CALCULUS
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Problem: A cube is growing.

At a certain moment, ← t_0

its side length is 7 inches

and its volume is growing

at 4 cubic inches per second.

At that moment, ← t_0

what is the rate of growth in side length,

in inches per second? $4/(3\cdot 49)$ in/s

Notation: An overdot abbreviates d/dt

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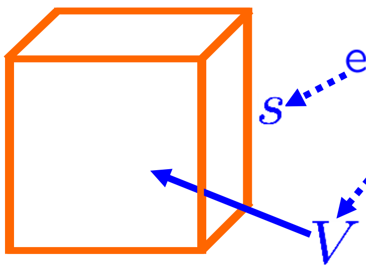
Sol'n: 12 step program. . . Admit you *have* a problem.

Read. Draw snapshot. Identify important quantities.

Notate drawing. Notate the requested rate.

Notate the other information given in the problem.



Know: $[s]_{t:\to t_0} = 7$

$[\dot{V}]_{t:\to t_0} = 4$

Want: $[\dot{s}]_{t:\to t_0} = ?$

$V = s^3$

$[\,\dot{V} = 3s^2\,\dot{s}\,]_{t:\to t_0}$

$4 = 3(7^2)(?)$

$? = 4/(3\cdot 49)$

RELATE THE QUANTITIES.

DIFFERENTIATE (w.r.t. t), RELATING THE RATES.

Plug in the information given in the problem.

Solve for the requested rate.

§6.2 Celebrate! (Responsibly!)

EXAMPLE: Air is being pumped into a spherical balloon so that its volume increases at a rate of 200 in^3/s. How fast is the radius of the balloon increasing when the diameter is 80 in?

1/(32π) in/s

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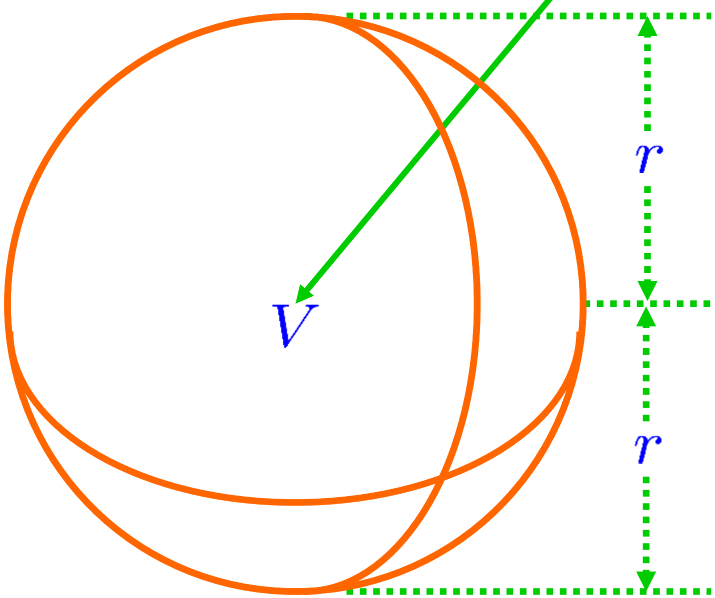
$$\dot{V} = 200$$
$$[2r]_{t:\to t_0} = 80$$
$$[r]_{t:\to t_0} = 40$$
$$[\dot{r}]_{t:\to t_0} = ?$$



RELATE THE QUANTITIES:

$$V = \frac{4}{3}\pi r^3$$

DIFFERENTIATE (w.r.t. t), RELATING THE RATES:

$$\left[\ \dot{V} = \frac{4}{3}\pi(3r^2)\,\dot{r}\ \right]_{t:\to t_0}$$

$$200 = 4\pi(40 \cdot 40)\,(?)$$

$$1/(32\pi) = 200/(4\pi \cdot 40 \cdot 5 \cdot 8) = ?$$

EXAMPLE: A ladder 15 ft long rests against vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 9 ft from the wall? 3/4 ft/s ■

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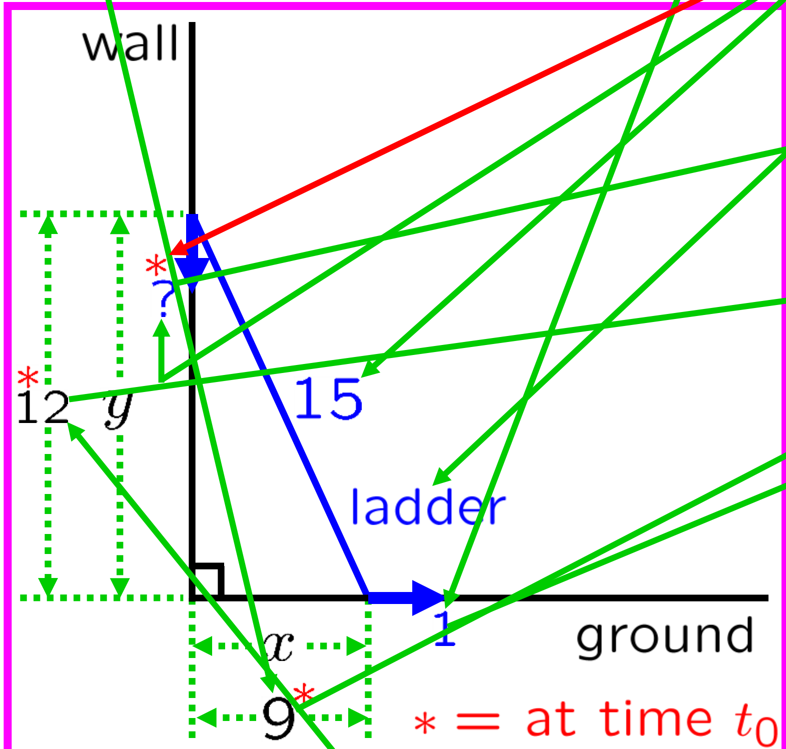

$\sqrt{15^2 - 9^2} = 12$

$[\dot{y}]_{t:\to t_0} = -?$

$[x]_{t:\to t_0} = 9$ $[y]_{t:\to t_0} = 12$

$\dot{x} = 1$

RELATE THE QUANTITIES:

$$x^2 + y^2 = 225$$

DIFFERENTIATE (w.r.t. t), RELATING THE RATES:

$$[\ 2x\dot{x} + 2y\dot{y} = 0\ ]_{t:\to t_0}$$

$$? = 3/4 \Leftarrow 2(9)(1) + 2(12)(-?) = 0$$

EXAMPLE: The height of a rectangle is increasing at a rate of 9 cm/s and its width is increasing at a rate of 2 cm/s. At some moment in time, the height is 30 cm and the width is 15 cm.

How fast is its area increasing at that moment?

195 cm^2/s ■

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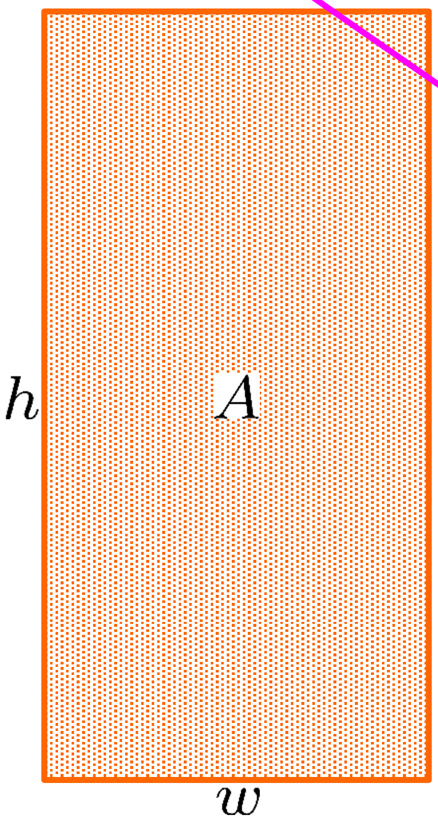

$[\dot{A}]_{t:\to t_0} = ?$

$\dot{h} = 9$

$\dot{w} = 2$

$[h]_{t:\to t_0} = 30$

$[w]_{t:\to t_0} = 15$

$? = (9)(15) + (30)(2) = 195$

$[\dot{A} = \dot{h}w + h\dot{w}]_{t:\to t_0}$

$A = hw$

EXAMPLE: A tank has the shape of an inverted circular cone with base radius 2 yards and height 4 yards. Suppose water is being pumped into the tank at a rate of 189 ft^3/min. Find the rate at which the water level is rising when the water is 6 ft deep. $7/\pi$ yards/min ■

7 yards^3/min

2 yards

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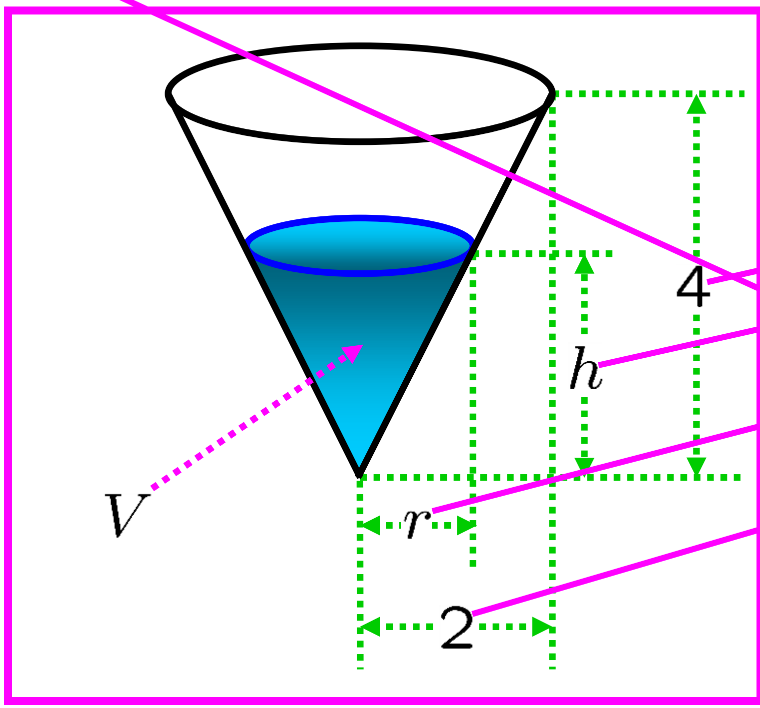


$$[\dot{h}]_{t:\to t_0} = ?$$

$$r/2 = h/4$$

$$[h]_{t:\to t_0} = 2$$

$$r = h/2$$

$$\dot{V} = 7$$

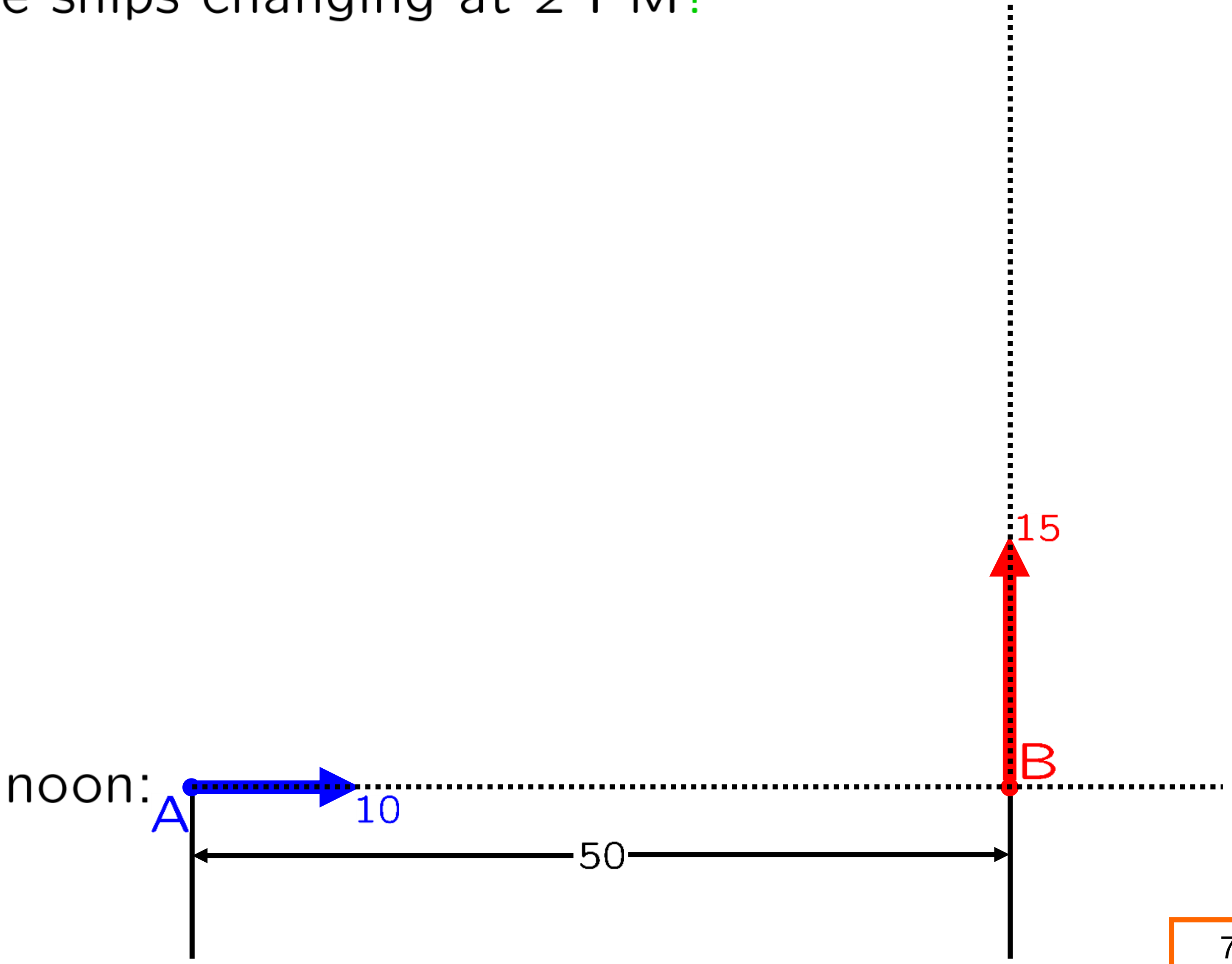
$$V = \pi r^2 h/3 = \pi(h/2)^2 h/3$$
$$= \pi h^3/12$$

$$\left[\begin{aligned} \dot{V} &= 3\pi h^2 \dot{h}/12 \\ &= \pi h^2 \dot{h}/4 \end{aligned} \right]_{t:\to t_0}$$

$$7 = \pi(2^2)(?)/4$$

$$? = 7/\pi$$

EXAMPLE: At noon, ship A is 50 miles west of ship B. Ship A is traveling east at 10 miles/hr and ship B is traveling north at 15 miles/hr. How fast is the distance between the ships changing at 2 PM?



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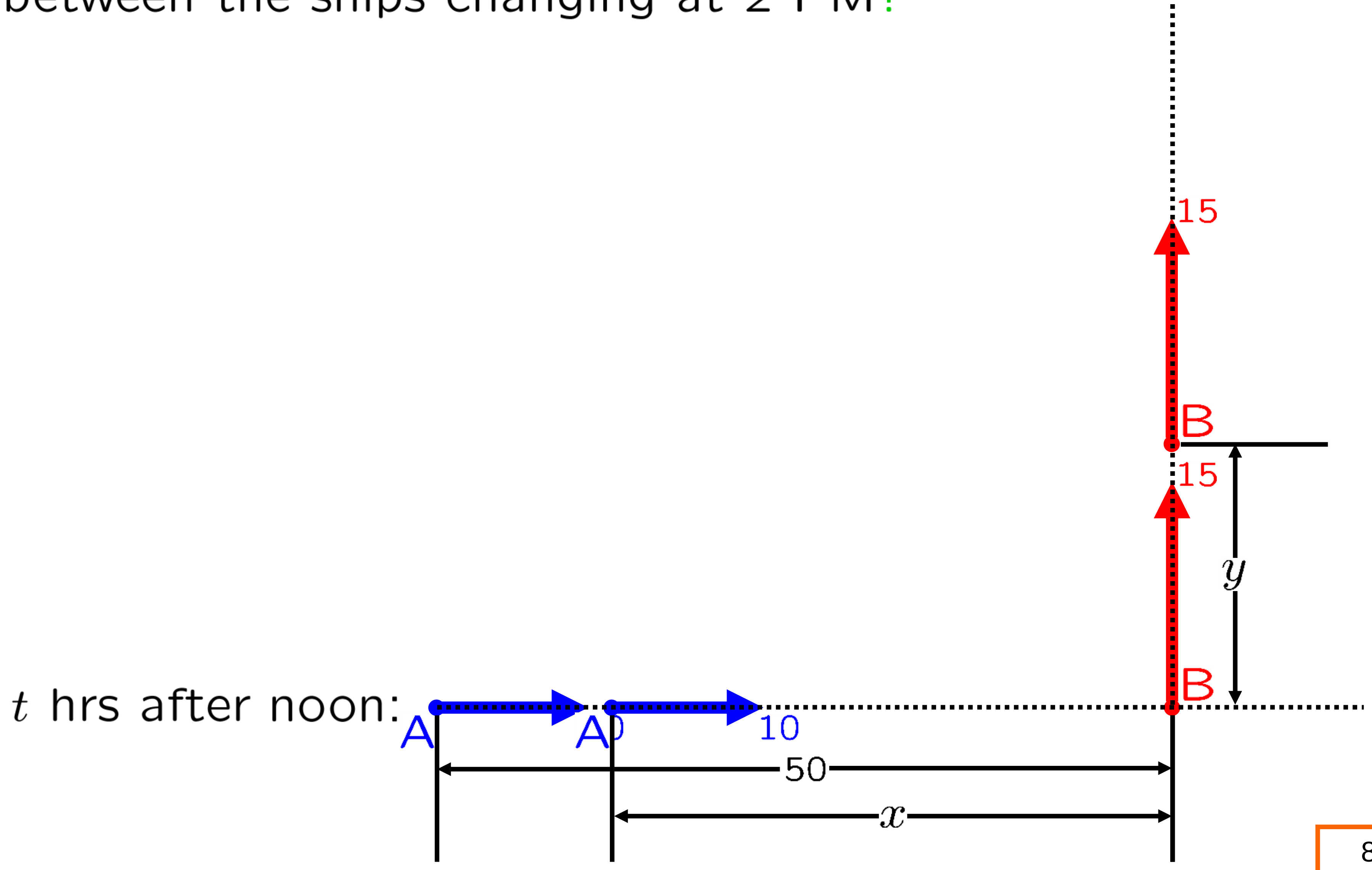


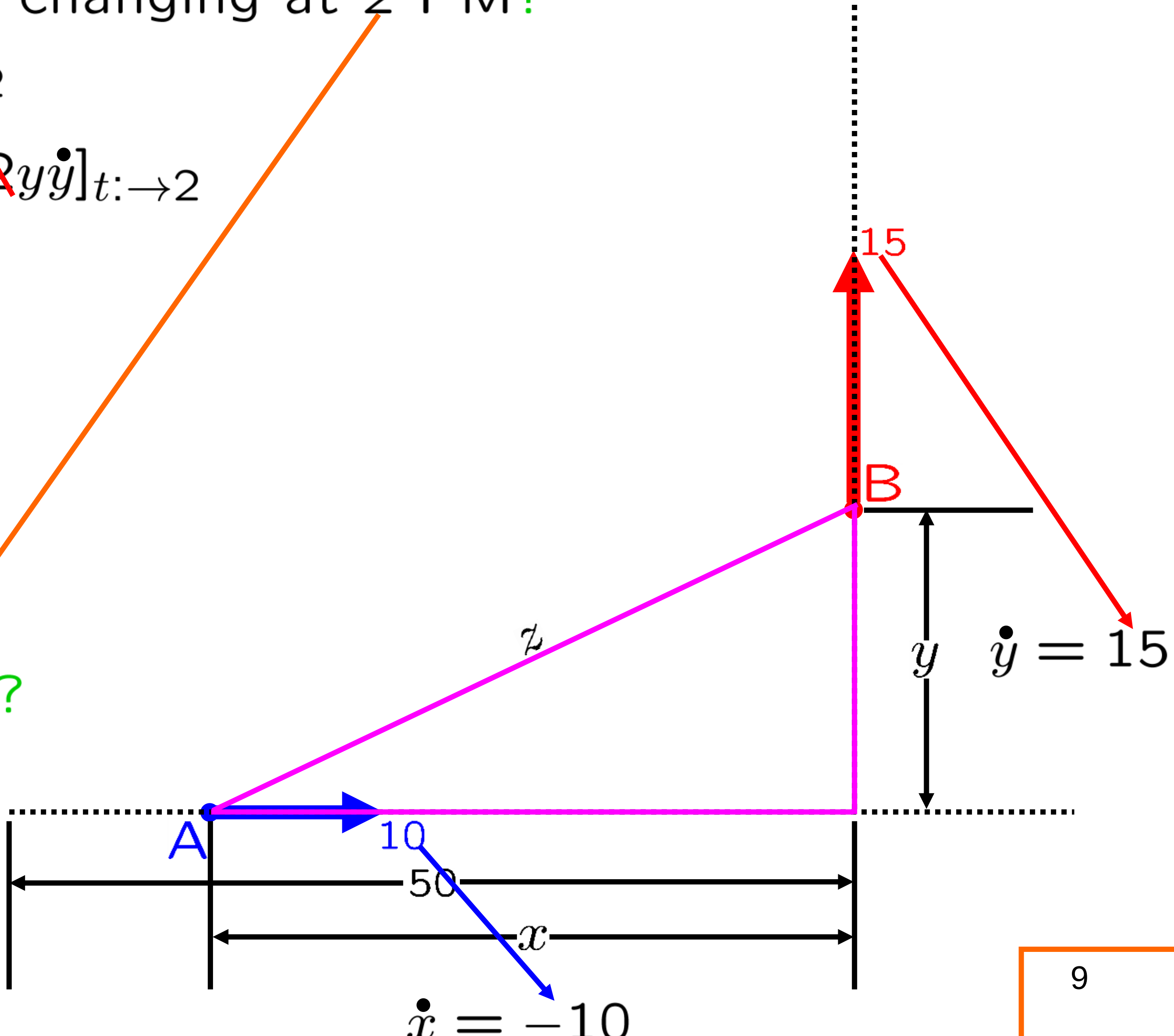
EXAMPLE: At noon, ship A is 50 miles west of ship B. Ship A is traveling east at 10 miles/hr and ship B is traveling north at 15 miles/hr. How fast is the distance between the ships changing at 2 PM?

$$z^2 = x^2 + y^2$$

$$[2z\dot{z} = 2x\dot{x} + 2y\dot{y}]_{t:\to 2}$$

$$[\dot{z}]_{t:\to 2} = ?$$

t hrs after noon:



EXAMPLE: At noon, ship A is 50 miles west of ship B. Ship A is traveling east at 10 miles/hr and ship B is traveling north at 15 miles/hr. How fast is the distance between the ships changing at 2 PM?

$$z^2 = x^2 + y^2$$

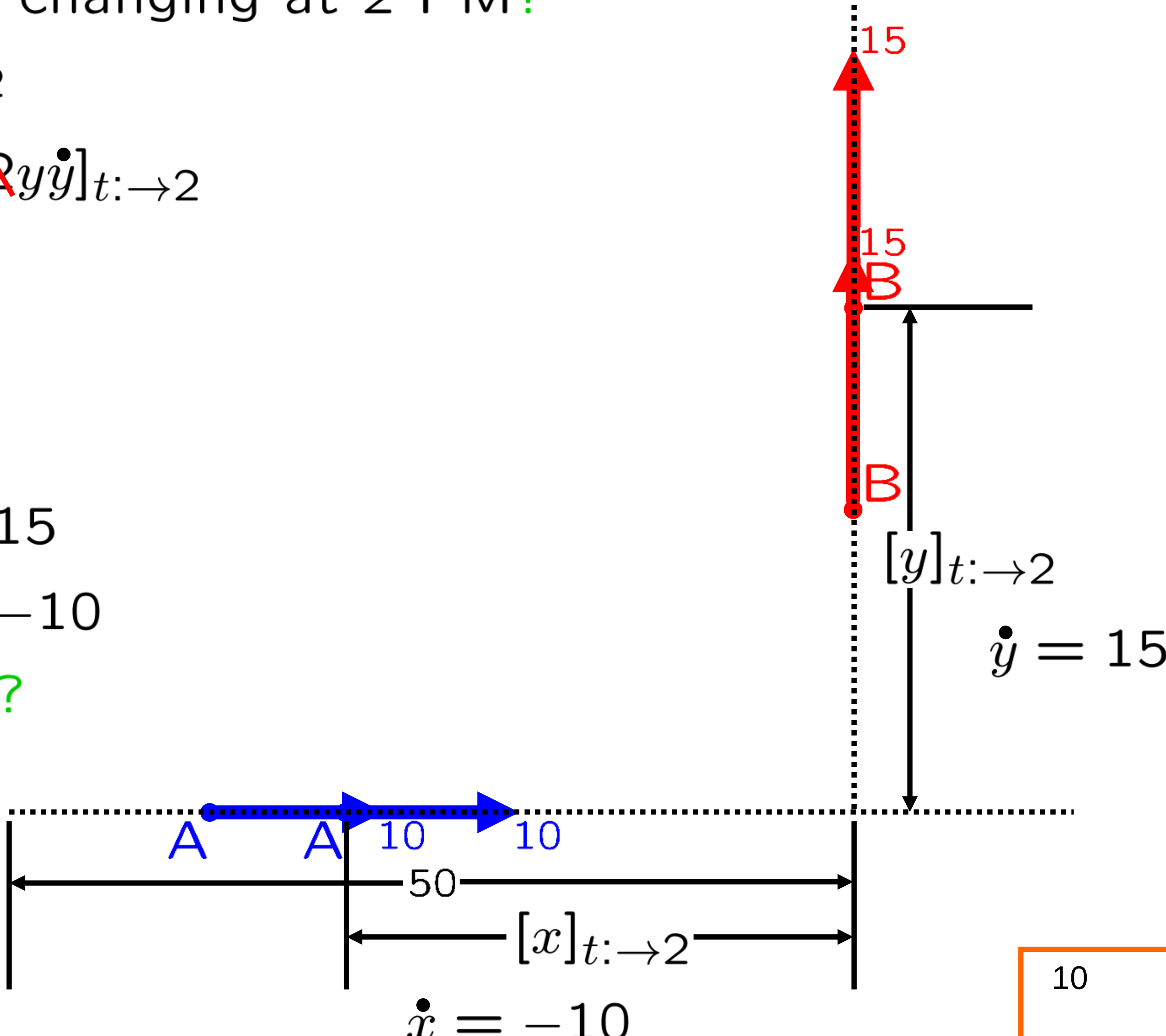
$$[\not{2} z \dot{z} = \not{2} x \dot{x} + \not{2} y \dot{y}]_{t:\to 2}$$

$$\dot{y} = 15$$

$$\dot{x} = -10$$

$$[\dot{z}]_{t:\to 2} = ?$$

2 hrs after noon:



EXAMPLE: At noon, ship A is 50 miles west of ship B. Ship A is traveling east at 10 miles/hr and ship B is traveling north at 15 miles/hr. How fast is the distance between the ships changing at 2 PM?

$5/\sqrt{2}$ mph ■

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$$z^2 = x^2 + y^2$$

$$[\not{2} z\dot{z} = \not{2} x\dot{x} + \not{2} y\dot{y}]_{t:\to 2}$$

$$\not{30}\sqrt{2}\,? = (\not{30})(-10) + (\not{30})(15)$$

$$\sqrt{2}\,? = 5$$

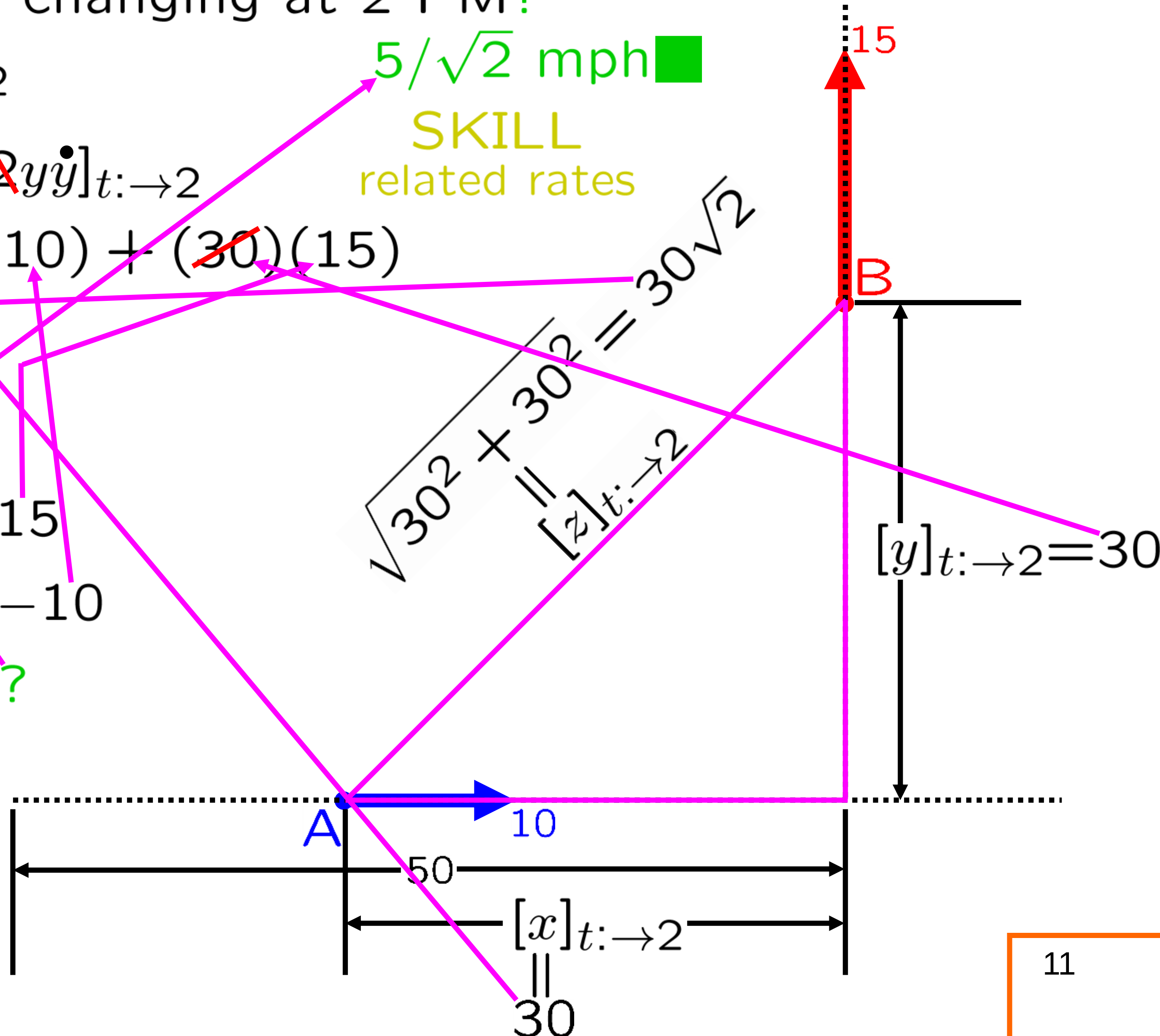
$$? = 5/\sqrt{2}$$

$$\dot{y} = 15$$

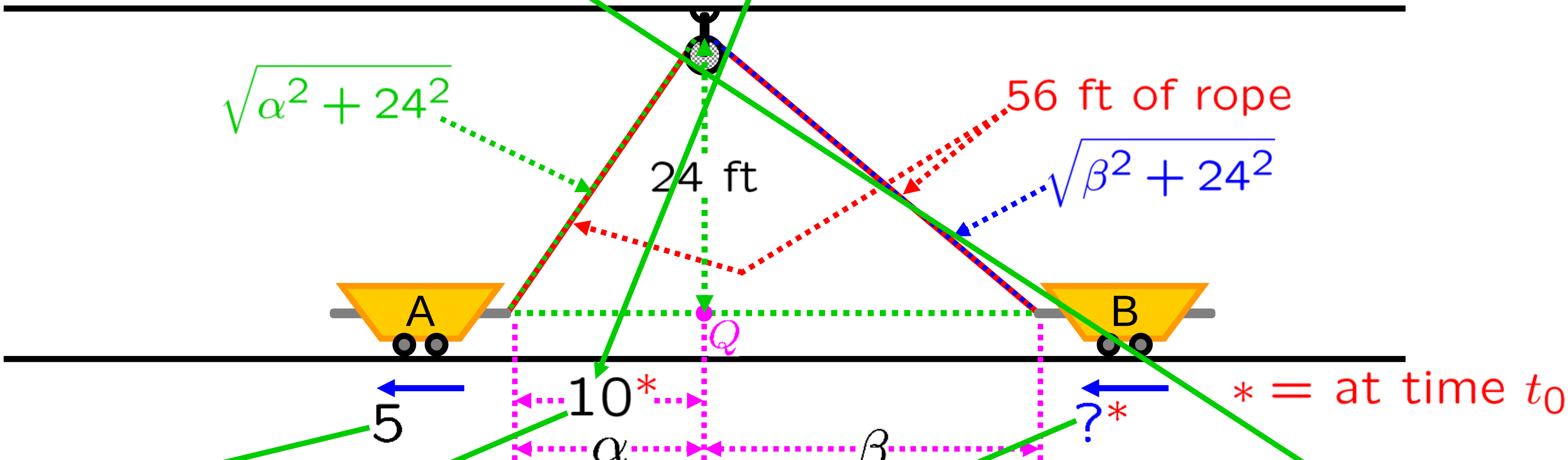
$$\dot{x} = -10$$

$$[\dot{z}]_{t:\to 2} = ?$$

2 hrs after noon:



EXAMPLE: Two carts, labeled "A" and "B", are connected by a rope 56 ft long that passes over a pulley (see below). The point Q is located 24 ft directly beneath the pulley between the carts. Cart A is being pulled away from Q at a speed of 5 ft/sec. How fast is cart B moving toward Q at the moment when cart A is 10 ft from Q?



$\dot{\alpha} = 5$

$[\alpha]_{t:\to t_0} = 10$

$$\sqrt{\alpha^2+24^2} + \sqrt{\beta^2+24^2} = 56$$

$$\left(\frac{2\alpha\dot{\alpha}}{2\sqrt{\alpha^2+24^2}} + \frac{2\beta\dot{\beta}}{2\sqrt{\beta^2+24^2}} = 0 \right)_{t:\to t_0}$$

$[\dot{\beta}]_{t:\to t_0} = -?$

EXAMPLE: Two carts, labeled "A" and "B", are connected by a rope 56 ft long that passes over a pulley (see below). The point Q is located 24 ft directly beneath the pulley between the carts. Cart A is being pulled away from Q at a speed of 5 ft/sec. How fast is cart B moving toward Q at the moment when cart A is 10 ft from Q?

$$\sqrt{\alpha^2 + 24^2} + \sqrt{\beta^2 + 24^2} = 56$$

$$\dot{\alpha} = 5$$

$$[\alpha]_{t:\to t_0} = 10$$

$$\sqrt{\alpha^2 + 24^2} + \sqrt{\beta^2 + 24^2} = 56$$

$$\left[\frac{2\alpha\dot{\alpha}}{2\sqrt{\alpha^2+24^2}} + \frac{2\beta\dot{\beta}}{2\sqrt{\beta^2+24^2}} = 0\right]_{t:\to t_0}$$

$$[\dot{\beta}]_{t:\to t_0} = -?$$

EXAMPLE: Two carts, labeled "A" and "B", are connected by a rope 56 ft long that passes over a pulley (see below). The point Q is located 24 ft directly beneath the pulley between the carts. Cart A is being pulled away from Q at a speed of 5 ft/sec. How fast is cart B moving toward Q at the moment when cart A is 10 ft from Q?

$$\sqrt{10^2 + 24^2} + \sqrt{??^2 + 24^2} = 56 \qquad \sqrt{10^2 + 24^2} = 26$$

$$\left[\sqrt{\alpha^2 + 24^2} + \sqrt{\beta^2 + 24^2} = 56\right]_{t:\to t_0}$$

$$[\beta]_{t:\to t_0} = ??$$

$$\dot{\alpha} = 5$$

$$[\alpha]_{t:\to t_0} = 10$$

$$\left[\frac{\not{2}\alpha\dot{\alpha}}{\not{2}\sqrt{\alpha^2 + 24^2}} + \frac{\not{2}\beta\dot{\beta}}{\not{2}\sqrt{\beta^2 + 24^2}} = 0\right]_{t:\to t_0}$$

$$[\dot{\beta}]_{t:\to t_0} = -?$$

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$$\sqrt{10^2 + 24^2} + \sqrt{??^2 + 24^2} = 56 \qquad \sqrt{10^2 + 24^2} = 26$$

$$26 + \sqrt{??^2 + 24^2} = 56$$

$$\sqrt{??^2 + 24^2} = 30 \qquad \sqrt{18^2 + 24^2} = 30$$

$$[\beta]_{t:\to t_0} = ?? = 18$$

$$\dot{\alpha} = 5$$

$$[\alpha]_{t:\to t_0} = 10$$

$$[\dot{\beta}]_{t:\to t_0} = -?$$

$$\left[\frac{2\alpha\dot{\alpha}}{2\sqrt{\alpha^2 + 24^2}} + \frac{2\beta\dot{\beta}}{2\sqrt{\beta^2 + 24^2}} = 0\right]_{t:\to t_0}$$

EXAMPLE: Two carts, labeled "A" and "B", are connected by a rope 56 ft long that passes over a pulley (see below). The point Q is located 24 ft directly beneath the pulley between the carts. Cart A is being pulled away from Q at a speed of 5 ft/sec. How fast is cart B moving toward Q at the moment when cart A is 10 ft from Q?

$$\sqrt{10^2+24^2}=26$$

$$\sqrt{10^2+24^2}=26$$

$$\frac{10\cdot 5}{\sqrt{10^2+24^2}}+\frac{18\cdot(-?)}{\sqrt{18^2+24^2}}=0$$

$$\sqrt{18^2+24^2}=30$$

$$\sqrt{18^2+24^2}=30$$

$$[\beta]_{t:\to t_0}=??=18$$

$$\dot{\alpha}=5$$

$$[\alpha]_{t:\to t_0}=10$$

$$\left[\frac{2\alpha\dot{\alpha}}{2\sqrt{\alpha^2+24^2}}+\frac{2\beta\dot{\beta}}{2\sqrt{\beta^2+24^2}}=0\right]_{t:\to t_0}$$

$$[\dot{\beta}]_{t:\to t_0}=-?$$

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$$\sqrt{10^2+24^2}=26$$

$$\frac{10\cdot 5}{\sqrt{10^2+24^2}}+\frac{18\cdot(-?)}{\sqrt{18^2+24^2}}=0$$

$$\sqrt{18^2+24^2}=30$$

$$\frac{25}{13}-\frac{3\cdot ?}{5}=\frac{50}{26}-\frac{18\cdot ?}{30}=\frac{10\cdot 5}{26}+\frac{18\cdot(-?)}{30}=0$$

$$\frac{25}{13}=\frac{3\cdot ?}{5}$$

$$?=\left[\frac{5}{3}\right]\left[\frac{25}{13}\right]\doteq 3.205$$

EXAMPLE: Two carts, labeled "A" and "B", are connected by a rope 56 ft long that passes over a pulley (see below). The point Q is located 24 ft directly beneath the pulley between the carts. Cart A is being pulled away from Q at a speed of 5 ft/sec. How fast is cart B moving toward Q at the moment when cart A is 10 ft from Q?

approx. 3.205 ft/s ■

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$$\frac{10\cdot 5}{\sqrt{10^2+24^2}}+\frac{18\cdot(-?)}{\sqrt{18^2+24^2}}=0$$

$$\frac{25}{13}-\frac{3\cdot ?}{5}=\frac{50}{26}-\frac{18\cdot ?}{30}=\frac{10\cdot 5}{26}+\frac{18\cdot(-?)}{30}=0$$

$$\frac{25}{13}=\frac{3\cdot ?}{5}$$

$$? = \left[\frac{5}{3}\right]\left[\frac{25}{13}\right] \doteq 3.205$$

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Whitman problems

§6.2, p. 124–127, #1-25

STOP