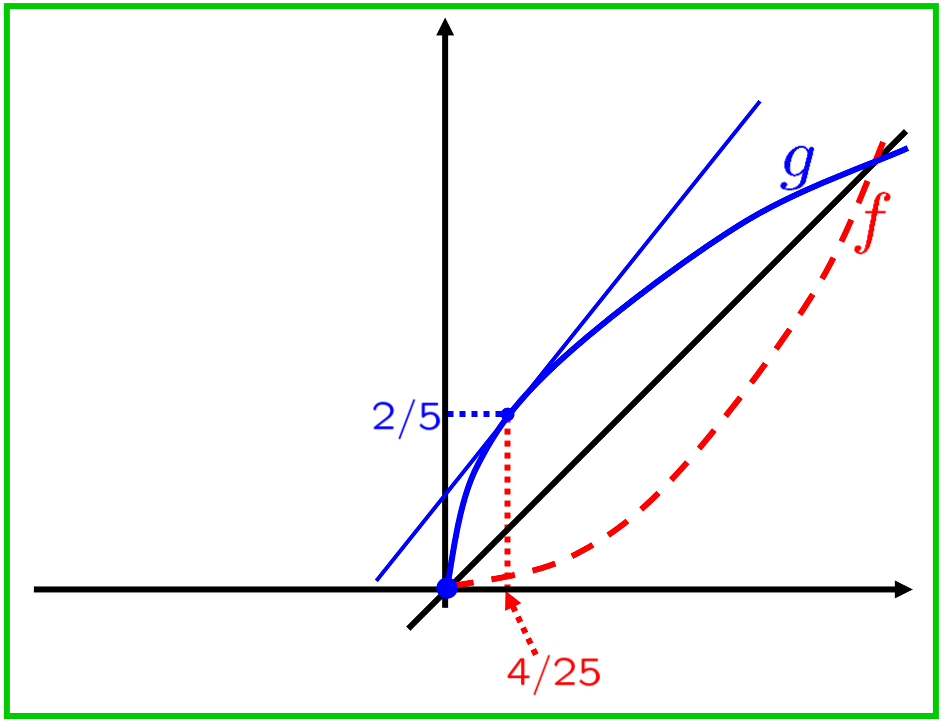
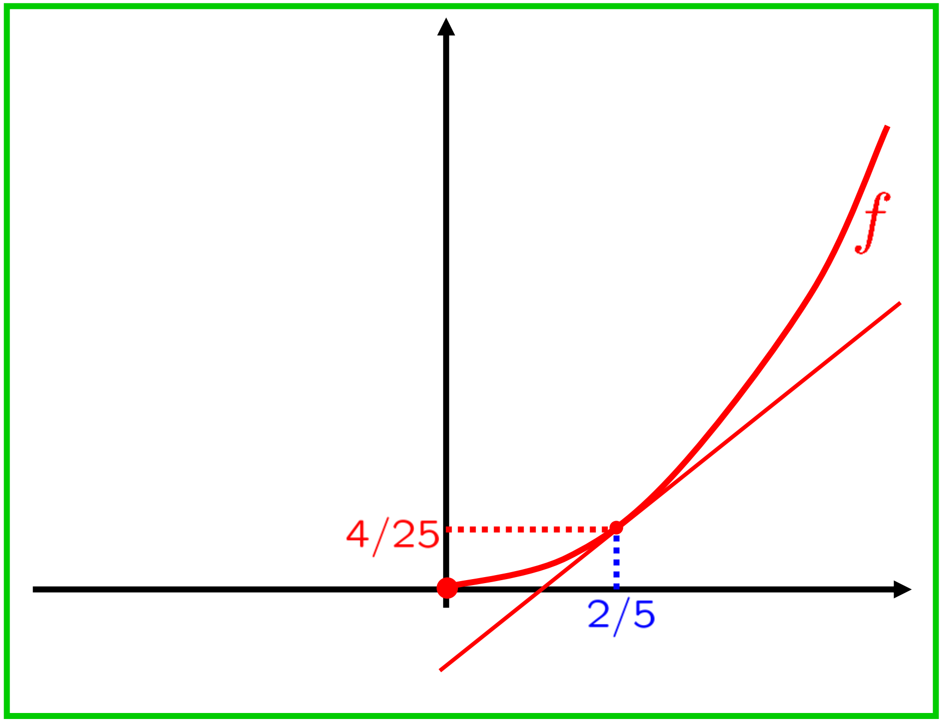
CALCULUS
Derivatives of inverse functions
(The Inverse Function Theorem)

DERIVATIVES OF INVERSE FUNCTIONS, AN EXAMPLE

$g := f^{-1} = \sqrt{\bullet}$ $f := (\bullet)^2 | [0, \infty)$





$f' := (2\bullet) | (0, \infty)$

$g(4/25) = 2/5$ $f(2/5) = 4/25$

$f'(2/5) = ???$

$f(-1)$ is undefined; $f(0) = 0$.
$f'(-1)$ and $f'(0)$ are undefined.

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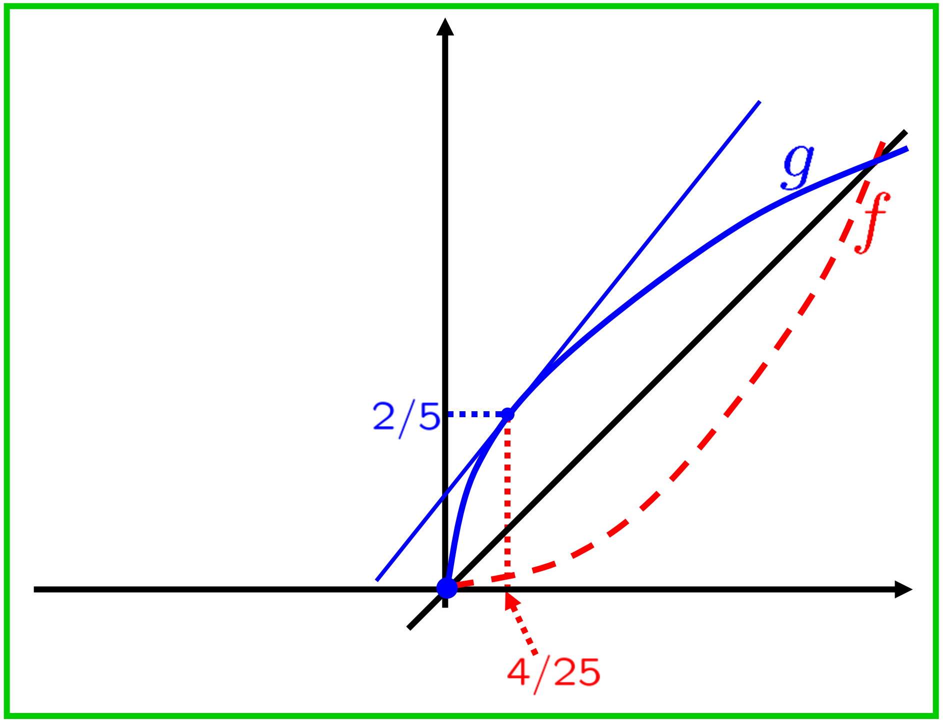

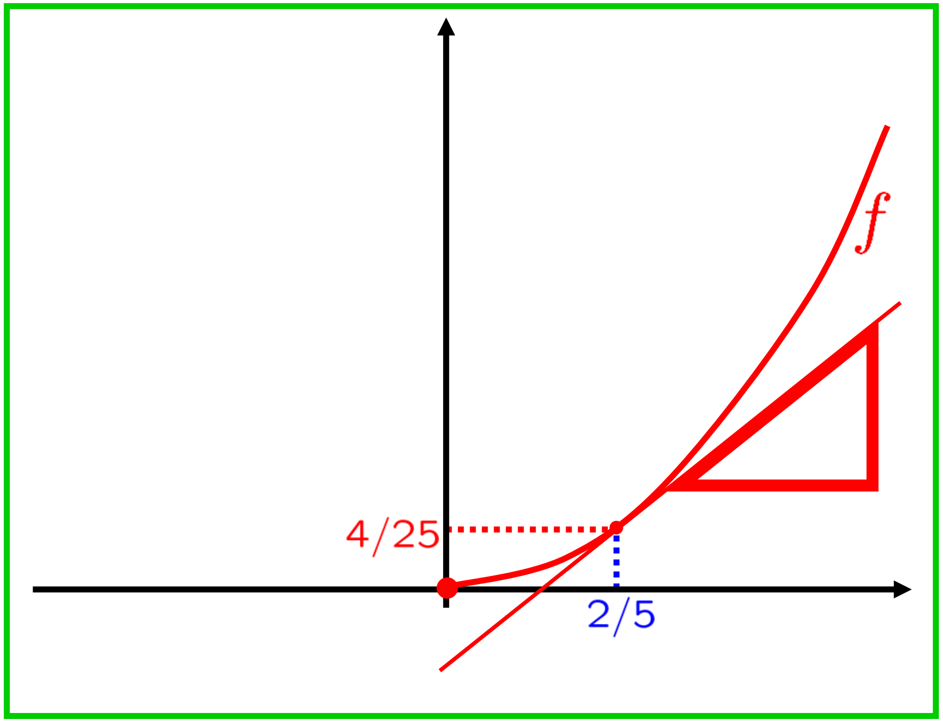

$$f' := (2\bullet) | (0, \infty)$$

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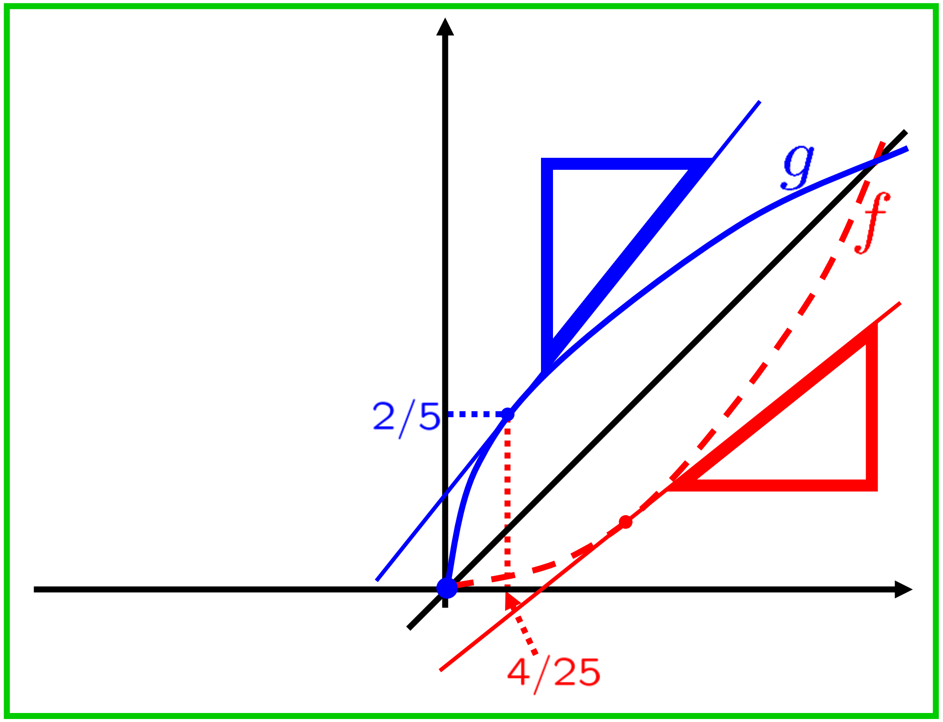

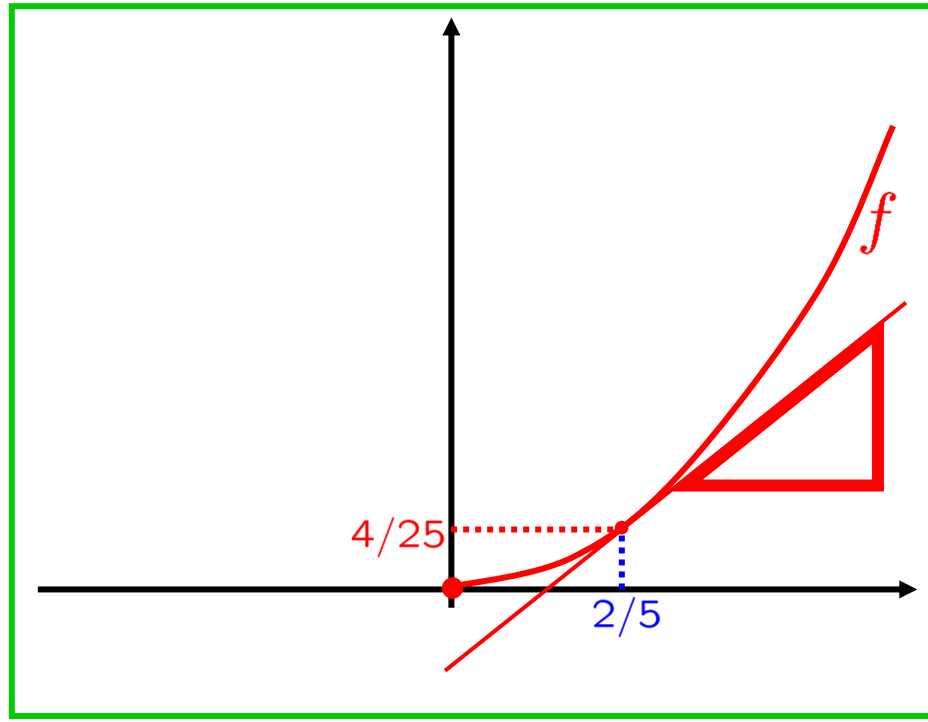

$f' := (2\bullet)|(0,\infty)$

$g(4/25) = 2/5$ $\qquad$ $f(2/5) = 4/25$

$g'(4/25) = ???$ $\qquad$ $f'(2/5) = 4/5$

$\frac{\text{RISE}}{\text{RUN}} = \frac{5}{4}$ $\qquad$ $\frac{\text{RISE}}{\text{RUN}} = \frac{4}{5}$

DERIVATIVES OF INVERSE FUNCTIONS, AN EXAMPLE

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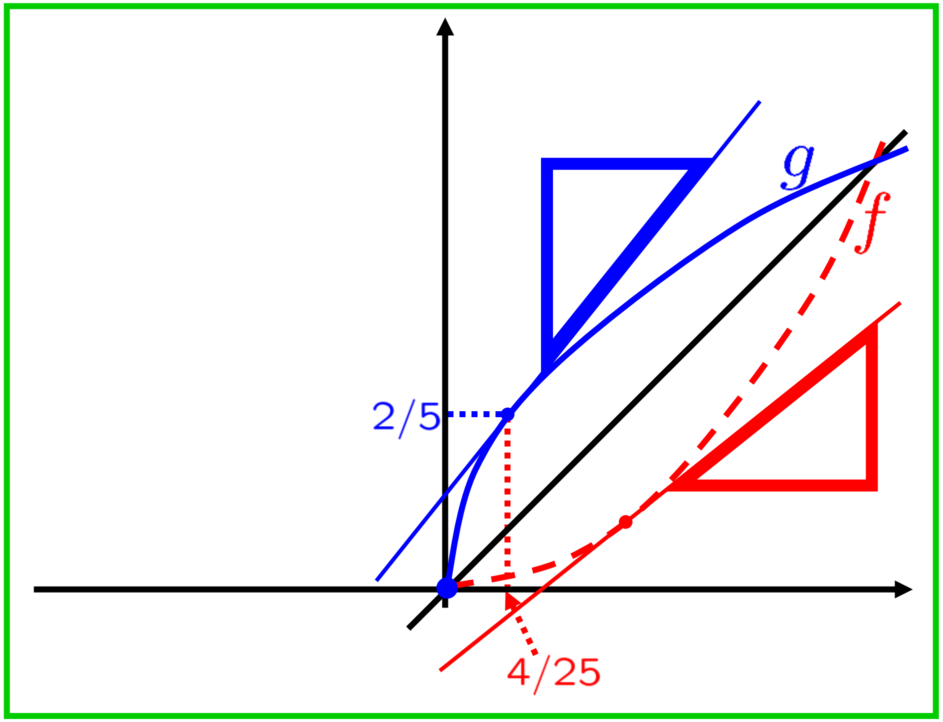

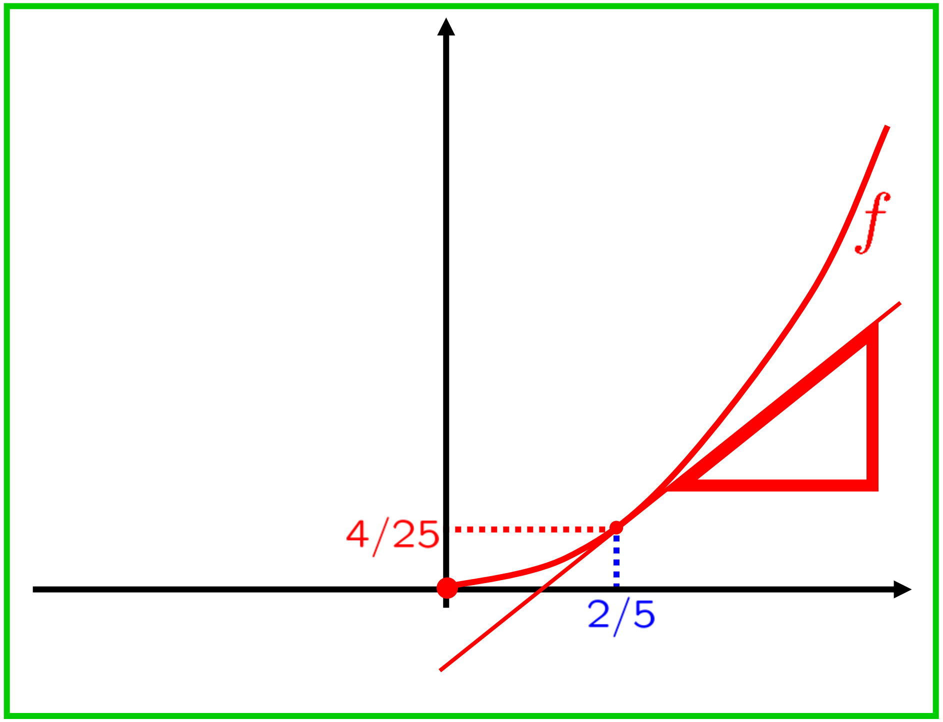

Next: A "symbolic" approach

$f' := (2\bullet) | (0, \infty)$

$g(4/25) = 2/5$

$g'(4/25) = 5/4$

$\frac{\text{RISE}}{\text{RUN}} = \frac{5}{4}$

$f(2/5) = 4/25$

$f'(2/5) = 4/5$

$\frac{\text{RISE}}{\text{RUN}} = \frac{4}{5}$

DERIVATIVES OF INVERSE FUNCTIONS, AN EXAMPLE

Let $f : [0,\infty) \to [0,\infty)$ be the restricted squaring function.

$f' := (2\bullet)|(0,\infty)$ Let $g := f^{-1} : [0,\infty) \to [0,\infty)$, so $g = \sqrt{\bullet}$.

$f(2/5) = 4/25$ $g(4/25) = 2/5$

$f'(2/5) = 4/5$ $g'(4/25) = ???$

$$f(y) = f(g(x))$$

$$f(y) = x$$

Want: $g'(x)$

DERIVATIVES OF INVERSE FUNCTIONS, AN EXAMPLE

Let $f : [0,\infty) \to [0,\infty)$ be the restricted squaring function.

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$f(2/5) = 4/25$ $g(4/25) = 2/5$

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$y = g(x)$

IMPLICIT DIFF. . . .

$$\frac{d}{dx}[f(y)] = \frac{d}{dx}[\,x\,]$$

CHAIN RULE

$y' = g'(x)$

$$[f'(y)][y'] = 1$$

$$g'(x) = y' = \frac{1}{f'(y)} = \frac{1}{f'(g(x))}$$

Want: $g'(x)$

$x :\to 4/25$

$$g'(4/25) = \frac{1}{f'(g(4/25))} = \frac{1}{f'(2/5)} = \frac{1}{4/5}$$

DERIVATIVES OF INVERSE FUNCTIONS, AN EXAMPLE

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$$f(2/5) = 4/25 \qquad g(4/25) = 2/5$$

$$f'(2/5) = 4/5 \qquad g'(4/25) = 5/4$$

$$y = g(x)$$

$$\frac{d}{dx}[f(y)] = \frac{d}{dx}[\,x\,]$$

$$y' = g'(x)$$

$$[f'(y)][y'] = 1$$

GENERAL FORMULA . . .

$$(f^{-1})'(x) = g'(x) = y' = \frac{1}{f'(y)} = \frac{1}{f'(g(x))} = \frac{1}{f'(f^{-1}(x))}$$

$x :\to 4/25$

$$g'(4/25) = \frac{1}{f'(g(4/25))} = \frac{1}{f'(2/5)} = \frac{1}{4/5}$$

THE INVERSE FUNCTION THEOREM

Let f be a 1-1 function.

If $f'(f^{-1}(x))$ exists and is nonzero,

then $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$.

$$(f^{-1})'(x) \qquad = \frac{1}{f'(f^{-1}(x))}$$

THE INVERSE FUNCTION THEOREM

Let f be a 1-1 function.

If $f'(f^{-1}(x))$ exists and is nonzero,

then $$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

You don't need to remember the IFT.
You can quickly rederive it, by implicit differentiation . . .

$$y = f^{-1}(x) \qquad y' = (f^{-1})'(x)$$

$$f(y) = x$$

$$[f'(y)]y' = 1$$

Want: y' $$y' = \frac{1}{f'(y)}$$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

§4.10, p. 89 (DERIVATIVES OF) INVERSE TRIGONOMETRIC FUNCTIONS

$$y = \arcsin x$$

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§4.10, p. 89 (DERIVATIVES OF) INVERSE TRIGONOMETRIC FUNCTIONS

$$y = \arcsin x$$

$$\frac{d}{dx}[\sin y] = \frac{d}{dx}[x] \qquad -1 \le x \le 1$$

$$[\cos y][y'] = [\sin' y][y'] = 1$$

$$\frac{d}{dx}[\arcsin x] = \frac{dy}{dx} = y' = \frac{1}{\cos y} = \sec(y) = \sec(\arcsin x)$$

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§4.10, p. 89 (DERIVATIVES OF) INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\arcsin x] = \sec(\arcsin x)$$

correct, but not preferred...

$-1 < x < 1$
UNNEEDED

$$\frac{d}{dx}[\arcsin x] = \sec(\arcsin x)$$

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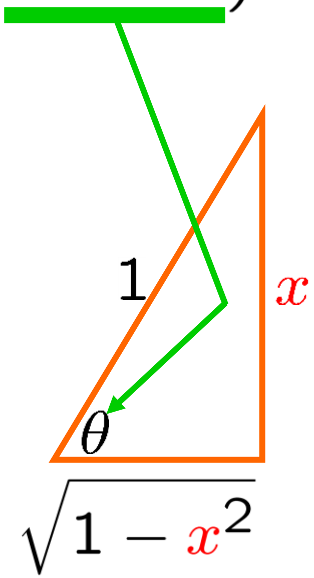
§4.10, p. 89 (DERIVATIVES OF) INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\arcsin x] = \sec(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\sec(\arcsin x) = \sec(\theta) = \frac{1}{\sqrt{1-x^2}}$$



Want: $\sin\theta = x$

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§4.10, p. 89 (DERIVATIVES OF) INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\arcsin x] = \sec(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

Correct: $\frac{d}{dx}[\arcsin x] = \sec(\arcsin x)$

Preferred: $\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$

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§4.10, p. 89 (DERIVATIVES OF) INVERSE TRIGONOMETRIC FUNCTIONS

$$y = \arccos x$$

$$\frac{d}{dx}[\cos y] = \frac{d}{dx}[x] \qquad -1 \le x \le 1$$

$$[-\sin y][y'] = [\cos' y][y'] = 1$$

$$\frac{d}{dx}[\arccos x] = \frac{dy}{dx} = y' = \frac{1}{-\sin y} = -\csc(y) = -\csc(\arccos x)$$

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§4.10, p. 89 (DERIVATIVES OF) INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\arccos x] = -\csc(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$$

correct, but not precise

$-1 < x < 1$

UNNEEDED

$$\frac{d}{dx}[\arccos x] \qquad = -\csc(\arccos x)$$

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§4.10, p. 89 (DERIVATIVES OF) INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\arccos x] = \frac{-1}{\sqrt{1-x^2}} \qquad = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

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§4.10, p. 89 (DERIVATIVES OF) INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\arccos x] = \frac{-1}{\sqrt{1-x^2}}$$

THESE ADD TO 0

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

RECALL:

Fact: $\forall x \in [-1,1],\ [\arcsin x] + [\arccos x] = \frac{\pi}{2}$

THEREFORE:

$$\forall x \in (-1,1),\ \frac{d}{dx}[\arcsin x] + \frac{d}{dx}[\arccos x] = \frac{d}{dx}\left[\frac{\pi}{2}\right] = 0$$

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§4.10, p. 89 (DERIVATIVES OF) INVERSE TRIGONOMETRIC FUNCTIONS

$$y = \arctan x$$

$$\frac{d}{dx}[\tan y] = \frac{d}{dx}[x]$$

$$[\sec^2 y][y'] = [\tan' y][y'] = 1$$

$$\frac{d}{dx}[\arctan x] = \frac{dy}{dx} = y' = \frac{1}{\sec^2 y} = \cos^2(y) = \cos^2(\arctan x)$$

You don't need to remember the IFT.
You can quickly rederive it, by implicit differentiation ...

§4.10, p. 89 (DERIVATIVES OF) INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\arctan x] = \cos^2(\arctan x) = \frac{1}{1+x^2}.$$

correct, but not exercise

$$\frac{d}{dx}[\arctan x] \qquad = \cos^2(\arctan x)$$

You don't need to remember the IFT.
You can quickly rederive it, by implicit differentiation . . .

§4.10, p. 89 (DERIVATIVES OF) INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\text{arctan}\, x] = \frac{1}{1+x^2} \qquad = \frac{1}{1+x^2}$$

Exercise: $\frac{d}{dx}[\text{arccot}\, x] = \frac{-1}{1+x^2}$

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§4.10, p. 89 (DERIVATIVES OF) INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\text{arctan}\, x] = \frac{1}{1+x^2}$$

THESE ADD TO 0

$$\frac{d}{dx}[\text{arccot}\, x] = \frac{-1}{1+x^2}$$

RECALL:

Fact: $\forall x \in \mathbb{R},\ [\text{arctan}\, x] + [\text{arccot}\, x] = \frac{\pi}{2}$

THEREFORE:

$$\forall x \in \mathbb{R},\ \frac{d}{dx}[\text{arctan}\, x] + \frac{d}{dx}[\text{arccot}\, x] = \frac{d}{dx}\left[\frac{\pi}{2}\right] = 0$$

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§4.10, p. 89 (DERIVATIVES OF) INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\arcsin x]] = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}[\mathrm{arccot}\, x] = \frac{-1}{1 + x^2}$$

$$\frac{d}{dx}[\arccos x] = \frac{-1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}[\mathrm{arccot}\, x] = \frac{-1}{1 + x^2}$$

§4.10, p. 89 (DERIVATIVES OF) INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$$

MEMORIZE THESE

$$\frac{d}{dx}[\arccos x] = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\operatorname{arccot} x] = \frac{-1}{1+x^2}$$

GET THESE FOR FREE

NOTE: Differences of opinion about how to define arcsec, see STEWART §3.6, p. 214 Homework #58. We'll leave arcsec and arccsc undefined in this course.

EXAMPLE: Find the deriv. of $y = \arctan\left(3x - \sqrt{1+4x^2}\right)$.

$$\frac{dy}{dx} = \frac{3 - (1/2)(1+4x^2)^{-1/2}(8x)}{1 + \left(3x - (1+4x^2)^{1/2}\right)^2}$$

EXAMPLE: Find the derivative of $F(\theta) = \arcsin\left(\sqrt[3]{\sin\theta}\right)$.

$$\frac{d}{d\theta}[F(\theta)] = \frac{(1/3)(\sin\theta)^{-2/3}(\cos\theta)}{\sqrt{1-((\sin\theta)^{1/3})^2}}$$

EXAMPLE: Find the derivative of $y = \arctan\sqrt{\frac{1-3x}{1+x^2}}$.

$$\frac{dy}{dx} = \frac{[1/2]\left[\left(\frac{1-3x}{1+x^2}\right)^{-1/2}\right]\left[\frac{(1+x^2)(-3)-(1-3x)(2x)}{(1+x^2)^2}\right]}{1+\left(\left(\frac{1-3x}{1+x^2}\right)^{\boxed{1/2}}\right)^2}$$

SKILL

inverse trig diff

Whitman problems

§4.10, p. 91–92, #1-12