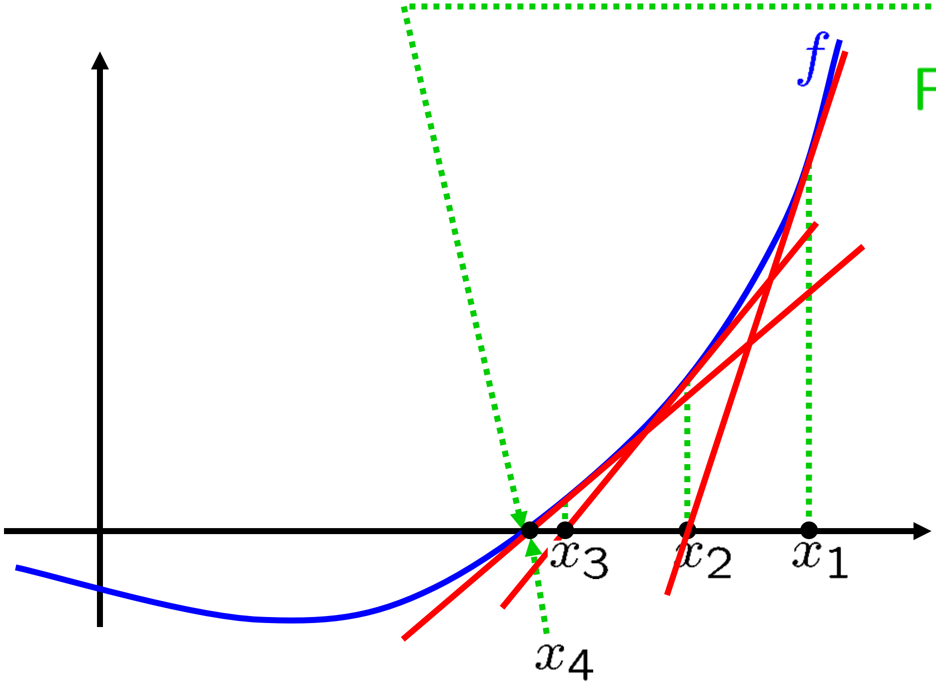
CALCULUS
Newton's method

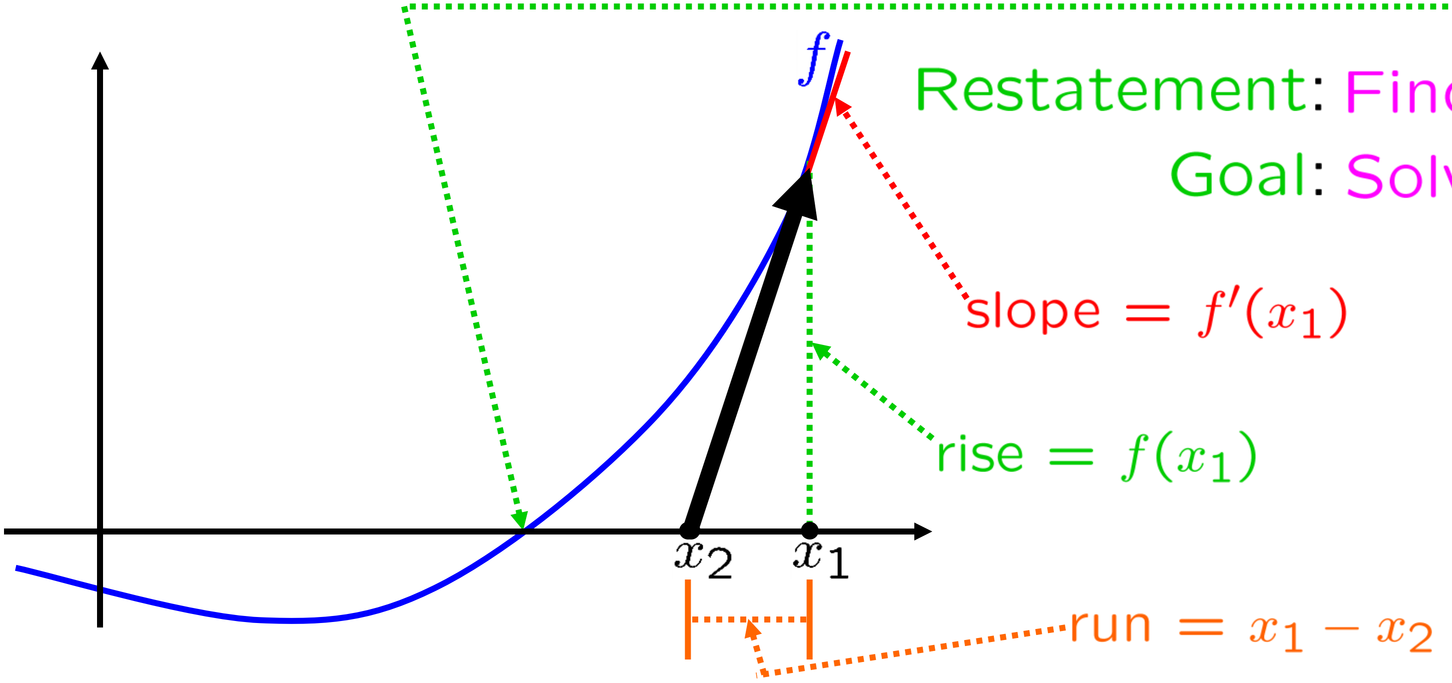


Restatement: Find this number.

Goal: Solve $f(x) = 0$.

Idea: Make an "initial guess".
Find it on the x-axis.
Go up to graph.
Follow tangent line back down to x-axis.
Repeat.

Can we find a formula for x_2 in terms of x_1?



Restatement: Find this number.

Goal: Solve $f(x) = 0$.

Can we solve for x_2?

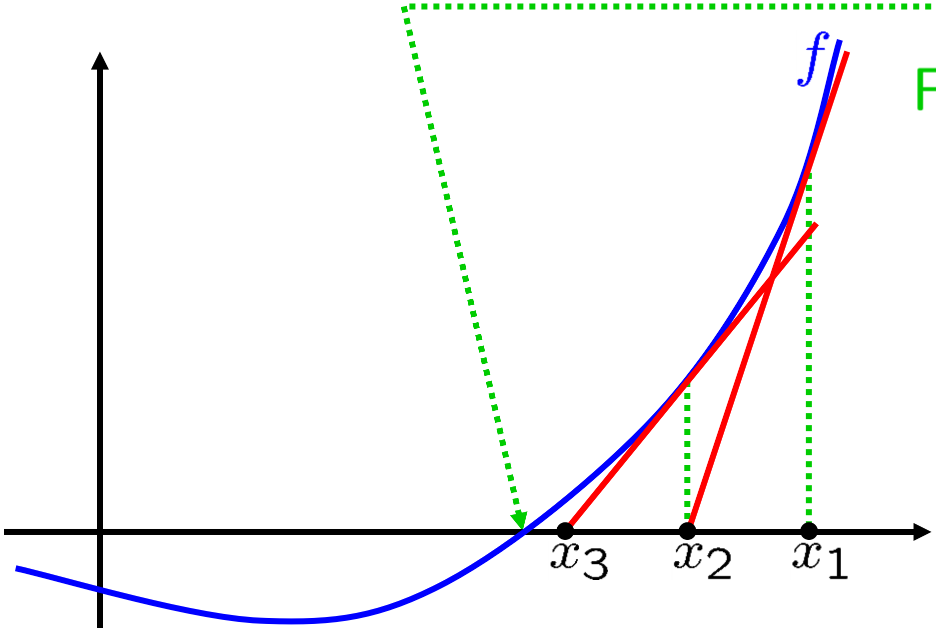
$$\frac{f(x_1)}{x_1 - x_2} = f'(x_1)$$

YES!

$$\frac{f(x_1)}{f'(x_1)} = x_1 - x_2$$

Can we find a formula for x_2 in terms of x_1?

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$



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Goal: Solve $f(x) = 0$.

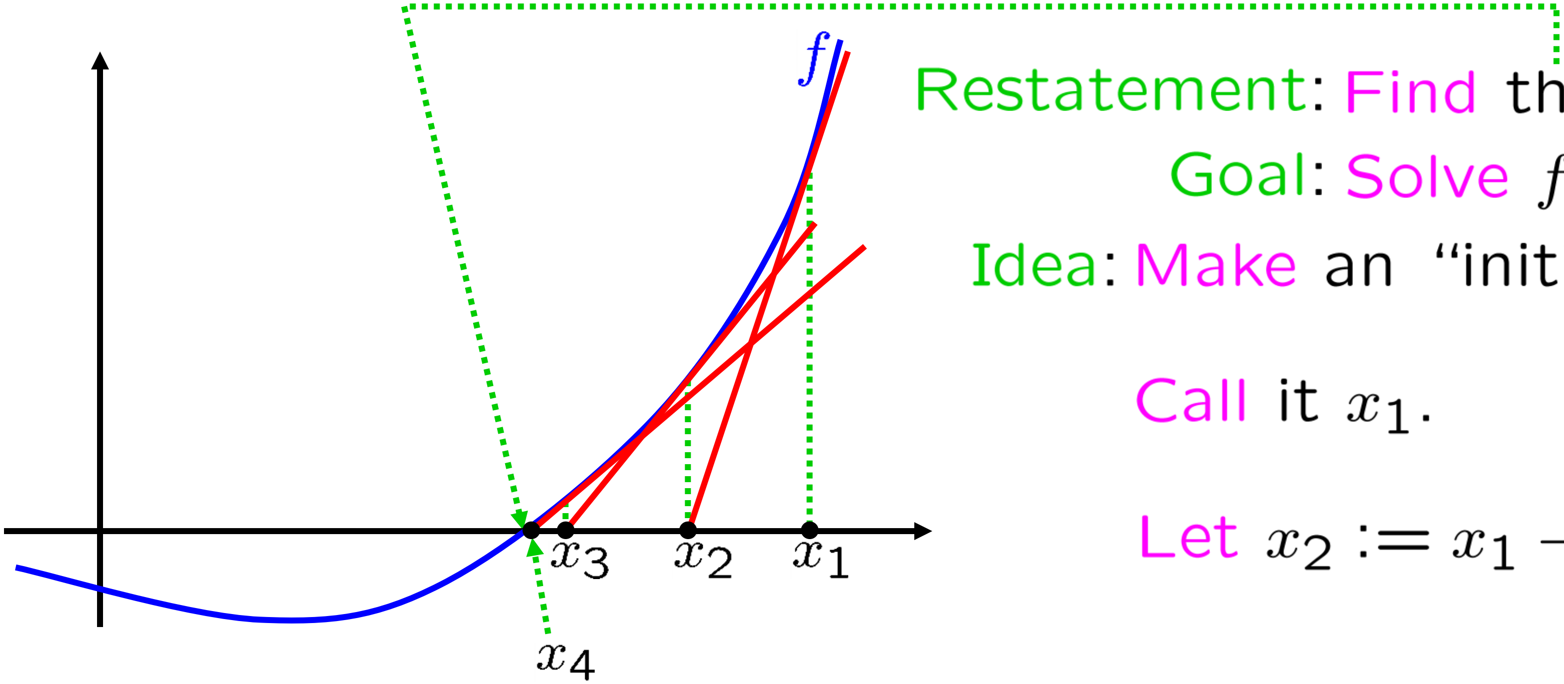
Idea: Make an "initial guess".

Call it x_1.

Let $x_2 := x_1 - \dfrac{f(x_1)}{f'(x_1)}$

Let $x_3 := x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$



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Goal: Solve $f(x) = 0$.

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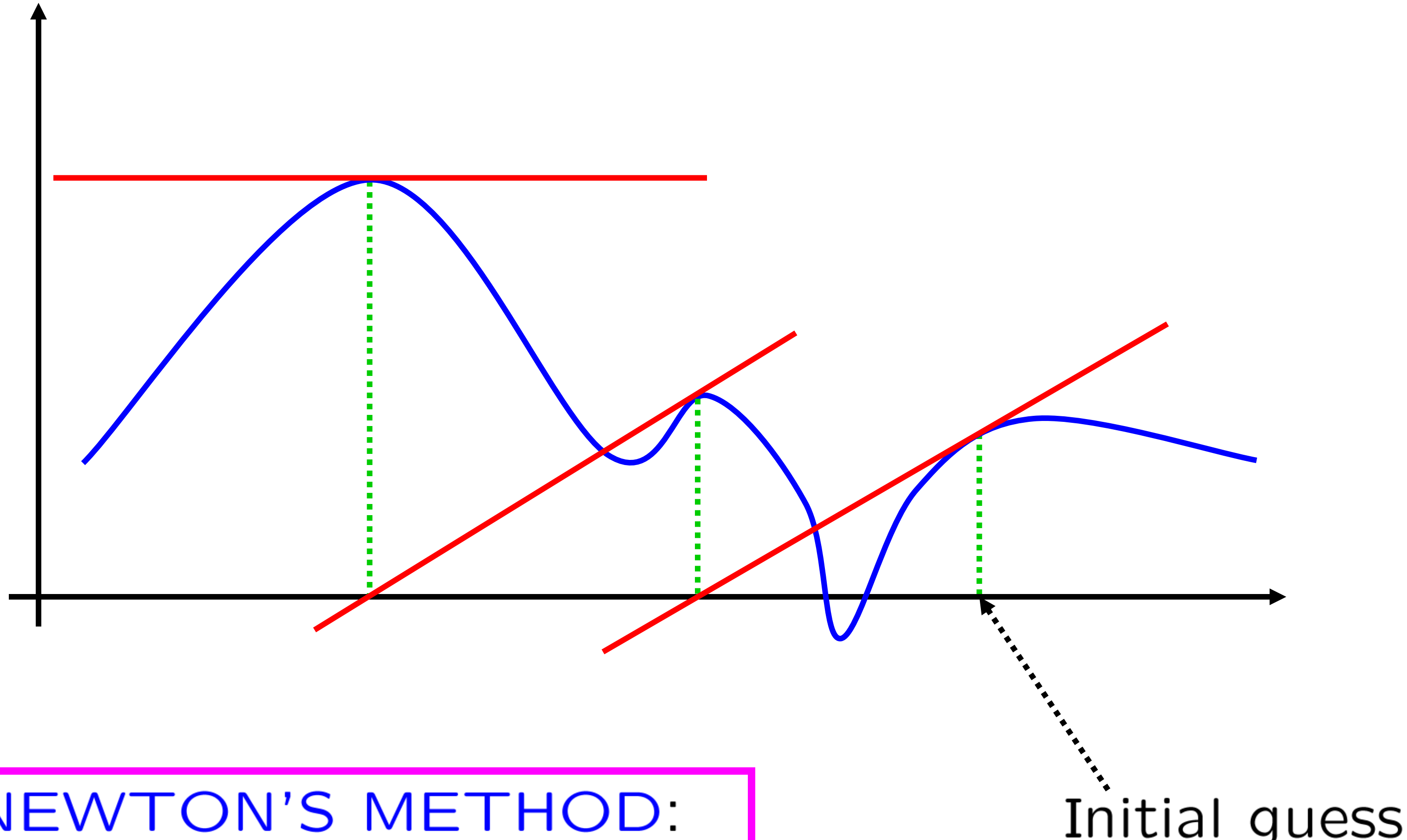
Let $x_3 := x_2 - \frac{f(x_2)}{f'(x_2)}$

Let $x_4 := x_3 - \frac{f(x_3)}{f'(x_3)}$

NEWTON'S METHOD:
$\forall$integers $n \geq 1$,

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$$

WARNING: Doesn't always work!



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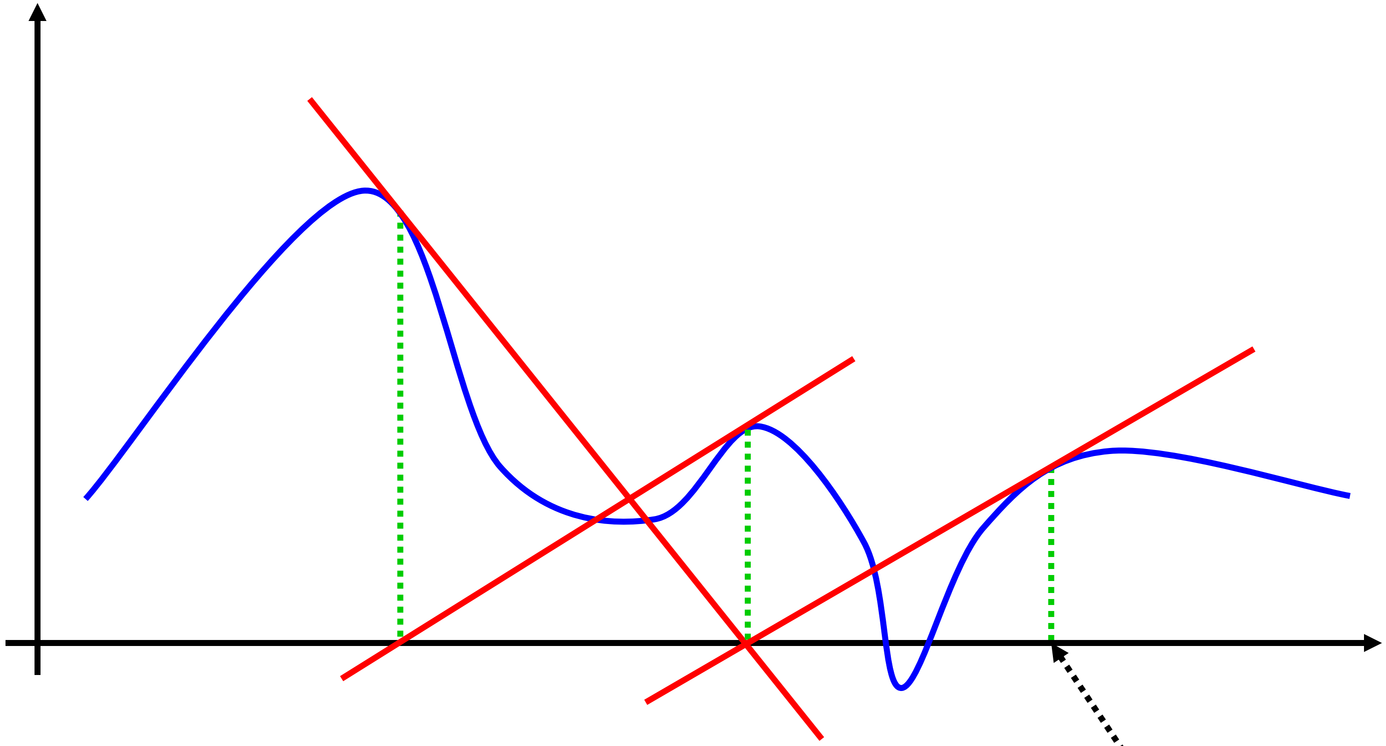

Initial guess

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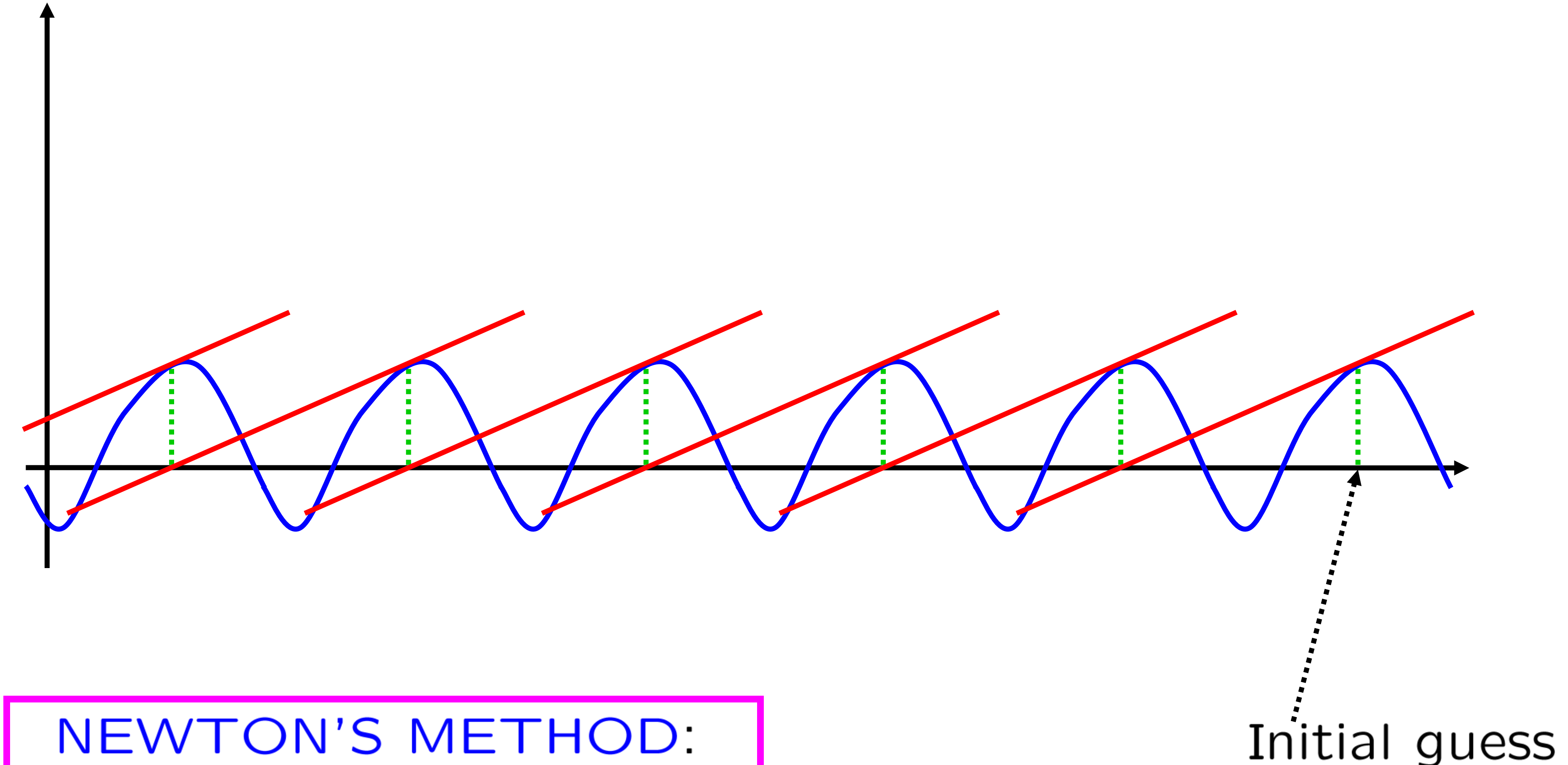

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§6.3

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Newton's method works incredibly well. . .

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However, for many (if not most) functions,

Newton's method works incredibly well. . .

For example, to compute $\sqrt{2}$,

we can find roots of $f(x) = x^2 - 2$.

In this instance, the number of decimal places of accuracy

DOUBLES with each iteration!

So, if we start with an inital guess of 1.414,

then, after 20 iterations,

we'll have more than one million

decimal places of accuracy!

More on this later. . .

NEWTON'S METHOD:

$\forall$integers $n > 1$,

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$$

Advice: Formulas are hard to remember; perhaps algorithms are easier. . .

To get x_{n+1},

take the logarithmic derivative of $f(x)$ w.r.t. x,

reciprocate

evaluate at x_n,

and then subtract from x_n.

$$\frac{d}{dx}[\,\mathrm{la}(f(x))] = \frac{f'(x)}{f(x)} \qquad \boxed{\frac{f(x)}{f'(x)}} \qquad x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$$

the **reciprocated logarithmic derivative of $f(x)$ w.r.t. x**

EXAMPLE: Starting at $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 4x + 2 = 0$.

$$\frac{x^3 - 4x + 2}{3x^2 - 4}$$

RECIPROCATED
LOGARITHMIC
DERIVATIVE
(w.r.t. x)

EVALUATE
at $x :\to x_n$

$$\frac{d}{dx}[\,\mathrm{la}(f(x))] = \frac{f'(x)}{f(x)} \qquad \boxed{\frac{f(x)}{f'(x)}} \qquad x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$$

the **reciprocated logarithmic derivative of $f(x)$ w.r.t. x**

EXAMPLE: Starting at $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 4x + 2 = 0$.

SUBTRACT FROM x_n

$$x_{n+1} := x_n - \frac{x_n^3 - 4x_n + 2}{3x_n^2 - 4}$$

$n :\to 1$

$$x_2 := x_1 - \frac{x_1^3 - 4x_1 + 2}{3x_1^2 - 4}$$

$$= 2 - \frac{2^3 - 4(2) + 2}{3(2)^2 - 4} = 1.75$$

$$\frac{d}{dx}[\,\mathrm{la}(f(x))] = \frac{f'(x)}{f(x)} \qquad \boxed{\frac{f(x)}{f'(x)}} \qquad x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$$

the **reciprocated logarithmic derivative of $f(x)$ w.r.t. x**

EXAMPLE: Starting at $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 4x + 2 = 0$.

$$x_{n+1} := x_n - \frac{x_n^3 - 4x_n + 2}{3x_n^2 - 4}$$

$n :\to 2$

$$x_2 := x_1 - \frac{x_1^3 - 4x_1 + 2}{3x_1^2 - 4}$$

$$= 2 - \frac{2^3 - 4(2) + 2}{3(2)^2 - 4} = 1.75$$

$$x_3 := x_2 - \frac{x_2^3 - 4x_2 + 2}{3x_2^2 - 4}$$

SKILL
Newton's method

$$= 1.75 - \frac{(1.75)^3 - 4(1.75) + 2}{3(1.75)^2 - 4} \doteq 1.6807$$

EXAMPLE: Find $\sqrt{7}$.

$$x^2 - 7 = 0$$

$$\frac{x^2 - 7}{2x}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 7}{2x_n}$$

$$= x_n - \frac{x_n^2}{2x_n} + \frac{+7}{2x_n}$$

$$= x_n - \frac{x_n}{2} + \frac{7}{2x_n}$$

$$= x_n\left(1 - \frac{1}{2}\right) + \frac{7}{2x_n} = \frac{1}{2}x_n + \frac{1}{2}\left(\frac{7}{x_n}\right)$$

$$= \frac{1}{2}\left(x_n + \frac{7}{x_n}\right)$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{7}{x_n}\right)$$

x_{n+1} Average together

$$x_n \text{ and } \frac{7}{x_n}.$$

$$= \frac{1}{2}\left(x_n + \frac{7}{x_n}\right)$$

EXAMPLE: Find $\sqrt{7}$.

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{7}{x_n}\right)$$

Average together x_n and $\frac{7}{x_n}$.

This is probably how your calculator's $\sqrt{\bullet}$ button works.

NOTE: $\frac{7}{\sqrt{7}} = \sqrt{7}$

So, if $x_n \boxed{=} \sqrt{7}$,

then $x_{n+1} \boxed{=} \sqrt{7}$.

EXAMPLE: Find $\sqrt{7}$.

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{7}{x_n}\right)$$

Average together x_n and $\frac{7}{x_n}$.

This is probably how your calculator's $\sqrt{\bullet}$ button works.

NOTE: $\frac{7}{\sqrt{7}} = \sqrt{7}$

If $x_n \approx \sqrt{7}$, then $x_{n+1} \approx \sqrt{7}$.

Number of decimals of accuracy *DOUBLES* with each iteration!

The method is "stable".

Exercise for you: Starting with $x_1 = 3$, do a few iterations. ■

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Newton's method

EXAMPLE: Use Newton's method to find $\sqrt[5]{3}$ correct to eight decimal places.

$$x^5 - 3 = 0$$

$$\frac{x^5 - 3}{5x^4}$$

$$x_{n+1} = x_n - \frac{x_n^5 - 3}{5x_n^4}$$

$$x_1 = 1$$
$$x_2 = 1.40000000$$
$$x_3 \doteq 1.27618492$$
$$x_4 \doteq 1.24715013$$
$$x_5 \doteq 1.24573426$$
$$x_6 \doteq 1.24573094$$
$$x_7 \doteq 1.24573094 \quad \blacksquare$$

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Newton's method

$$1.245730935 \leq \sqrt[5]{3} < 1.24573094$$

EXAMPLE: Find, correct to six decimal places, the root of the equation $\cos x = 2x$.

$$(\cos x) - 2x = 0$$

$$\frac{(\cos x) - 2x}{(-\sin x) - 2}$$

$$x_{n+1} = x_n + \frac{(\cos x_n) - 2x_n}{(+\sin x_n) + 2}$$

$$= x_n + \frac{(\cos x_n) - 2x_n}{(\sin x_n) + 2}$$

$x_1 = 1$

$x_2 \doteq 0.486288$

$x_3 \doteq 0.450419$

$x_4 \doteq 0.450184$

$x_5 \doteq 0.450184$ ■

$\exists x \in [0.4501835, 0.450184)$ s.t. $\cos x = 2x$.

SKILL
Newton's method

EXAMPLE: Use Newton's method with the initial approx. $x_1 = -3$ to find x_3, the third approximation to the root of $\underbrace{x^5 + 2x^2 - 1}_{g(x)} = 0$. (Give your answer to four decimal places.)

-1.9771 ■

$$x_{n+1} = x_n - \frac{x_n^5 + 2x_n^2 - 1}{5x_n^4 + 4x_n}$$

$$x_2 \doteq -2.424936387$$
$$x_3 \doteq -1.977061357$$
$$x_4 \doteq -1.635537781$$
$$x_5 \doteq -1.384022483$$
$$x_6 \doteq -1.208593933$$
$$x_7 \doteq -1.095922158$$
$$x_8 \doteq -1.032720787$$
$$x_9 \doteq -1.005964689$$
$$x_{10} \doteq -1.000263458$$
$$x_{11} \doteq -1.000000553$$

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Newton's method

$$g(-1) = 0$$

EXAMPLE: Use Newton's method to approximate $\sqrt[100]{150}$ correct to nine decimal places.

$$x^{100} - 150 = 0$$

$$x_{n+1} = x_n - \frac{x_n^{100} - 150}{100x_n^{99}} = (0.99)x_n + \frac{1.5}{x_n^{99}}$$

~~$x_1 = 1.000000000$~~
~~$x_2 = 2.490000000$~~
~~$x_3 = 2.465100000$~~
~~$x_4 = 2.440449000$~~

TOO SLOW!!

$$(1.1)^{100} \doteq 13780.61234$$

$$(1.1)^{100} > 150$$

$$1.1 > \sqrt[100]{150}$$

$x_1 = 1.100000000$
$x_2 \doteq 1.089119733$
$x_3 \doteq 1.078548871$
$x_4 \doteq 1.068604677$
$x_5 \doteq 1.060023384$
$x_6 \doteq 1.054099017$
$x_7 \doteq 1.051701916$
$x_8 \doteq 1.051387651$
$x_9 \doteq 1.051382909$
$x_{10} \doteq 1.051382908$
$x_{11} \doteq 1.051382908$

$$\sqrt[100]{150} \doteq 1.051382908 \blacksquare$$

EXAMPLE: Explain why Newton's method fails when applied to the eq'n $\sqrt[3]{x} = 0$ with any initial approximation $x_1 \neq 0$. Illustrate your explanation with a sketch.

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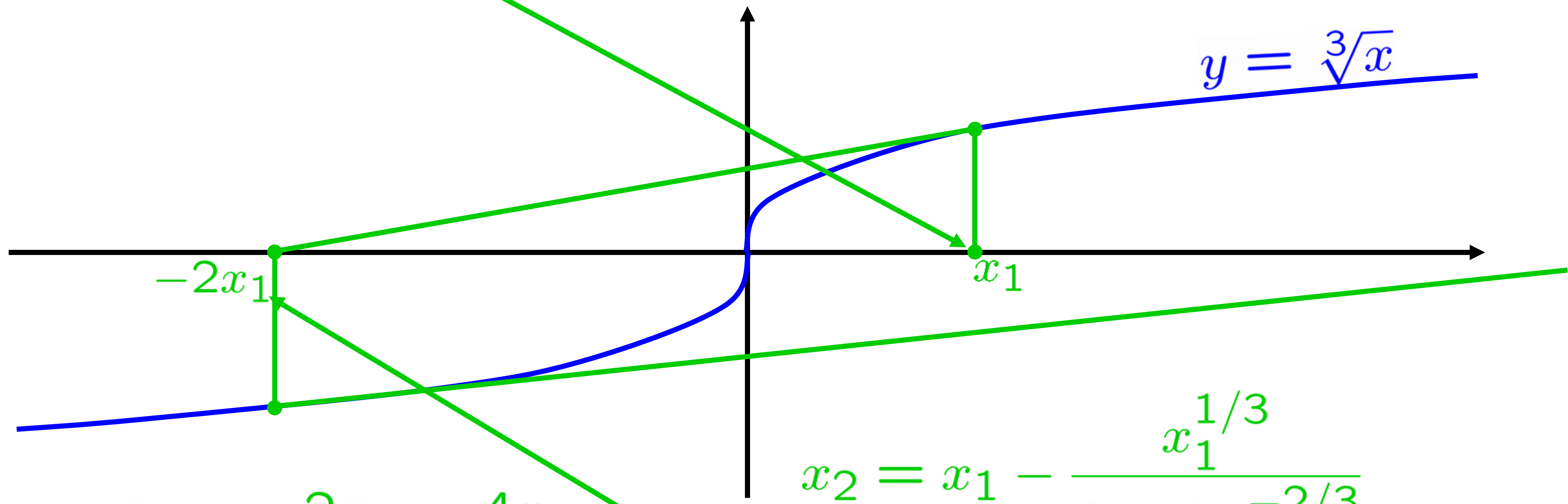
Newton's method

Each answer is twice as far from the correct answer (which is 0) as the one before it.



$$x_2 = x_1 - \frac{x_1^{1/3}}{(1/3)x_1^{-2/3}}$$

$$= x_1 - 3x_1$$

$$= -2x_1$$

$$x_3 = -2x_2 = 4x_1$$

etc.

SKILL
Newton's method

Whitman problems
§6.3, p. 130, #1-4

STOP