\,2\underset{0}{x}\,)+2\underset{0}{x}(\underset{0}{x}^2-3)$$

$$9(\,-6m\,)+\cancel{(-6m)(9-9)}=0(\,0\,)+\cancel{0\,(0-3)}$$

$$-54m=0$$

$$m=0$$

$$y-(-3)=0(x-0)$$

$$y=-3$$ ■

EXAMPLE: Find an equation of the tangent line to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0).

DIVIDE BY 2

$$\frac{\cancel{2}x}{a^2} - \frac{\cancel{2}yy'}{b^2} = 0$$

$$x :\longrightarrow x_0, y :\longrightarrow y_0, y' :\longrightarrow m$$

$$\frac{x_0}{a^2} - \boxed{\frac{y_0 m}{b^2}} = 0$$

SOLVE FOR m

$$\frac{x_0}{a^2} = \frac{y_0 m}{b^2}$$

$$\frac{b^2 x_0}{a^2 y_0} = m$$

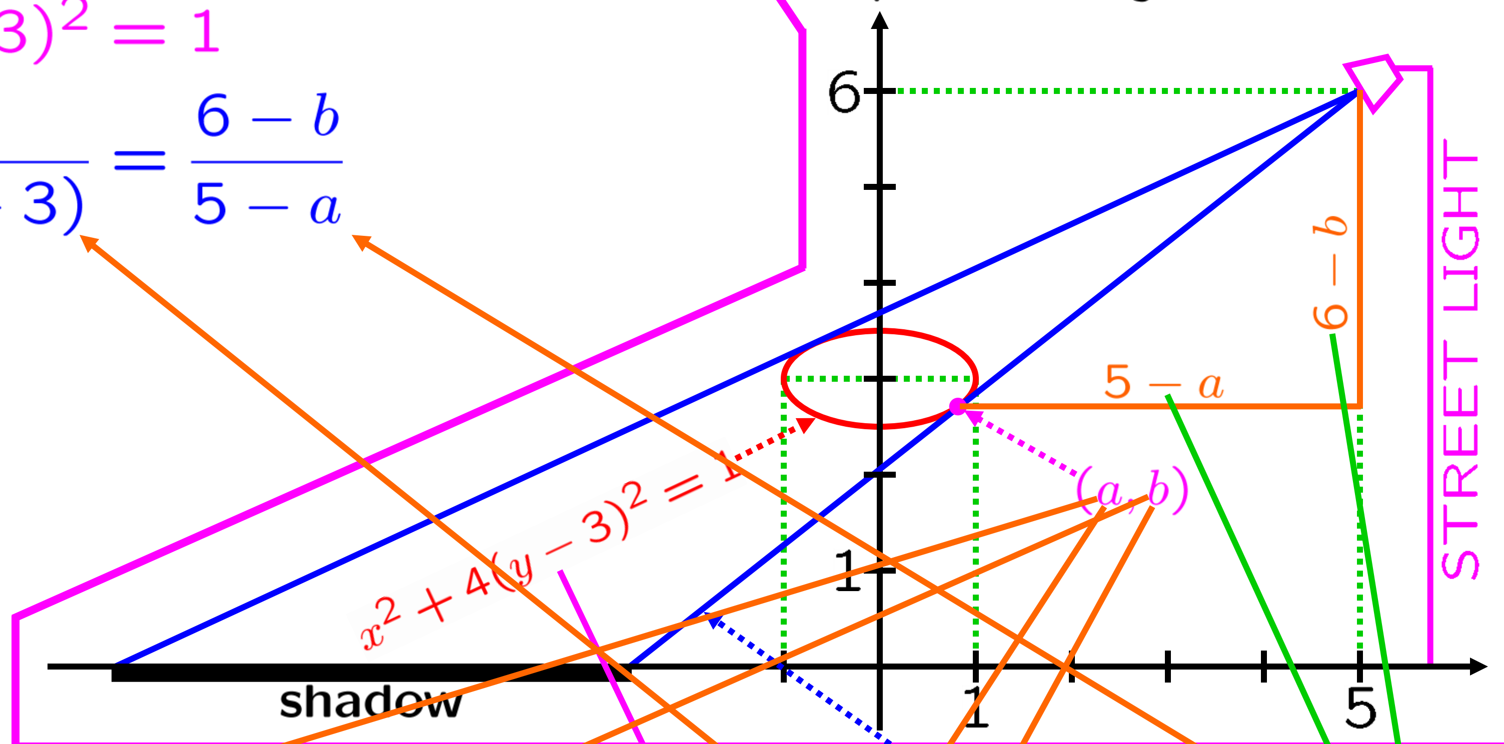
$$y - y_0 = \overset{?}{m}(x - x_0)$$

$$y - y_0 = \frac{b^2 x_0}{a^2 y_0}(x - x_0)$$

EXAMPLE: How long is the **shadow** cast by the elliptical object below?

$$a^2 + 4(b-3)^2 = 1$$

$$-\frac{a}{4(b-3)} = \frac{6-b}{5-a}$$



$x :\to a$ $\quad y :\to b$

$$d/dx \longrightarrow x^2 + 4(y-3)^2 = 1$$

$$2x + 8(y-3)y' = 0$$

$x :\to a$

$$y' = -\frac{x}{4(y-3)}$$

$y = b$

$$\text{slope} = \frac{6-b}{5-a}$$

$$\| $$

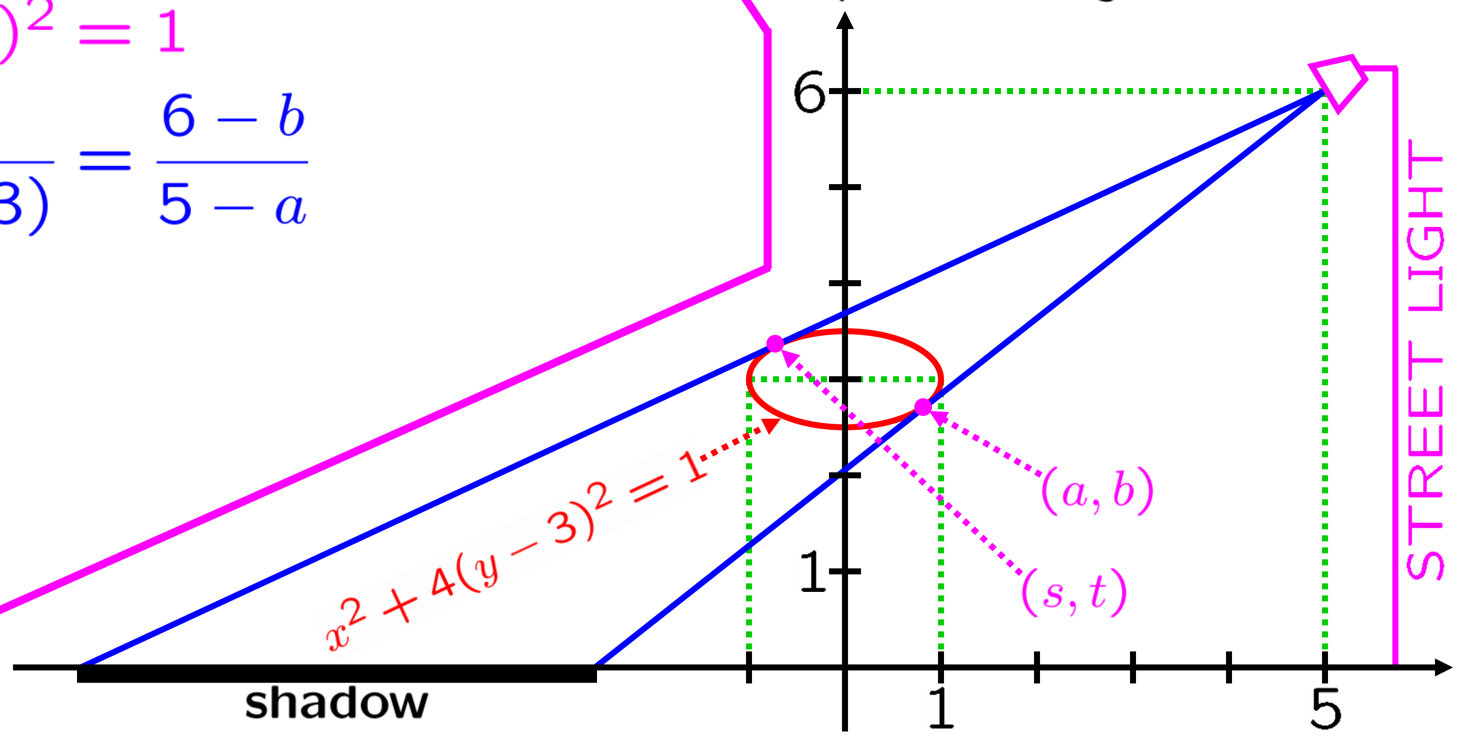
$$-\frac{a}{4(b-3)}$$

EXAMPLE: How long is the **shadow** cast by the elliptical object below?

LET'S WORK ON THESE FIRST:

$$a^2 + 4(b-3)^2 = 1$$

$$-\frac{a}{4(b-3)} = \frac{6-b}{5-a}$$



$a :\to s$
$b :\to t$

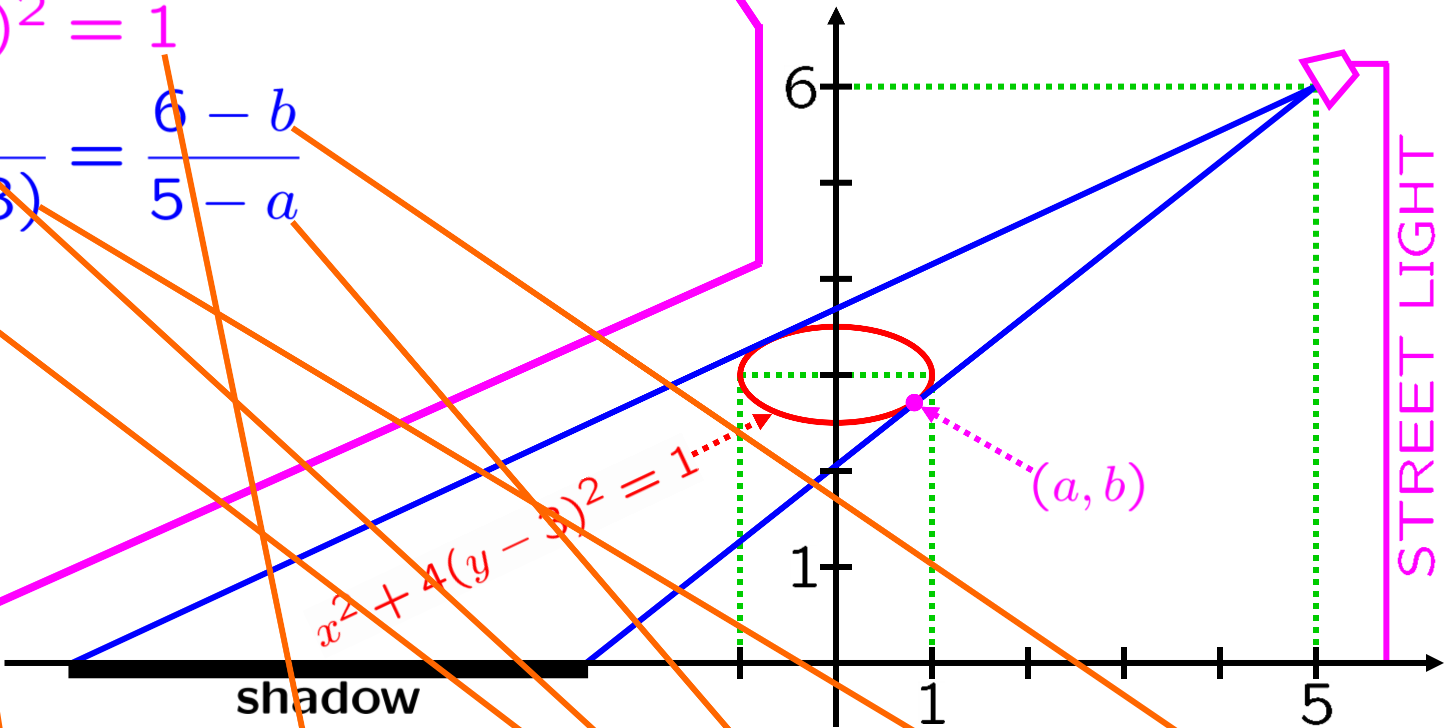
$$s^2 + 4(t-3)^2 = 1$$

$$-\frac{s}{4(t-3)} = \frac{6-t}{5-s}$$

EXAMPLE: How long is the **shadow** cast by the elliptical object below?

$$a^2+4(b-3)^2=1$$

$$-\frac{a}{4(b-3)}=\frac{6-b}{5-a}$$



EXPAND

$$a^2+4(b^2-6b+9)=1$$

$$a^2+4b^2-24b+\boxed{36}=1$$

$$\cancel{a^2}+\cancel{4b^2}\qquad -24b=-35$$

$$-\cancel{a^2}-\cancel{4b^2}+5a+36b=72$$

ADD

$$0+0+5a+12b=37$$

$$-a(5-a)=4(b-3)(6-b)$$

$$-a(a-5)=4(b-3)(b-6)$$

$$-a^2+5a=4(b^2-9b+18)$$

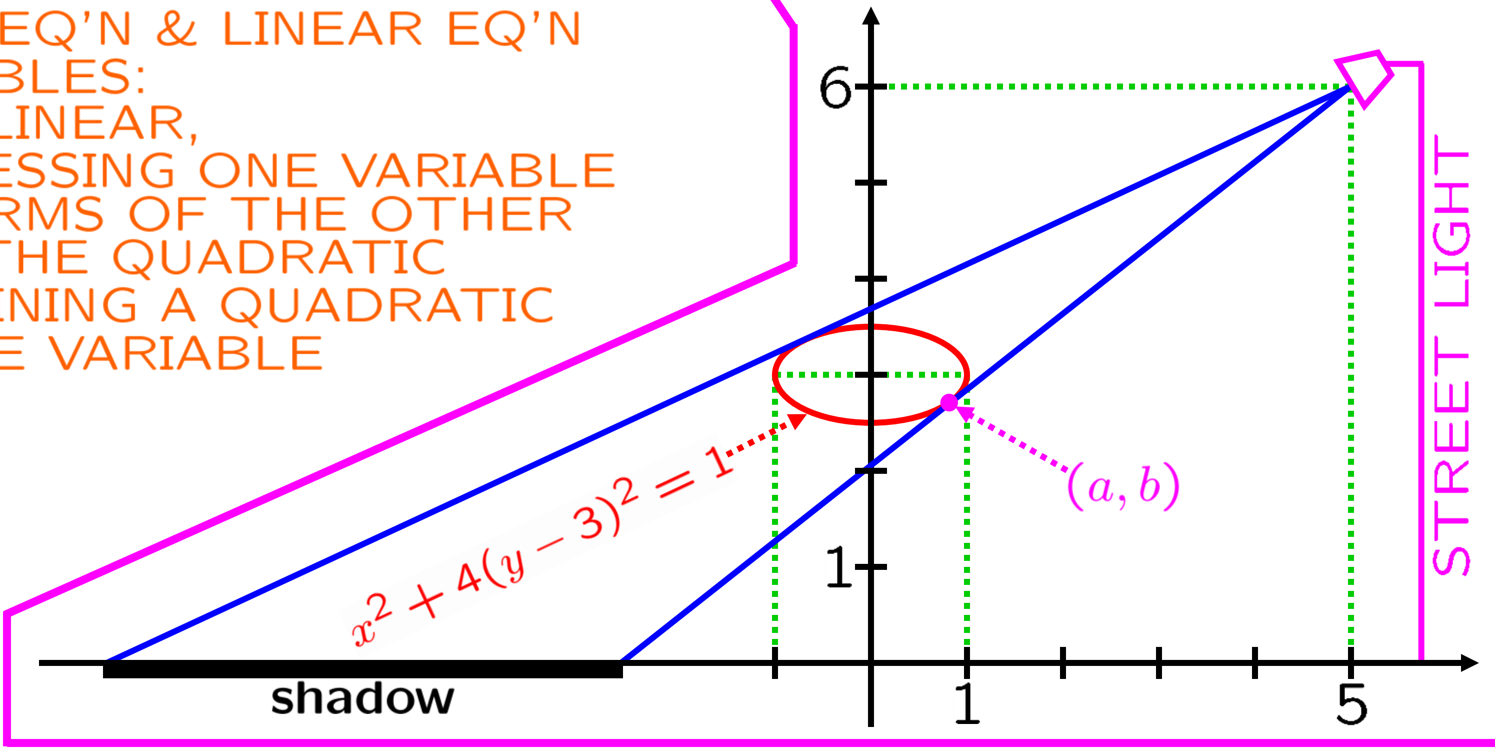
$$-a^2+5a=\boxed{4b^2}-\boxed{36b}+72$$

2 QUADRATICS IN 2 VARIABLES

§4.9 HOPE FOR CANCELING QUADRATIC PARTS

EXAMPLE: How long is the **shadow** cast by the elliptical object below?

TO SOLVE
QUADRATIC EQ'N & LINEAR EQ'N
IN TWO VARIABLES:
SOLVE THE LINEAR,
EXPRESSING ONE VARIABLE
IN TERMS OF THE OTHER
PLUG INTO THE QUADRATIC
OBTAINING A QUADRATIC
IN ONE VARIABLE



QUADRATIC EQ'N & LINEAR EQ'N
IN TWO VARIABLES:

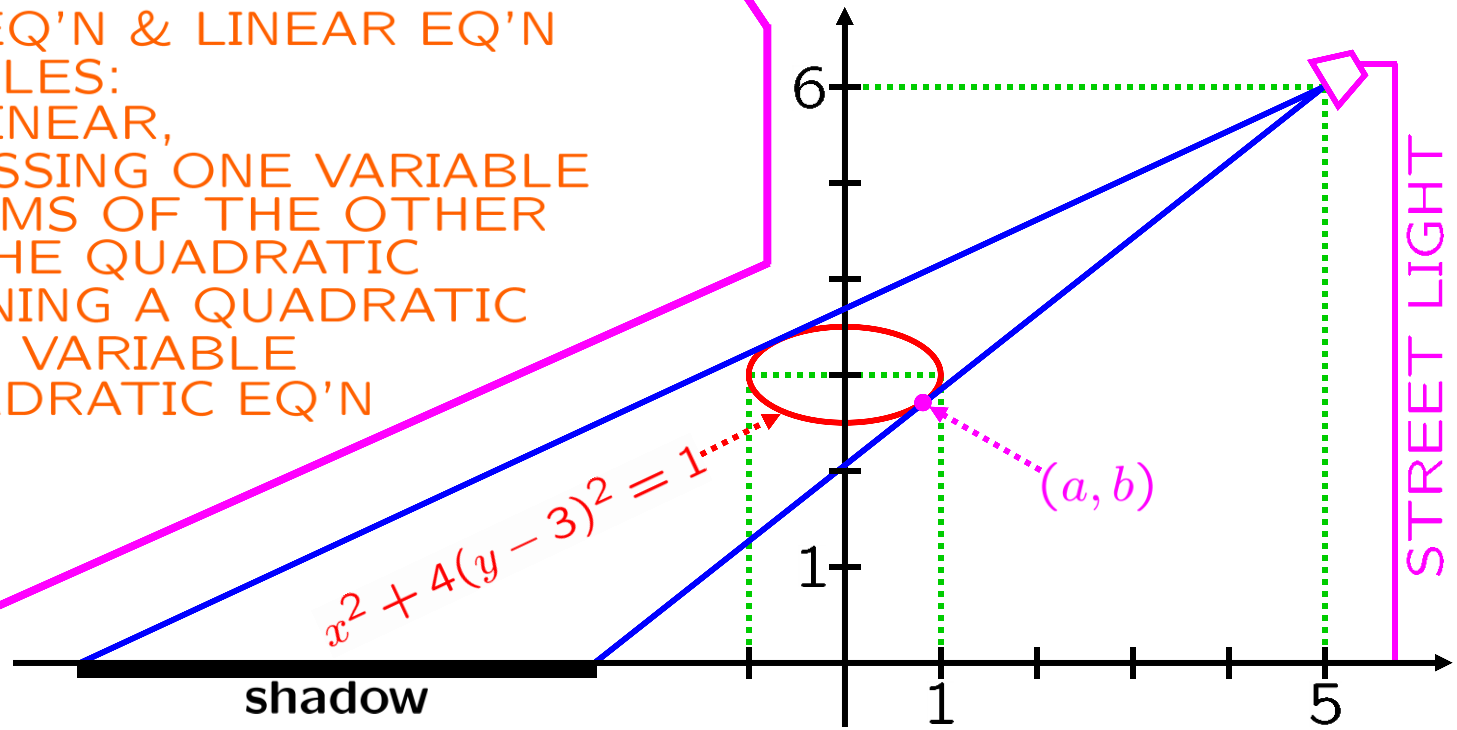
$$a^2 + 4b^2 \qquad - 24b = -35$$

$$5a + 12b = 37$$

$$a = \frac{37 - 12b}{5}$$

EXAMPLE: How long is the **shadow** cast by the elliptical object below?

TO SOLVE
QUADRATIC EQ'N & LINEAR EQ'N
IN TWO VARIABLES:
SOLVE THE LINEAR,
EXPRESSING ONE VARIABLE
IN TERMS OF THE OTHER
PLUG INTO THE QUADRATIC
OBTAINING A QUADRATIC
IN ONE VARIABLE
USE THE QUADRATIC EQ'N



$$\left(\frac{37-12b}{5}\right)^2 + 4b^2 - 24b = -35$$

$$a^2 + 4b^2 \qquad - 24b = -35$$

$$5a + 12b = 37$$

$$a = \boxed{\frac{37-12b}{5}}$$

EXAMPLE: How long is the **shadow** cast by the elliptical object below?

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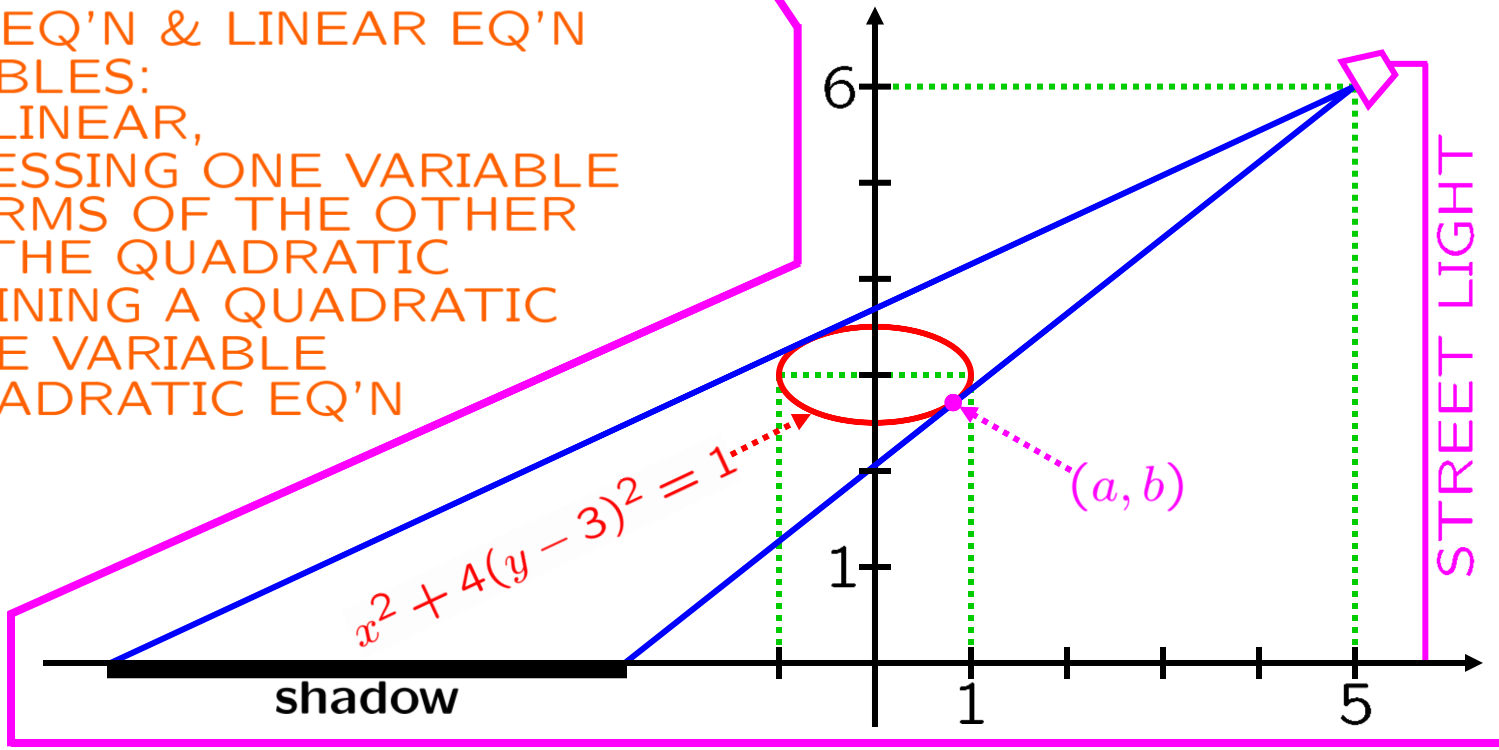


DISTRIBUTE SQUARING OVER DIVISION

$$\left(\frac{37-12b}{5}\right)^2 + 4b^2 - 24b = -35$$

$$\left[\frac{(37-12b)^2}{25} + 4b^2 - 24b = -35\right] \times 25$$

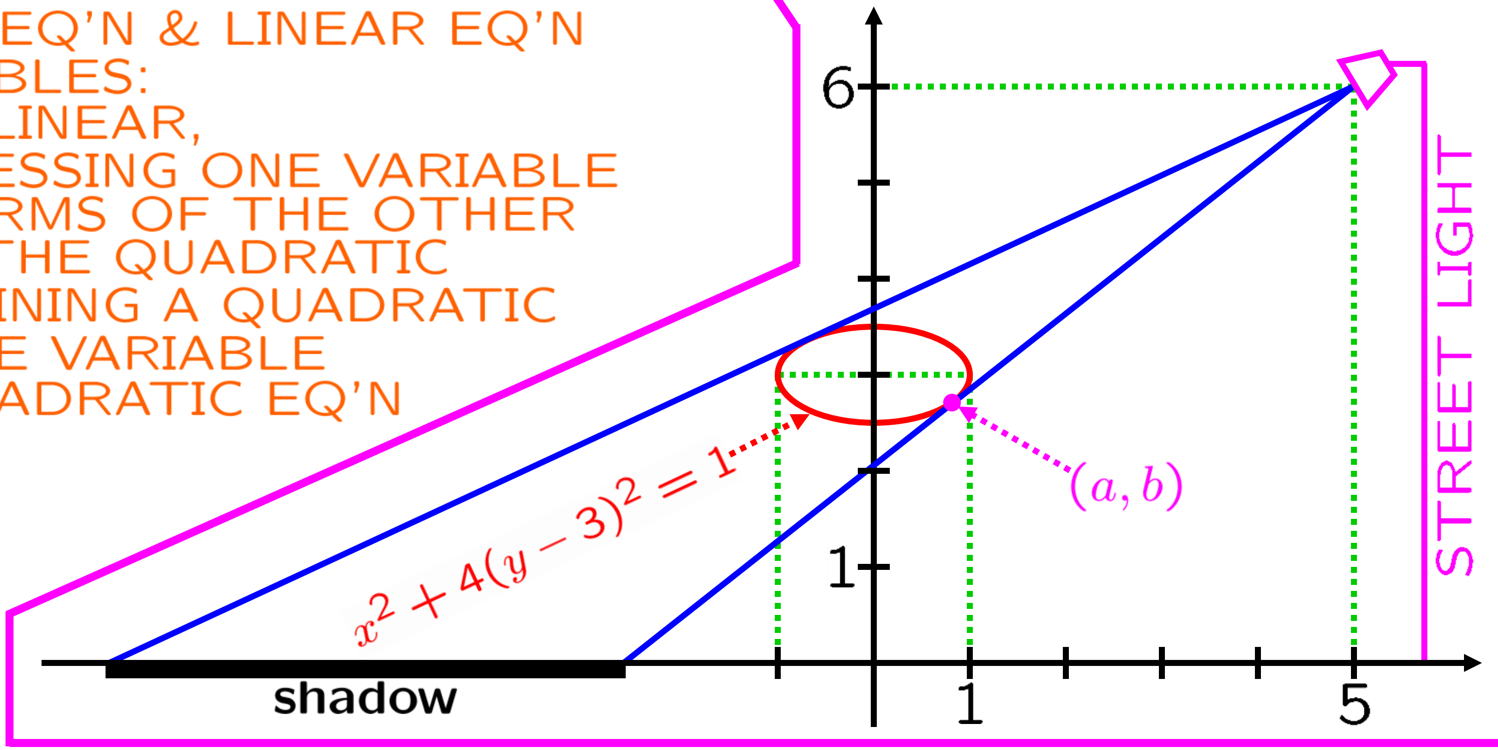
EXPAND

$$(37-12b)^2 + 100b^2 - 600b = -875$$

$$a = \frac{37-12b}{5}$$

EXAMPLE: How long is the **shadow** cast by the elliptical object below?

TO SOLVE
QUADRATIC EQ'N & LINEAR EQ'N
IN TWO VARIABLES:
SOLVE THE LINEAR,
EXPRESSING ONE VARIABLE
IN TERMS OF THE OTHER
PLUG INTO THE QUADRATIC
OBTAINING A QUADRATIC
IN ONE VARIABLE
USE THE QUADRATIC EQ'N



$$a = \frac{37 - 12b}{5}$$

$$\left(\frac{37 - 12b}{5}\right)^2 + 4b^2 - 24b = -35$$

$$\frac{(37 - 12b)^2}{25} + 4b^2 - 24b = -35$$

$$(37 - 12b)^2 + 100b^2 - 600b = -875$$

EXPAND

$$(1369 - 888b \cdot \quad 00b^2 - 600b = -875$$

$$a = \frac{37 - 12b}{5}$$

:T TERMS

EXAMPLE: How long is the **shadow** cast by the elliptical object below?

TO SOLVE
QUADRATIC EQ'N & LINEAR EQ'N
IN TWO VARIABLES:
SOLVE THE LINEAR,
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IN TERMS OF THE OTHER
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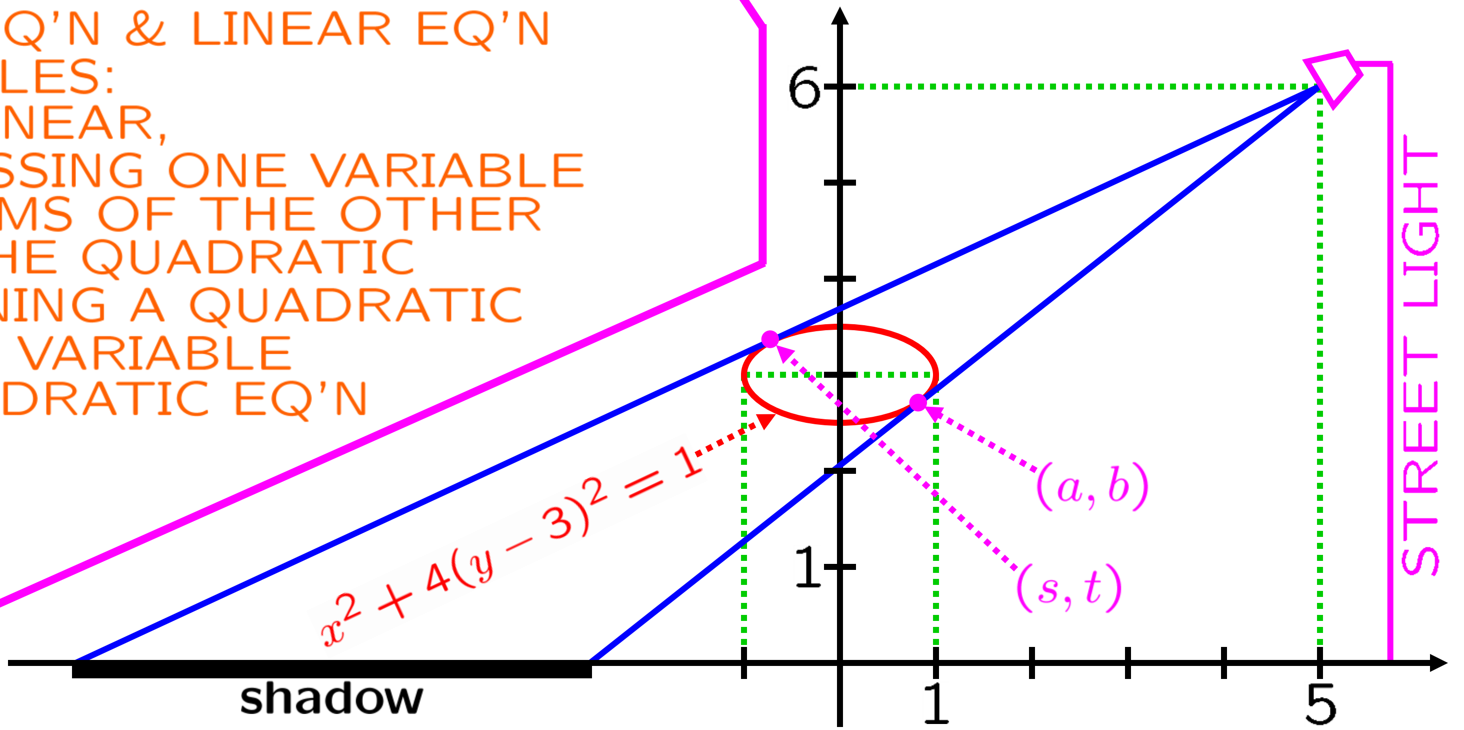
OBTAINING A QUADRATIC
IN ONE VARIABLE
USE THE QUADRATIC EQ'N



$$a = \frac{37-12b}{5}$$

$$(144+100)b^2-(888+600)b+(1369+875)=0$$

$$(1/4)\times\ \left[244b^2-1488b+2244\right]=0$$

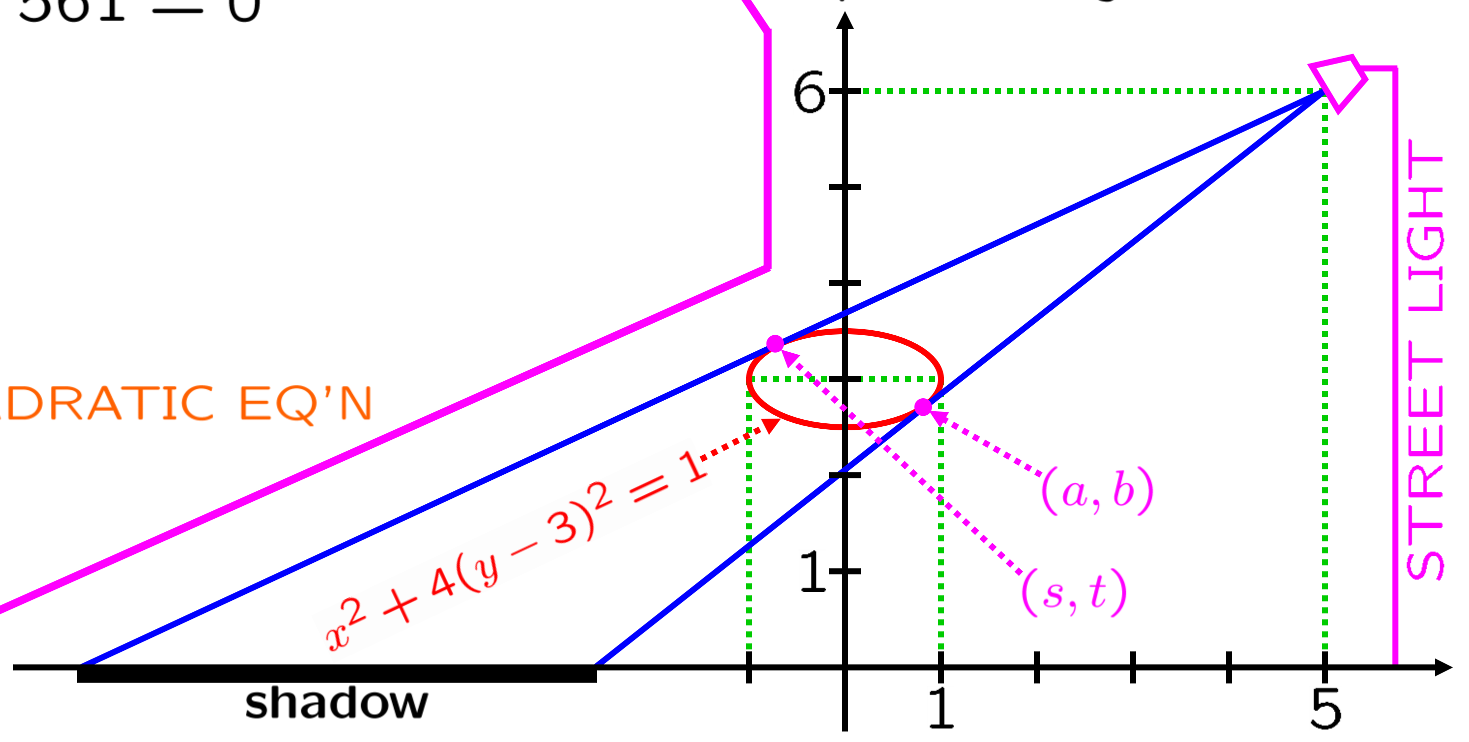
$$61b^2-372b+561=0$$

$$(1369-888b+144b^2)+100b^2-600b=-875$$

COLLECT TERMS

EXAMPLE: How long is the **shadow** cast by the elliptical object below?

$$61t^2 - 372t + 561 = 0$$



$$s = \frac{37 - 12t}{5}$$

USE THE QUADRATIC EQ'N

$$a = \frac{37 - 12b}{5}$$

USE THE QUADRATIC EQ'N

$$61b^2 - 372b + 561 = 0$$

$a :\to s$
$b :\to t$

$$61b^2 - 372b + 561 = 0$$

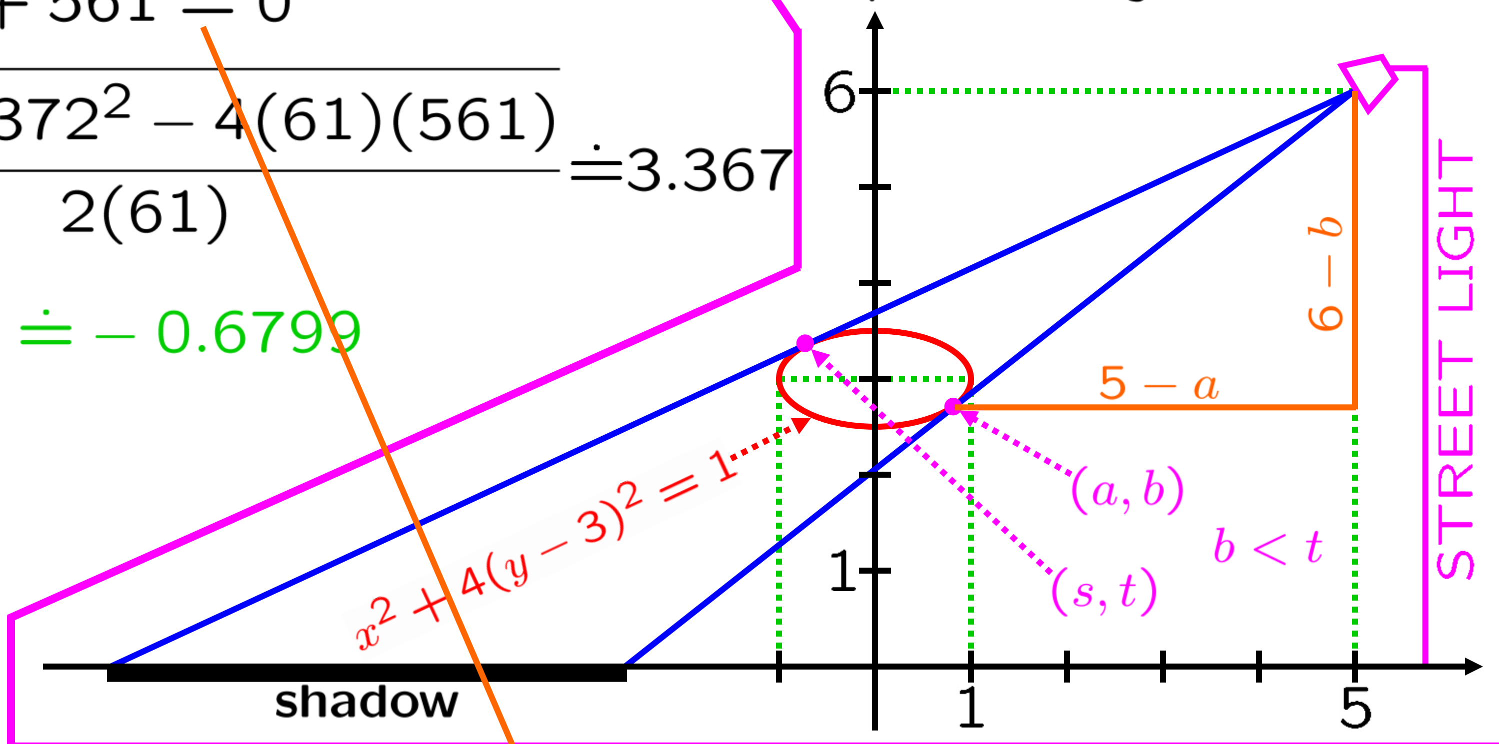
$$a = \frac{37 - 12b}{5}$$

EXAMPLE: How long is the **shadow** cast by the elliptical object below?

$$61t^2 - 372t + 561 = 0$$

$$t = \frac{372 + \sqrt{372^2 - 4(61)(561)}}{2(61)} \doteq 3.367$$

$$s = \frac{37 - 12t}{5} \doteq -0.6799$$



USE THE QUADRATIC EQ'N

$$61b^2 - 372b + 561 = 0$$

$$b = \frac{372 - \sqrt{372^2 - 4(61)(561)}}{2(61)} \doteq 2.732$$

$$a = \frac{37 - 12b}{5} \doteq 0.8439$$

$61y^2 - 372y + 561 = 0$
has solutions $y = b$ and $y = t$.

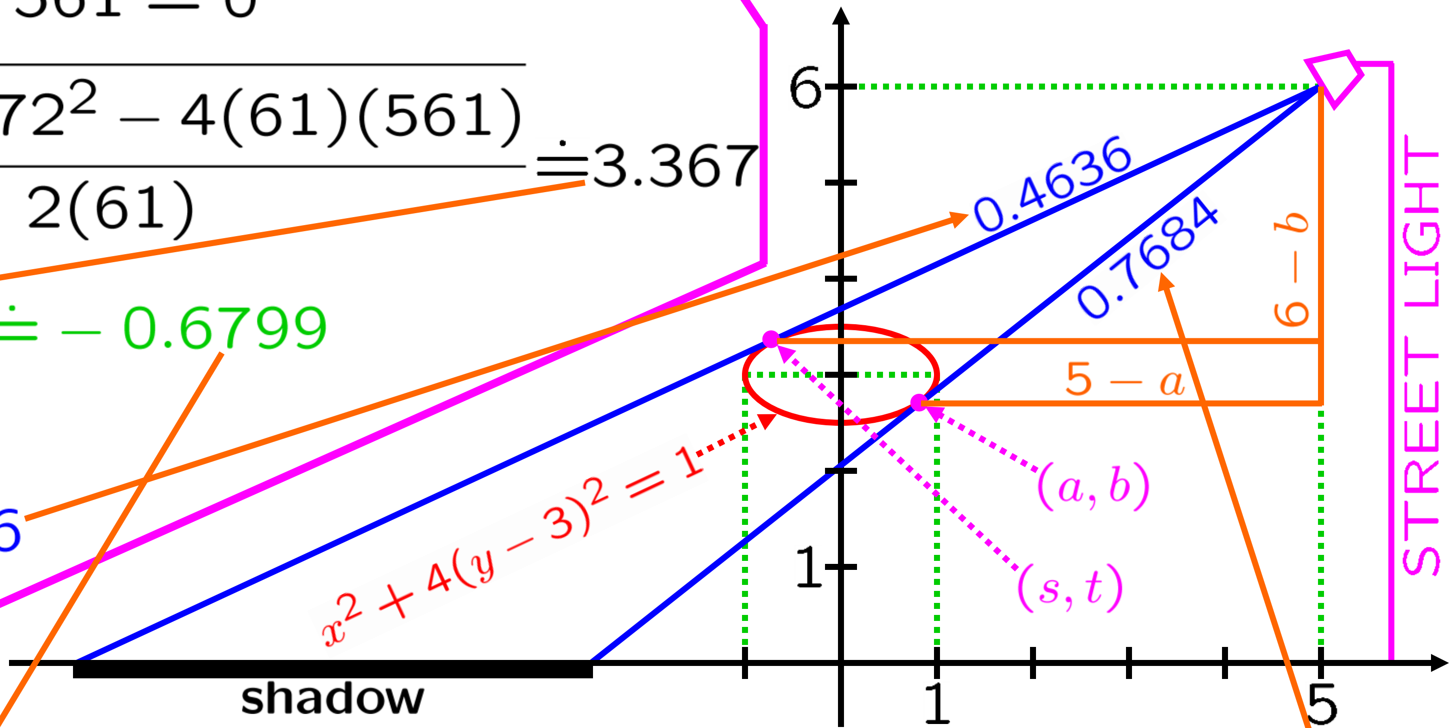
EXAMPLE: How long is the **shadow** cast by the elliptical object below?

$$61t^2 - 372t + 561 = 0$$

$$t = \frac{372 + \sqrt{372^2 - 4(61)(561)}}{2(61)} \doteq 3.367$$

$$s = \frac{37 - 12t}{5} \doteq -0.6799$$

$$\frac{6-t}{5-s} \doteq 0.4636$$



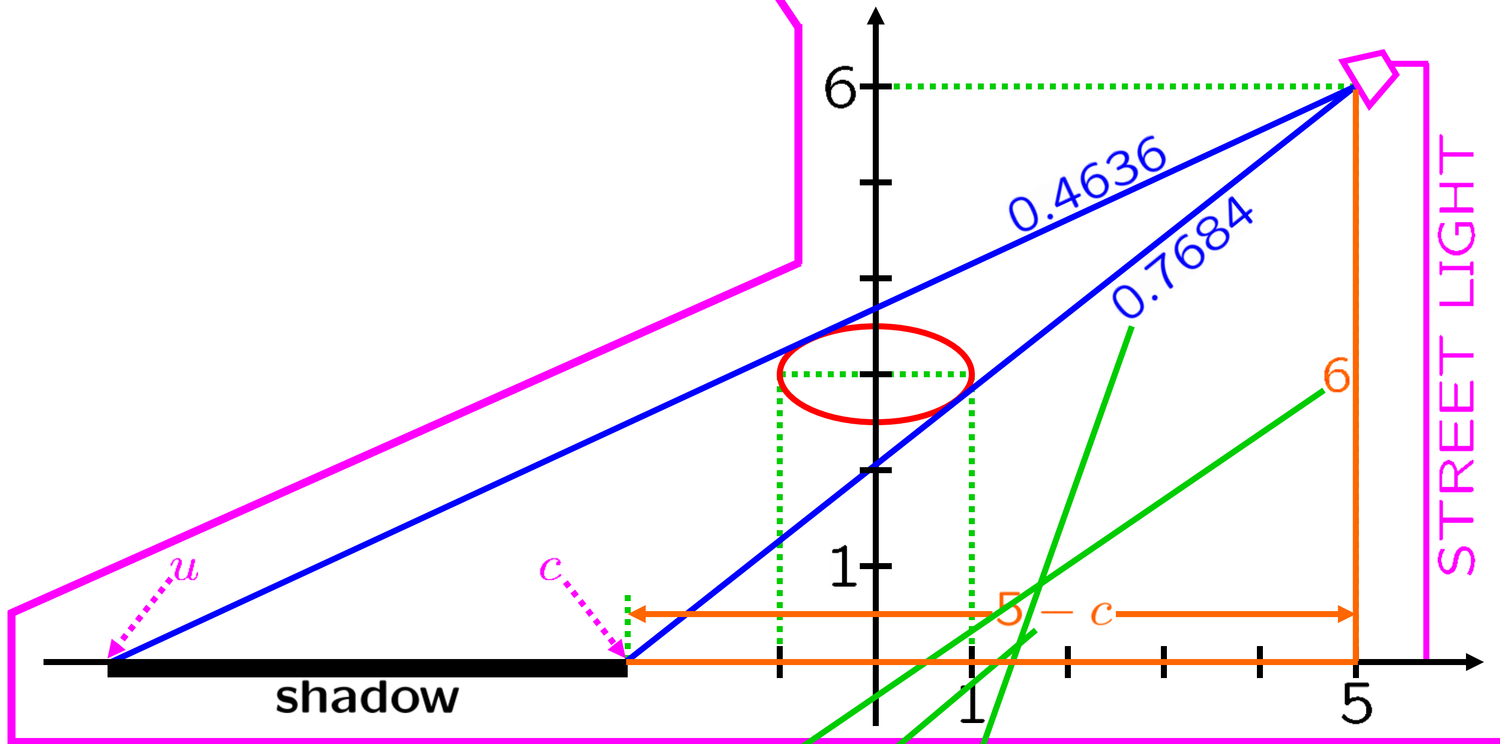
$$\frac{6-b}{5-a} \doteq \frac{6-2.732}{5-0.8439} \doteq 0.7684$$

$$61b^2 - 372b + 561 = 0$$

$$b = \frac{372 - \sqrt{372^2 - 4(61)(561)}}{2(61)} \doteq 2.732$$

$$a = \frac{37 - 12b}{5} \doteq 0.8439$$

EXAMPLE: How long is the **shadow** cast by the elliptical object below?



$$\frac{6}{5-c} \doteq 0.7684$$

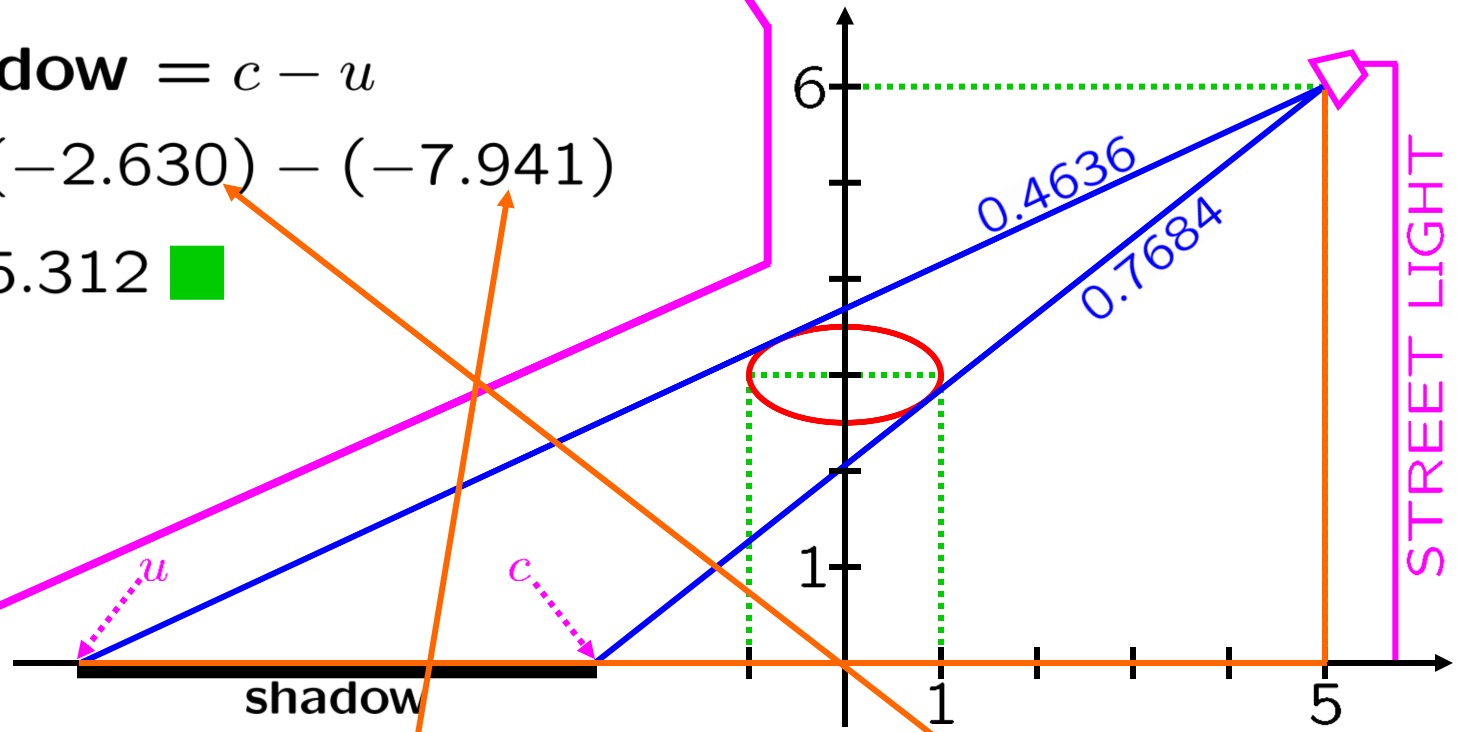
$$c \doteq 5 - \frac{6}{0.7684} \doteq -2.630$$

EXAMPLE: How long is the **shadow** cast by the elliptical object below?

length of **shadow** $= c - u$

$$\doteq (-2.630) - (-7.941)$$

$$\doteq 5.312 \quad ■$$



$$\frac{6}{5-u} \doteq 0.4636 \qquad \frac{6}{5-c} \doteq 0.7684$$

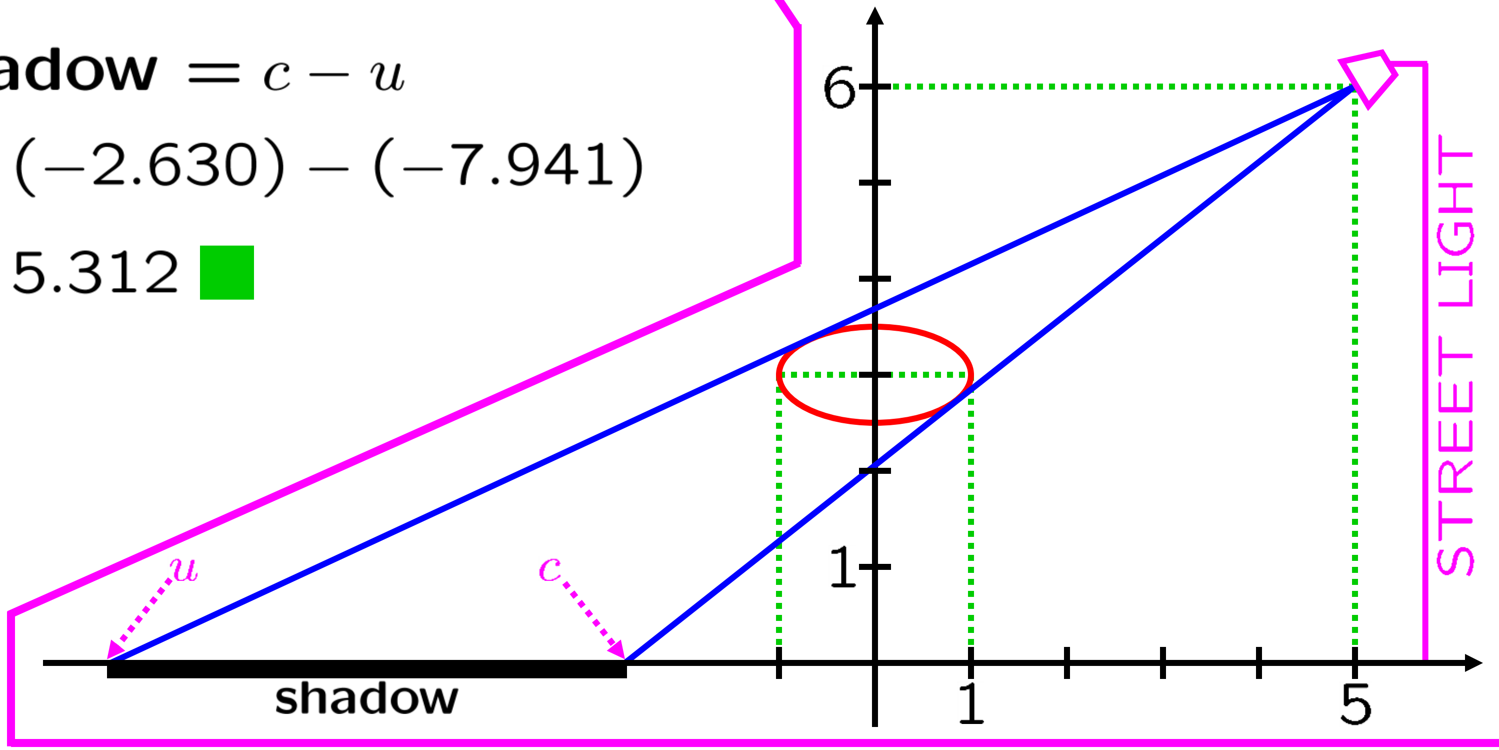
$$u \doteq 5 - \frac{6}{0.4636} \doteq -7.941 \qquad c \doteq 5 - \frac{6}{0.7684} \doteq -2.630$$

EXAMPLE: How long is the **shadow** cast by the elliptical object below?

length of **shadow** $= c - u$

$\doteq (-2.630) - (-7.941)$

$\doteq 5.312$ ∎



Note: You can also work backward – from the length of the shadow, you can compute the height of the street light.

SKILL

implicit diff

Whitman problems

§4.9, p. 88–89, #1-20