CALCULUS
More graphing problems
EXAMPLE: Sketch the graph of $y = 2x - (\sin x)$.

**A. Symmetry** odd (over $[0, \infty)$; reflect through origin)

(i) even function: $f(-x) = f(x)$

(ii) odd function: $f(-x) = -(f(x))$

(iii) periodic function: $f(x + p) = f(x)$
EXAMPLE: Sketch the graph of \( y = 2x - (\sin x) \).

**A. Symmetry** odd (over \([0, \infty)\); reflect through origin)

**B. Intervals of Positivity or Negativity, and**

(i) domain \( \subseteq [0, \infty) \)

(ii) \( x, y \)-intercepts \( \bullet (0,0) \)

(iii) vertical, horizontal asymptotes no asymptotes

**C. Intervals of Increase or Decrease**

\[ [\sin x]_{x: \to 0} = 0 = [x]_{x: \to 0} \]

\( \forall x \geq 0, \frac{d}{dx} [\sin x] \leq \frac{d}{dx} [x] \)

\( \forall x \geq 0, \sin x \leq x \)

\( \forall x > 0, x < 2x \)

\( \forall x > 0, \sin x \leq x < 2x, \text{ so } 2x - (\sin x) > 0. \)

\[ y = 2x - (\sin x) \]
EXAMPLE: Sketch the graph of \( y = 2x - (\sin x) \).

A. Symmetry odd (over \([0, \infty)\); reflect through origin)

B. Intervals of Positivity or Negativity, and
   (i) domain \( \subseteq [0, \infty) \)
   (ii) \( x,y \)-intercepts \( (0, 0) \)
   (iii) vertical, horizontal asymptotes no asymptotes

C. Intervals of Increase or Decrease

D. Concavity and Points of Inflection

\[
y = 2x - (\sin x)
\]

\[
\frac{dy}{dx} = 2 - (\cos x) > 0, \quad \forall x
\]
EXAMPLE: Sketch the graph of $y = 2x - (\sin x)$.

A. Symmetry odd (over $[0, \infty)$; reflect through origin)

B. Intervals of Positivity or Negativity, and
   (i) domain $\supseteq [0, \infty)$
   (ii) $x,y$-intercepts $\bullet (0, 0)$
   (iii) vertical, horizontal asymptotes no asymptotes

C. Intervals of Increase or Decrease $\uparrow [0, \infty)$

D. Concavity and Points of Inflection

\[
\frac{dy}{dx} = 2 - (\cos x)
\]

\[
\frac{d^2y}{dx^2} = \sin x - (\cos x)
\]
EXAMPLE: Sketch the graph of \( y = 2x - (\sin x) \).

A. Symmetry odd (over \([0, \infty)\); reflect through origin)

B. Intervals of Positivity or Negativity, and
   
   (i) domain \( \supseteq [0, \infty) \)
   
   (ii) \( x, y \)-intercepts \( \bullet (0, 0) \uparrow \)
   
   (iii) vertical, horizontal asymptotes \text{no asymptotes}

C. Intervals of Increase or Decrease \( \uparrow [0, \infty) \)

D. Concavity and Points of Inflection

\( \cup [0, \pi], \cap [\pi, 2\pi], \cup [2\pi, 3\pi], \cap [3\pi, 4\pi], \cup [4\pi, 5\pi], \cap [5\pi, 6\pi], \ldots \)

\[
\frac{dy}{dx} = 2 - (\cos x)
\]

\[
\frac{d^2y}{dx^2} = \sin x
\]

\( \frac{d^2y}{dx^2} \) \( 0 \) pos \( 0 \) neg \( 0 \) pos \( 0 \) neg \( 0 \) pos \( 0 \) neg \( 0 \) pos \( 0 \) neg \( 0 \) \( \ldots \)
EXAMPLE: Sketch the graph of $y = 2x - (\sin x)$.

domain $\supseteq [0, \infty)$

odd (over $[0, \infty)$; reflect through origin)

$\bullet (0,0) \notin \text{domain} \supseteq [0, \infty)$

$\uparrow [0, \infty)$

$\cup [0, \pi], \cap [\pi, 2\pi], \cup [2\pi, 3\pi], \cap [3\pi, 4\pi], \cup [4\pi, 5\pi], \cap [5\pi, 6\pi], \ldots$

$y = 2x - (\sin x)$
EXAMPLE: Sketch the graph of $y = 2x - (\sin x)$.

Odd (over $[0, \infty)$; reflect through origin)

domain $\supseteq [0, \infty)$

$\bullet (0, 0)$

$\cup [0, \pi], \cap [\pi, 2\pi], \cup [2\pi, 3\pi], \cap [3\pi, 4\pi], \cup [4\pi, 5\pi], \cap [5\pi, 6\pi], \ldots$

$\bullet (\pi, 2\pi)$  $\bullet (2\pi, 4\pi)$

$y = 2x - (\sin x)$
EXAMPLE: Sketch the graph of \( y = 2x - (\sin x) \).

odd (over \([0, \infty)\); reflect through origin)

\( \text{domain } \supseteq [0, \infty) \)

\( \bullet (0, 0) \)

\( \cup [0, \pi], \cap [\pi, 2\pi], \cup [2\pi, 3\pi], \cap [3\pi, 4\pi], \cup [4\pi, 5\pi], \cap [5\pi, 6\pi], \ldots \)

\( \bullet (\pi, 2\pi), \bullet (2\pi, 4\pi) \)

\[ y = 2x - (\sin x) \]
EXAMPLE: Sketch the graph of \( y = 2x - (\sin x) \).

Odd (over \([0, \infty)\); reflect through origin)

Domain \( \subseteq [0, \infty) \)

- \((0, 0) \neq \)

\( \cup [0, \pi], \cap [\pi, 2\pi], \cup [2\pi, 3\pi], \cap [3\pi, 4\pi], \cup [4\pi, 5\pi], \cap [5\pi, 6\pi], \ldots \)

- \((\pi, 2\pi)\)
- \((2\pi, 4\pi)\)

\[ y = 2x - (\sin x) \]
\[ x \rightarrow x - 2\pi \]
\[ y = 2(x - 2\pi) - (\sin(x - 2\pi)) \]
\[ = 2x - 4\pi - (\sin x) \]
EXAMPLE: Sketch the graph of $y = 2x - (\sin x)$. 

odd (over $[0, \infty)$; reflect through origin)

domain $\supseteq [0, \infty)$

$\bullet (0, 0)$

$\cup [0, \pi], \cap [\pi, 2\pi], \cup [2\pi, 3\pi], \cap [3\pi, 4\pi], \cup [4\pi, 5\pi], \cap [5\pi, 6\pi], \ldots$

$\bullet (\pi, 2\pi)$  $\bullet (2\pi, 4\pi)$

$y - 4\pi = 2x - 4\pi - (\sin x)$

$y = 2x - (\sin x)$

$y = 2(x - 2\pi) - (\sin(x - 2\pi))$

$= 2x - 4\pi - (\sin x)$

$\rightarrow y - 4\pi$

$y = 2x - (\sin x)$

$\rightarrow 8\pi$

$\rightarrow 6\pi$

$\rightarrow 4\pi$

$\rightarrow 2\pi$

$\rightarrow \pi$

$\rightarrow 2\pi$

$\rightarrow 3\pi$

$\rightarrow 4\pi$

$\rightarrow \pi$

$\rightarrow 2\pi$

$\rightarrow 3\pi$

$\rightarrow 4\pi$
EXAMPLE: Sketch the graph of \( y = 2x - (\sin x) \).

odd (over \([0, \infty)\); reflect through origin)

domain \( \supseteq [0, \infty) \)

\( (0, 0) \nleq \)

\( \bigcup [0, \pi], \cap [\pi, 2\pi], \bigcup [2\pi, 3\pi], \cap [3\pi, 4\pi], \bigcup [4\pi, 5\pi], \cap [5\pi, 6\pi], \ldots \)

\( (\pi, 2\pi) \quad (2\pi, 4\pi) \)
EXAMPLE: Sketch the graph of $y = 2x - (\sin x)$.

Domain $\subseteq [0, \infty)$

- $(0, 0)$
- $[0, \pi], \cap[\pi, 2\pi], \cup[2\pi, 3\pi], \cap[3\pi, 4\pi], \cup[4\pi, 5\pi], \cap[5\pi, 6\pi], \ldots$

- $(\pi, 2\pi)$
- $(2\pi, 4\pi)$

$y = 2x - (\sin x)$
EXAMPLE: Sketch the graph of \( y = 2x - (\sin x) \).

Domain \( \subseteq [0, \infty) \)
- \((0, 0) \uparrow [0, \infty) \)
- \(\cup [0, \pi], \cap [\pi, 2\pi], \cup [2\pi, 3\pi], \cap [3\pi, 4\pi], \cup [4\pi, 5\pi], \cap [5\pi, 6\pi], \ldots \)
- \((\pi, 2\pi) \quad (2\pi, 4\pi) \)

Recap: First, over \([0, 2\pi]\), then translate repeatedly, then reflect through origin.
EXAMPLE: Sketch the graph of $y = 2x - (\sin x)$.

- Domain $\subseteq [0, \infty)$
- Odd (over $[0, \infty)$; reflect through origin)
- $\cup [0, \pi], \cap [\pi, 2\pi], \cup [2\pi, 3\pi], \cap [3\pi, 4\pi], \cup [4\pi, 5\pi], \cap [5\pi, 6\pi], \ldots$
- $(0, 0) \not\in$
- $\uparrow [0, \infty)$

$y = 2x - (\sin x)$

Easier: First, gph over $[0, \pi]$, then reflect through origin, then translate repeatedly.

SKILL curve sketching
EXAMPLE: Sketch the graph of $y = \frac{x^3}{x^2 - 4}$.

**A. Symmetry** odd (over $[0, \infty)$; reflect through origin)

(i) even function: $f(-x) = f(x)$
(ii) odd function: $f(-x) = -(f(x))$
(iii) periodic function: $f(x + p) = f(x)$
EXAMPLE: Sketch the graph of \( y = \frac{x^3}{x^2 - 4} \).

A. Symmetry odd (over \([0, \infty)\); reflect through origin)

B. Intervals of Positivity or Negativity, and
   (i) domain \(\subseteq [0, \infty) \backslash \{2\}\)
   (ii) \(x,y\)-intercepts \(\bullet (0, 0)\)
   (iii) vertical, horizontal asymptotes \(\bullet (2, -\infty, \infty)\)

C. Intervals of Increase or Decrease

\[
y = \frac{x^3}{x^2 - 4} = \frac{x^3}{(x - 2)(x + 2)}
\]
EXAMPLE: Sketch the graph of \( y = \frac{x^3}{x^2 - 4} \).

**A. Symmetry** odd (over \([0, \infty)\); reflect through origin)

**B. Intervals of Positivity or Negativity**, and

(i) domain \(\supseteq [0, \infty) \setminus \{2\}\)

(ii) \(x,y\)-intercepts \(\bullet(0,0)\)

(iii) vertical, horizontal asymptotes \(\bullet(2, -\infty | \infty)\)

**C. Intervals of Increase or Decrease**

\[
y = \frac{x^3}{x^2 - 4}
\]

\[
\frac{dy}{dx} = \frac{(x^2 - 4)(3x^2) - (x^3)(2x)}{(x^2 - 4)^2} = \frac{(3x^4 - 12x^2) - (2x^4)}{(x^2 - 4)^2}
\]

\[
= \frac{x^2(x^2 - 12)}{(x^2 - 4)^2} = \frac{x^2(x^2 - 12)}{((x - 2)(x + 2))^2}
\]
EXAMPLE: Sketch the graph of $y = \frac{x^3}{x^2 - 4}$.

A. Symmetry odd (over $[0, \infty)$; reflect through origin)

B. Intervals of Positivity or Negativity, and

(i) domain $\subseteq [0, \infty) \setminus \{2\}$  \hspace{1cm} \text{neg}(0, 2), \text{ pos}(2, \infty)

(ii) $x, y$-intercepts $\bullet(0, 0)$  \hspace{1cm} $\bullet(\infty, \infty)$

(iii) vertical, horizontal asymptotes $\bullet(2, -\infty | \infty)$

C. Intervals of Increase or Decrease

\[
\frac{dy}{dx} = \frac{x^4 - 12x^2}{(x^2 - 4)^2} = \frac{x^2(x^2 - 12)}{((x - 2)(x + 2))^2}
\]

\[
\frac{dy}{dx} = \frac{x^4 - 12x^2}{(x^2 - 4)^2} = \frac{x^2(x^2 - 12)}{((x - 2)(x + 2))^2}
\]
EXAMPLE: Sketch the graph of \( y = \frac{x^3}{x^2 - 4} \).

A. Symmetry odd (over \([0, \infty)\); reflect through origin)

B. Intervals of Positivity or Negativity, and

(i) domain \( \subseteq [0, \infty) \setminus \{2\} \)
(ii) \( x, y \)-intercepts \( \bullet (0, 0) \)
(iii) vertical, horizontal asymptotes \( \bullet (2, -\infty | \infty) \)

C. Intervals of Increase or Decrease

\( \bullet (\alpha, \beta) \)
\( \downarrow [0, 2), \downarrow (2, \alpha], \uparrow [\alpha, \infty) \)

D. Concavity and Points of Inflection

\( \alpha := \sqrt{12} \approx 3.464 \)
\( \beta := \frac{\alpha^3}{\alpha^2 - 4} = \frac{\alpha^3}{8} \approx 5.196 \)

\[ \frac{dy}{dx} = \frac{x^4 - 12x^2}{(x^2 - 4)^2} = \frac{x^2(x^2 - 12)}{((x - 2)(x + 2))^2} = \frac{x^2(x - \alpha)(x + \alpha)}{(x - 2)^2(x + 2)^2} \]
EXAMPLE: Sketch the graph of \( y = \frac{x^3}{x^2 - 4} \).

A. Symmetry odd (over \([0, \infty)\); reflect through origin)

B. Intervals of Positivity or Negativity, and
   (i) domain \(\subseteq [0, \infty)\backslash\{2\}\)  
   (ii) \(x, y\)-intercepts \(\bullet (0, 0)\)  
   (iii) vertical, horizontal asymptotes \(\bullet (2, -\infty | \infty)\)

C. Intervals of Increase or Decrease
   \(\bullet (\alpha, \beta)\)  
   \(\downarrow [0, 2), \downarrow (2, \alpha], \uparrow [\alpha, \infty)\)
   \(\alpha \approx 3.464\)  
   \(\beta \approx 5.196\)

D. Concavity and Points of Inflection
   \[
   \frac{dy}{dx} = \frac{x^4 - 12x^2}{(x^2 - 4)^2}
   \]
   \[
   \frac{d^2y}{dx^2} = \frac{(x^2 - 4)^2(4x^3 - 24x) - (x^4 - 12x^2)(2(x^2 - 4)(2x))}{(x^2 - 4)^4}
   \]
   \[
   = \frac{(x^2 - 4)(4x^3 - 24x) - (x^4 - 12x^2)(4x)}{(x^2 - 4)^3}
   \]
EXAMPLE: Sketch the graph of \( y = \frac{x^3}{x^2 - 4} \).

**A. Symmetry** odd (over \([0, \infty)\); reflect through origin)

**B. Intervals of Positivity or Negativity,** and

- **(i)** domain \( \subseteq [0, \infty) \setminus \{2\} \)
- **(ii)** \( x, y \)-intercepts \( \bullet(0, 0) \rightarrow \bullet(\infty, \infty) \)
- **(iii)** vertical, horizontal asymptotes \( \bullet(2, -\infty \mid \infty) \)

**C. Intervals of Increase or Decrease**

- \( \bullet(\alpha, \beta) \rightarrow \downarrow [0, 2), \downarrow (2, \alpha], \uparrow [\alpha, \infty) \)

**D. Concavity and Points of Inflection**

\[
\frac{d^2 y}{dx^2} = \frac{(x^2 - 4)(4x^3 - 24x) - (x^4 - 12x^2)(4x)}{(x^2 - 4)^3}
\]

\[
\frac{d^2 y}{dx^2} = \frac{4x[(x^2 - 4)(x^2 - 6) - (x^4 - 12x^2)]}{(x^2 - 4)^3}
\]

\[
= \frac{(x^2 - 4)(4x^3 - 24x) - (x^4 - 12x^2)(4x)}{(x^2 - 4)^3}
\]
EXAMPLE: Sketch the graph of \( y = \frac{x^3}{x^2 - 4} \).

A. Symmetry odd (over \([0, \infty)\); reflect through origin)

B. Intervals of Positivity or Negativity, and

(i) domain \( \supseteq [0, \infty) \setminus \{2\} \)  \( \text{neg}(0, 2), \text{pos}(2, \infty) \)

(ii) \( x, y \)-intercepts \( \bullet(0, 0) \rightarrow \bullet(\infty, \infty) \)

(iii) vertical, horizontal asymptotes \( \bullet(2, -\infty \mid \infty) \)

C. Intervals of Increase or Decrease

\( \bullet(\alpha, \beta) \rightarrow \downarrow [0, 2), \downarrow (2, \alpha], \uparrow [\alpha, \infty) \)

\( \alpha \approx 3.464 \)

\( \beta \approx 5.196 \)

D. Concavity and Points of Inflection

\[
\frac{d^2 y}{dx^2} = \frac{(x^2 - 4)(4x^3 - 24x) - (x^4 - 12x^2)(4x)}{(x^2 - 4)^3}
\]

EXPAND

\[
= \frac{4x[(x^2 - 4)(x^2 - 6) - (x^4 - 12x^2)]}{(x^2 - 4)^3}
\]

\[
= \frac{4x[(x^4 - 10x^2 + 24) - (x^4 - 12x^2)]}{(x^2 - 4)^3}
\]
EXAMPLE: Sketch the graph of \( y = \frac{x^3}{x^2 - 4} \).

**A. Symmetry**  odd (over \([0, \infty)\); reflect through origin)

**B. Intervals of Positivity or Negativity, and**

(i) domain \(\subseteq [0, \infty) \setminus \{2\}\)  \(\neg(0, 2), \pos(2, \infty)\)

(ii) \(x, y\)-intercepts \(\bullet(0, 0)\rightarrow \bullet(\infty, \infty)\)

(iii) vertical, horizontal asymptotes \(\bullet(2, -\infty|\infty)\)

**C. Intervals of Increase or Decrease**  \(\alpha \approx 3.464\)

\(\bullet(\alpha, \beta)\rightarrow \downarrow [0, 2), \downarrow (2, \alpha], \uparrow [\alpha, \infty)\)  \(\beta \approx 5.196\)

**D. Concavity and Points of Inflection**

\[
\frac{d^2 y}{dx^2} = \frac{4x[(x^4 - 10x^2 + 24) - (x^4 - 12x^2)]}{(x^2 - 4)^3}
\]

\[= \frac{4x[(x^4 - 10x^2 + 24) - (x^4 - 12x^2)]}{(x^2 - 4)^3}\]
EXAMPLE: Sketch the graph of \( y = \frac{x^3}{x^2 - 4} \).

A. Symmetry odd (over \([0, \infty)\); reflect through origin)

B. Intervals of Positivity or Negativity, and
   (i) domain \( \geq [0, \infty) \setminus \{2\} \)
   (ii) \( x, y \)-intercepts \( \bullet(0, 0) \rightarrow \bullet(\infty, \infty) \)
   (iii) vertical, horizontal asymptotes \( \bullet(2, -\infty|\infty) \)

C. Intervals of Increase or Decrease
   \( \bullet(\alpha, \beta) \rightarrow \downarrow [0, 2), \downarrow (2, \alpha], \uparrow [\alpha, \infty) \)
   \( \alpha \approx 3.464 \)
   \( \beta \approx 5.196 \)

D. Concavity and Points of Inflection
   \( \frac{d^2y}{dx^2} = \frac{4x[(x^4 - 10x^2 + 24) - (x^4 - 12x^2)]}{(x^2 - 4)^3} \)
   \( = \frac{4x[2x^2 + 24]}{(x^2 - 4)^3} \)
   \( = \frac{4x[2(x^2 + 12)]}{((x - 2)(x + 2))^3} \)
   \( = \frac{8x(x^2 + 12)}{(x - 2)^3(x + 2)^3} \)
   \( \text{always positive} \)
EXAMPLE: Sketch the graph of \( y = \frac{x^3}{x^2 - 4} \).

Domain \( \subseteq [0, \infty) \setminus \{2\} \)

Neg(0, 2), Pos(2, \infty) (over \([0, \infty)\); reflect through origin)

\( (0, 0) \rightarrow \)

\( (\infty, \infty) \rightarrow \)

\( (2, -\infty | \infty) \rightarrow \)

\( (\alpha, \beta) \rightarrow \)

\( (\alpha, \beta) \rightarrow \)

\( \downarrow [0, 2), \downarrow (2, \alpha], \uparrow [\alpha, \infty) \)

\( \cap [0, 2), \cup (2, \infty) \)

\( \alpha \approx 3.464 \)

\( \beta \approx 5.196 \)

Ch. 5
EXAMPLE: Sketch the graph of \( y = \frac{x^3}{x^2 - 4} \).

**Domain**: \( \geq [0, \infty) \setminus \{2\} \)

**Negative (0, 2)**, **Positive (2, \infty)**

- \((0, 0)\to\) \(\alpha \approx 3.464\)
- \((\infty, \infty)\) \(\beta \approx 5.196\)
- \((2, -\infty|\infty)\)
- \((\alpha, \beta)\to\)

Odd (over \([0, \infty)\); reflect through origin)

\([0, 2)\), \([2, \alpha]\), \([\alpha, \infty)\)

\([0, 2)\), \(\cup(2, \infty)\)
EXAMPLE: Sketch the graph of \( y = \frac{x^3}{x^2 - 4} \).

Domain \( [0, \infty) \setminus \{2\} \)

Negative over \( (0, 2) \), positive over \( (2, \infty) \)

- \((0, 0)\) \(\rightarrow\) \(\alpha \approx 3.464\)
- \((\infty, \infty)\) \(\rightarrow\) \(\beta \approx 5.196\)
- \((2, -\infty) \setminus \infty\)
- \((\alpha, \beta)\) \(\rightarrow\)

Odd (over \( [0, \infty) \); reflect through origin)

\(\downarrow [0, 2), \downarrow (2, \alpha], \uparrow [\alpha, \infty) \cap [0, 2), \cup (2, \infty) \)

\(2\) \(\alpha\) \(4\)
EXAMPLE: Sketch the graph of \( y = \frac{x^3}{x^2 - 4} \).

**Domain:** \([0, \infty) \setminus \{2\}

- **Negative:** \((0, 2)\), **Positive:** \((2, \infty)\)
- \((0, 0)\) approaches \(\alpha \approx 3.464\)
- \((\infty, \infty)\)
- \((2, -\infty)\) approaches \(\beta \approx 5.196\)
- \((\alpha, \beta)\) approaches

**Odd:** (over \([0, \infty)\); reflect through origin)
- **Down:** \([0, 2)\), **Up:** \([2, \alpha)\), **Up:** \([\alpha, \infty)\)
- \(\cap [0, 2), \cup (2, \alpha)\)
- \(\cup (\alpha, \infty)\)

**SKILL:** curve sketching
SKILL
curve sketching
Whitman problems
§5.5, p. 103–104, #1-32