CALCULUS
Linear approximation

Definition: For any expression y of x,

$$\boxed{\triangle y} \quad := \quad \left([y]_{x:\to x+\triangle x}\right) \; - \; y,$$

which is an expression of x and $\triangle x$.

E.g., $$\triangle(x^4) \quad = \quad (x+\triangle x)^4 \; - \; x^4.$$

$$[\triangle(x^4)]_{x:\to 2, \triangle x:\to 0.03} \quad = \quad (2.03)^4 \; - \; 2^4$$
$$\doteq \quad 0.9818\,.$$

Note: $$[\triangle(x^4)]_{x:\to 2, \triangle x:\to 0.03} \quad = \quad [x^4]_{x:\to 2}^{x:\to 2.03}$$

Definition: For any expression y of x,

$$\boxed{dy} \quad := \quad \left[\frac{dy}{dx}\right] dx,$$

which is an expression of x and dx.

E.g., $$d(x^4) \quad = \quad [4x^3]\,dx.$$

$$[d(x^4)]_{x:\to 2, dx:\to 0.03} \quad = \quad \left[4 \cdot 2^3\right] \cdot 0.03$$
$$= \quad 0.96\,.$$

Definition: The **linearization of f at a** is the function L whose graph is the tangent line to f at $(a, f(a))$.

Problem: Find the linearization of $f(x) = 3x^2 - 4x + 1$ at $x = 2$.

Solution:

y-coordinate of point of tangency:

$$[3x^2 - 4x + 1]_{x:\to 2} = 3 \cdot 2^2 - 4 \cdot 2 + 1 = 12 - 8 + 1 = 5$$

point of tangency: $(2, 5)$

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Slope of tangent line:

$$[(d/dx)(3x^2 - 4x + 1)]_{x:\to 2} = [6x - 4]_{x:\to 2} = 6 \cdot 2 - 4 = 8$$

Equation of tangent line:

$$y - 5 = 8(x - 2), \text{ i.e., } y = 5 + 8(x - 2)$$

WRONG: ~~$L(x) = 8(x - 2)$~~

The graph of $y = L(x)$ should be the tangent line, but ...

$y = 8(x - 2)$ is NOT the tangent line to $y = 3x^2 - 4x + 1$ at $(2, 5)$.

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$f'(a)$

Equation of tangent line:

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Linearization: $L(x) = 5 + 8(x - 2)$ ■

For you: Graph f and L.

General formula: $L(x) = [f(a)] + [f'(a)][x - a]$

END OF PRELIMINARIES

Next: Weather Forecasting . . .

If the temperature at 8pm yesterday was 40 degrees,

guess what the temperature will be today at 8pm.

40 degrees

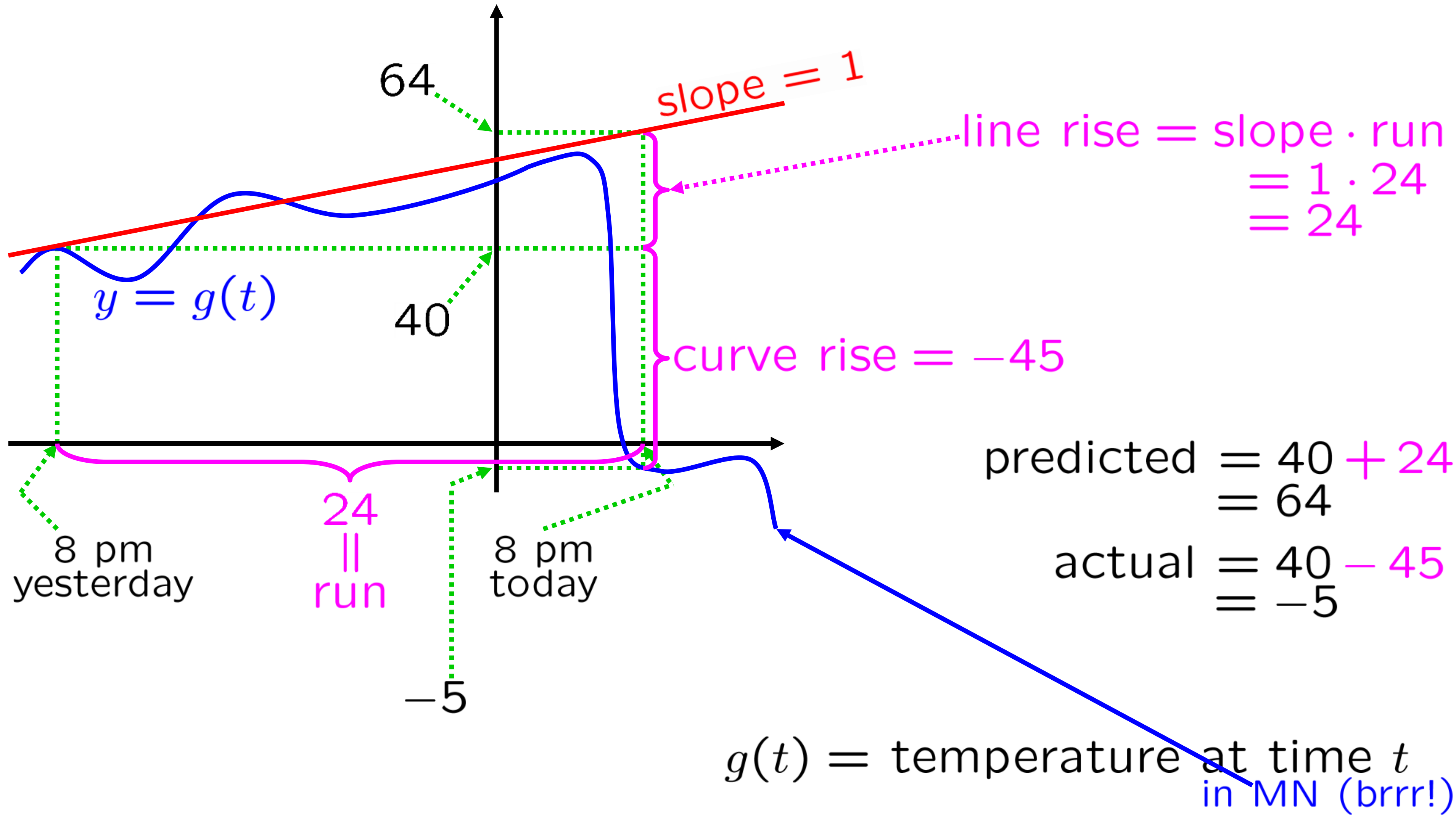
“0th order approximation”
“constant approximation”

If the temperature at 8pm yesterday was 40 degrees,
and was going up at 1 degree per hour,
guess what the temperature will be today at 8pm.

64 degrees

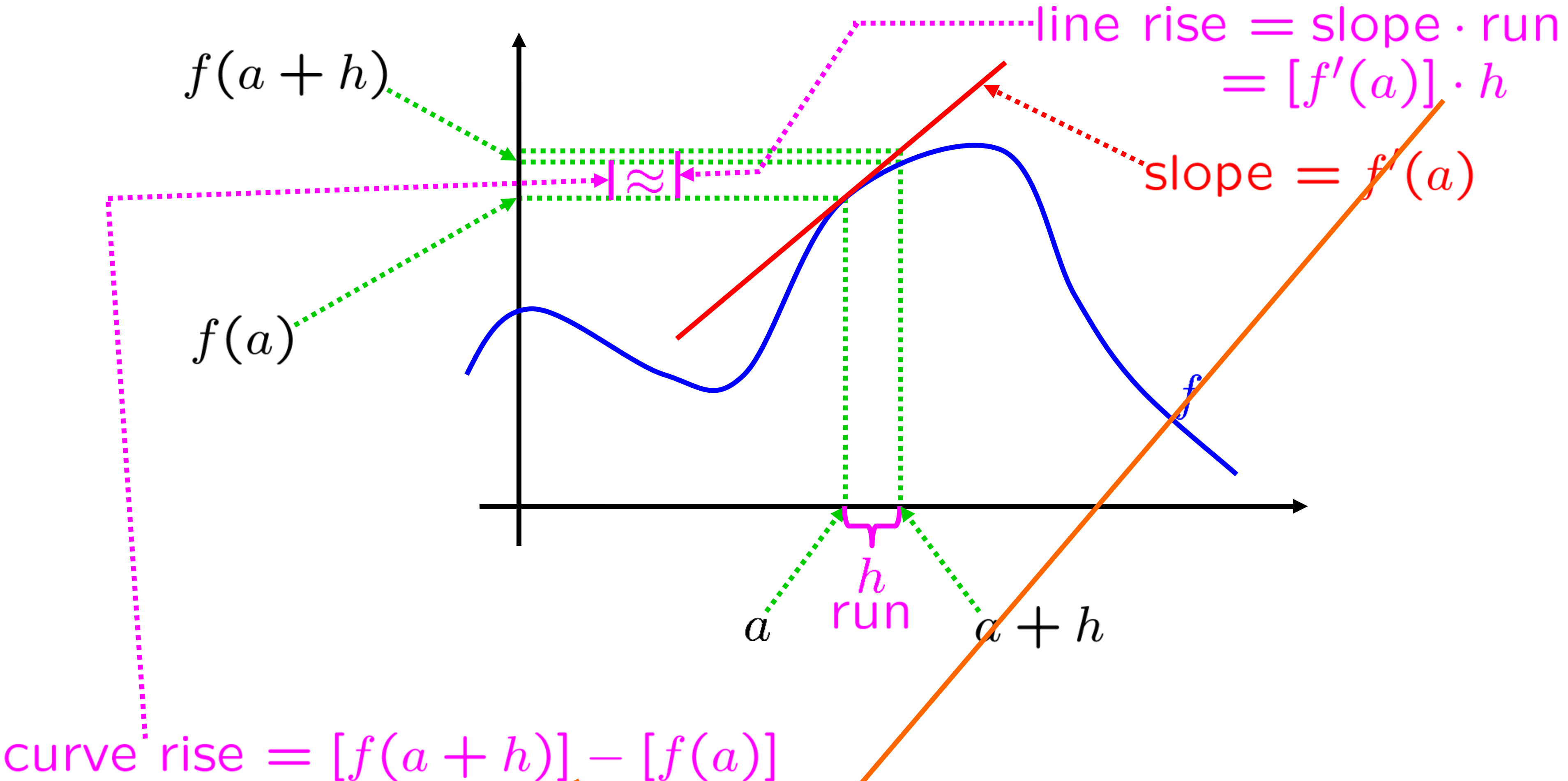
"1st order approximation"
"linear approximation"
"approximation by differentials"

Higher order approximations are called
Taylor approximations
and Maclaurin approximations,
see §10.11, p. 267–271.



Next: the general picture . . .

The tangent line hugs the curve,
for a little while.



PRINCIPLE OF LINEAR APPROXIMATION:

$$[f(a+h)] - [f(a)] \approx [f'(a)] \cdot h, \text{ for } h \text{ small}$$

Next: "linearization" . . .

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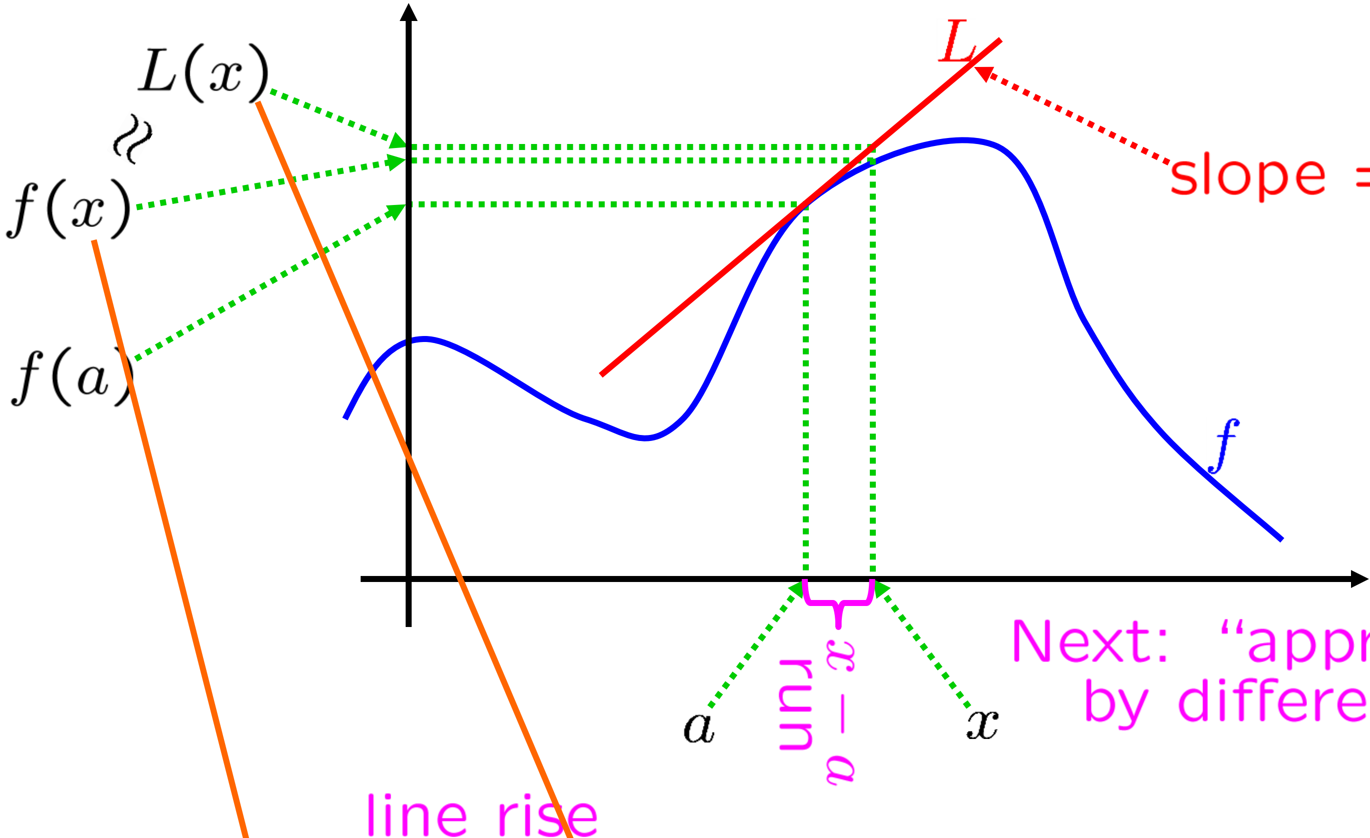

Next: "approximation by differentials" . . .

$L(x) = [f(a)] + [f'(a)][x - a]$ is the linearization of f at a.

PRINCIPLE OF LINEAR APPROXIMATION:

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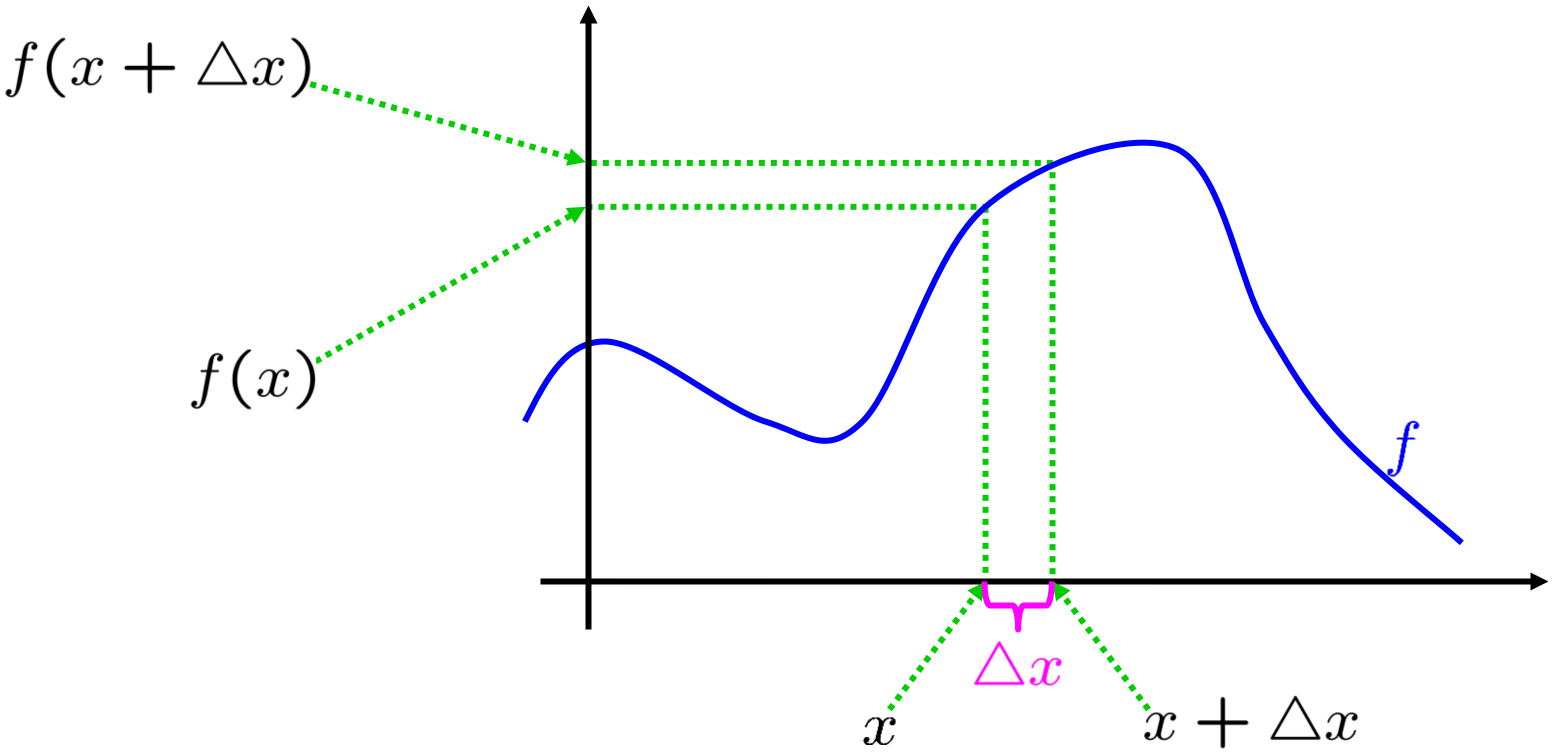

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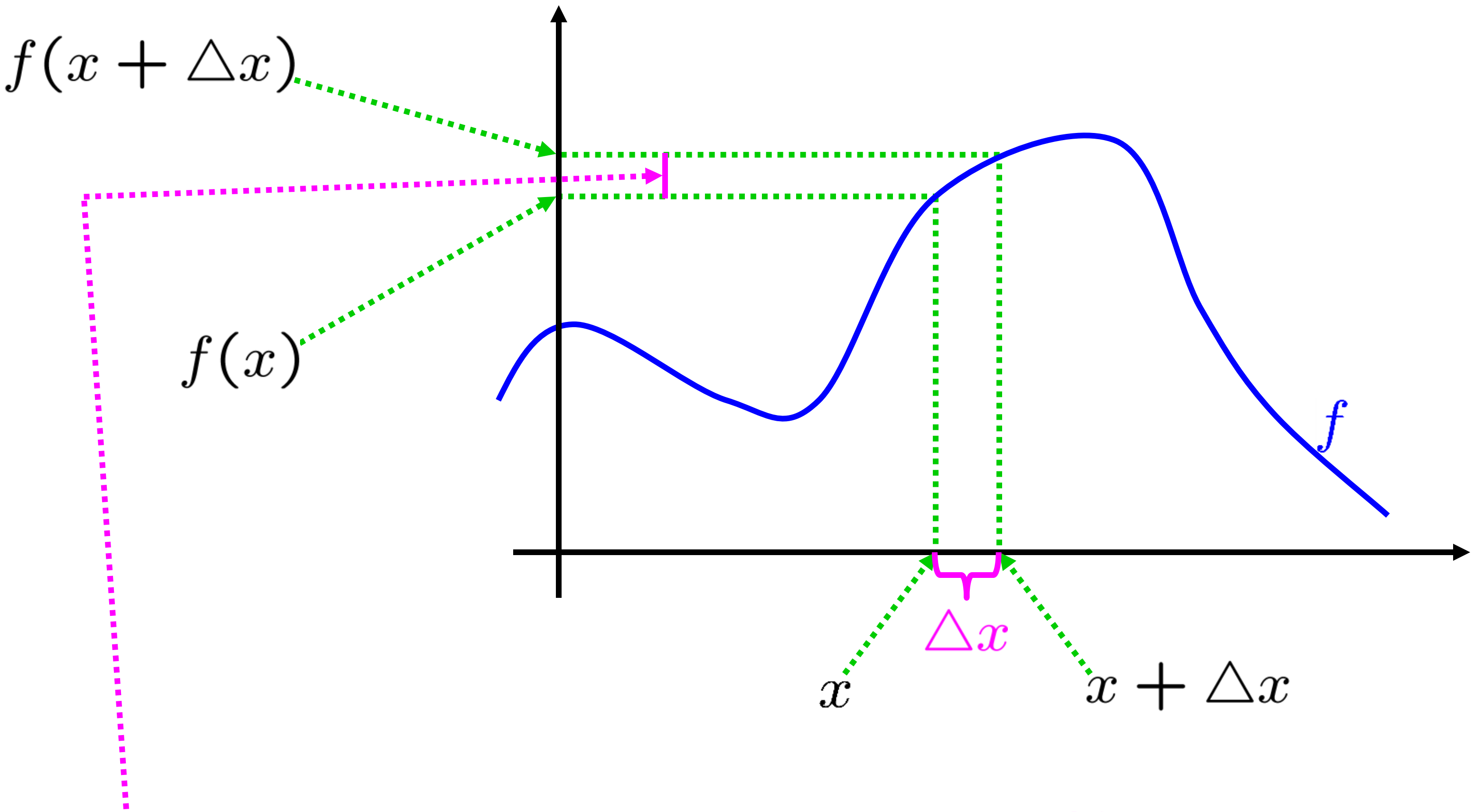
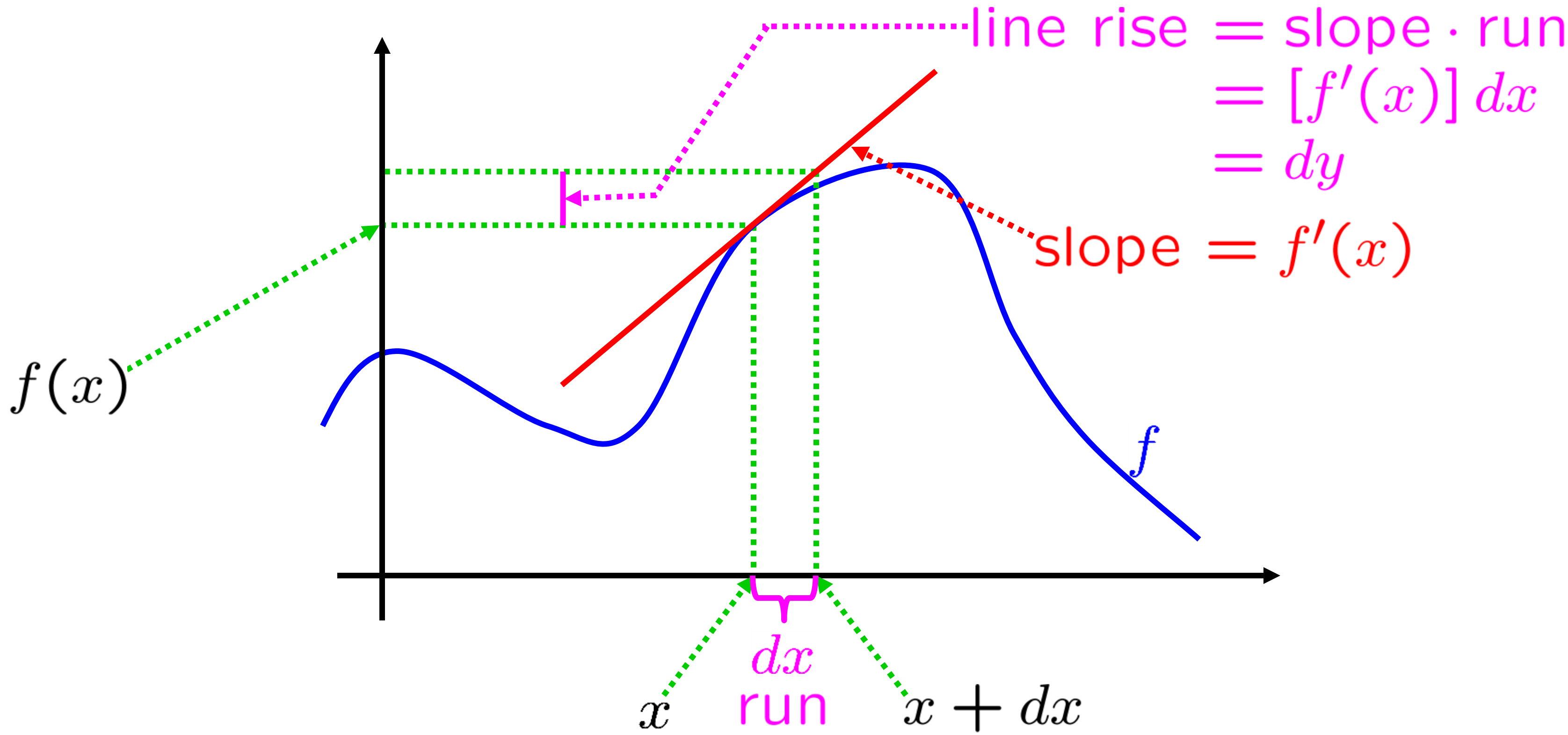


curve rise $= \triangle y$

Next: tangent line...

Let $y = f(x)$.

Definition: $\boxed{\triangle y} := \left([y]_{x:\to x+\triangle x} \right) - y = [f(x + \triangle x)] - [f(x)]$



Next: $\triangle x = dx$ . . .

Let $y = f(x)$.

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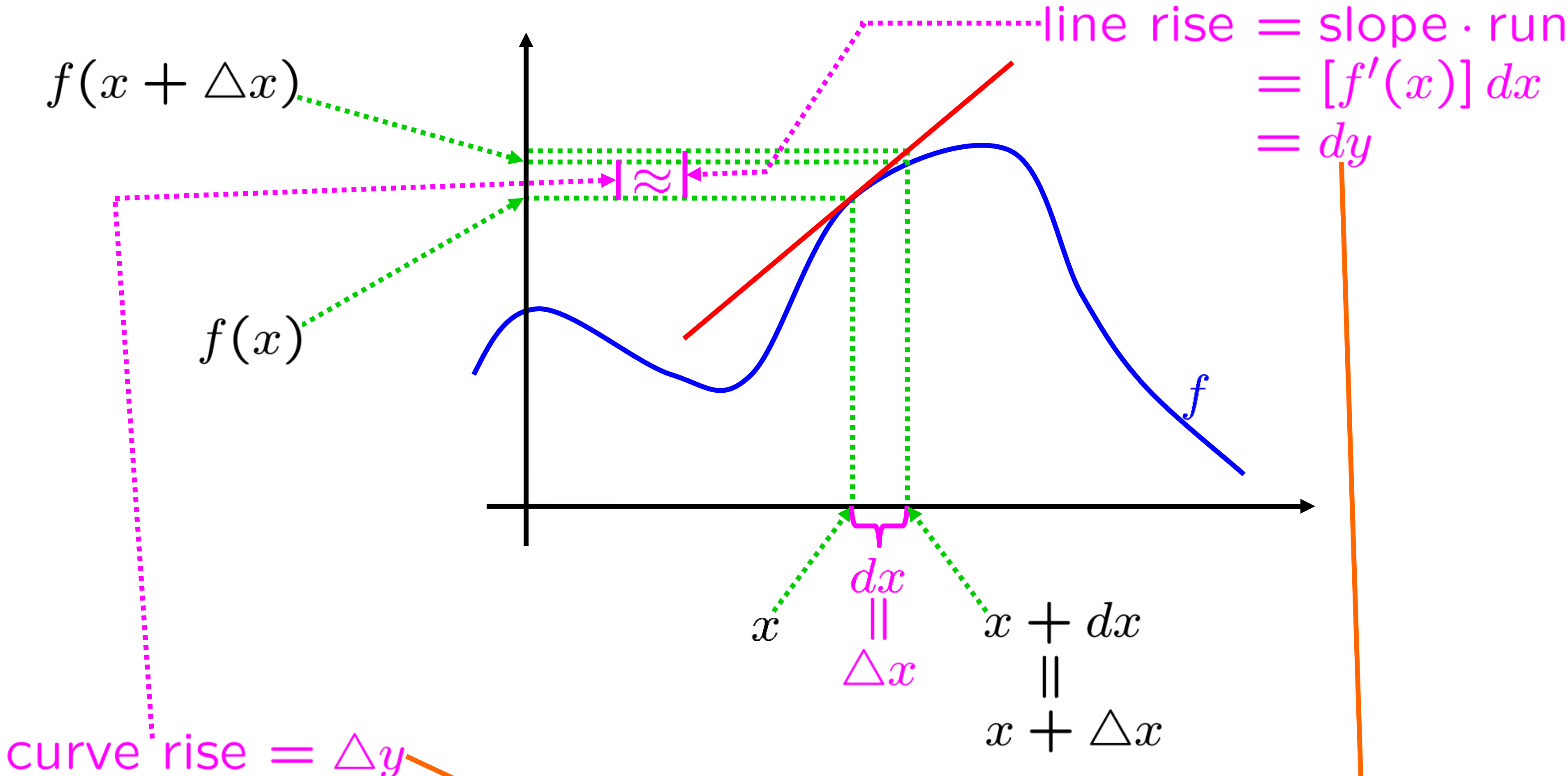

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If $\triangle x = dx$ is small, then $\cancel{\triangle x} \cdot \frac{\triangle y}{\cancel{\triangle x}} \approx \frac{dy}{\cancel{dx}} \cdot \cancel{dx}$,

i.e., $\triangle y \approx dy$. QED

If $\triangle x = dx$ is small, then

$$[f(x + \triangle x)] - [f(x)] \approx [f'(x)]\, dx.$$

$x :\to a$, $\triangle x = dx :\to h$

If h is small, then

$$[f(a + h)] - [f(a)] \approx [f'(a)] \cdot h.$$

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$$[f(a+h)] - \boxed{[f(a)]} \approx [f'(a)] \cdot h, \text{ for } h \text{ small}$$

$$f(a+h) \approx [f(a)] + [f'(a)] \cdot h, \text{ for } h \text{ small}$$

$h :\to x - a$

$$f(x) \approx [f(a)] + [f'(a)][x-a], \text{ for } x \text{ close to } a$$

If $\triangle x = dx$ is small, then

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Next: problems

$\boxed{f(x) \approx [f(a)] + [f'(a)][x-a], \text{ for } x \text{ close to } a}$

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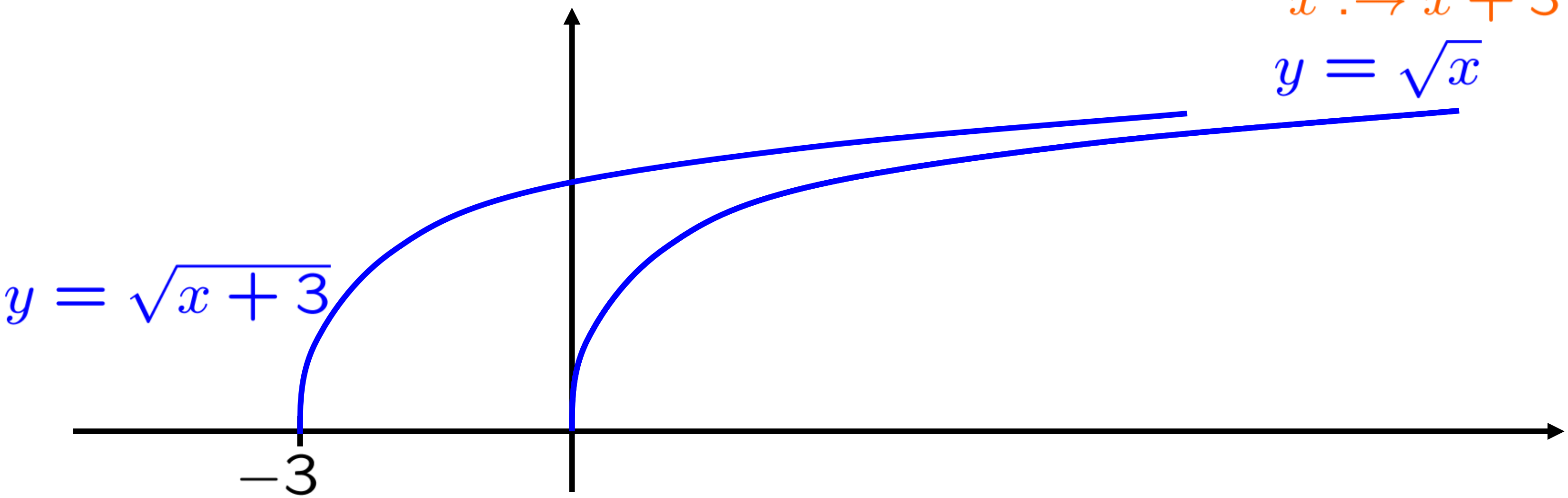
The tangent line hugs the curve,
for a little while.

EXAMPLE: Find the linearization of $f(x) = \sqrt{x+3}$ at $x = 1$ and use it to approximate $\sqrt{4.05}$.

Goal: Estimate $f(1.05)$.

$x :\to x + 3$



$y = \sqrt{x+3}$

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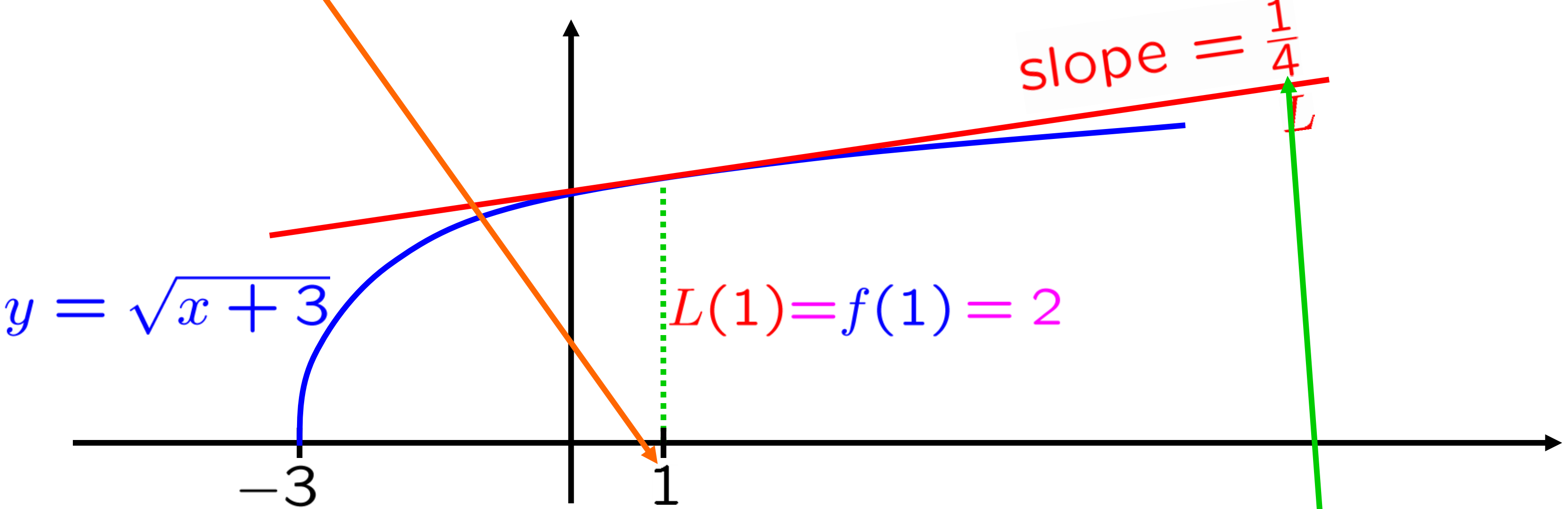

$$f'(x) = \tfrac{1}{2}(\quad x+3 \quad)^{-1/2}(1)$$

$$f'(1) = \tfrac{1}{2}(\quad 1+3 \quad)^{-1/2} = \frac{1}{2} \cdot \frac{1}{\sqrt{4}} = \frac{1}{4}$$

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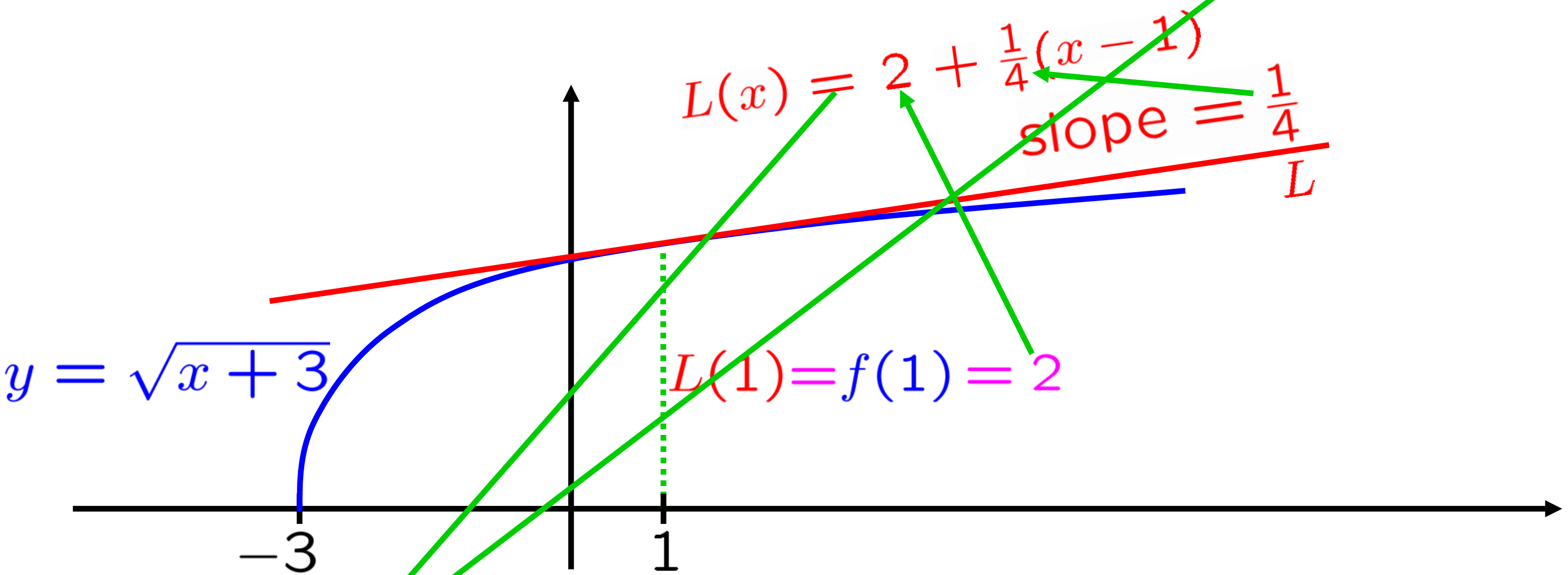

Another approach via $\triangle \approx d$ . . .

$f(1.05) = 2.01246\ldots$

$\wr\wr$

$L(1.05) = 2 + \frac{1}{4}[0.05] = 2 + 0.0125 = 2.0125$

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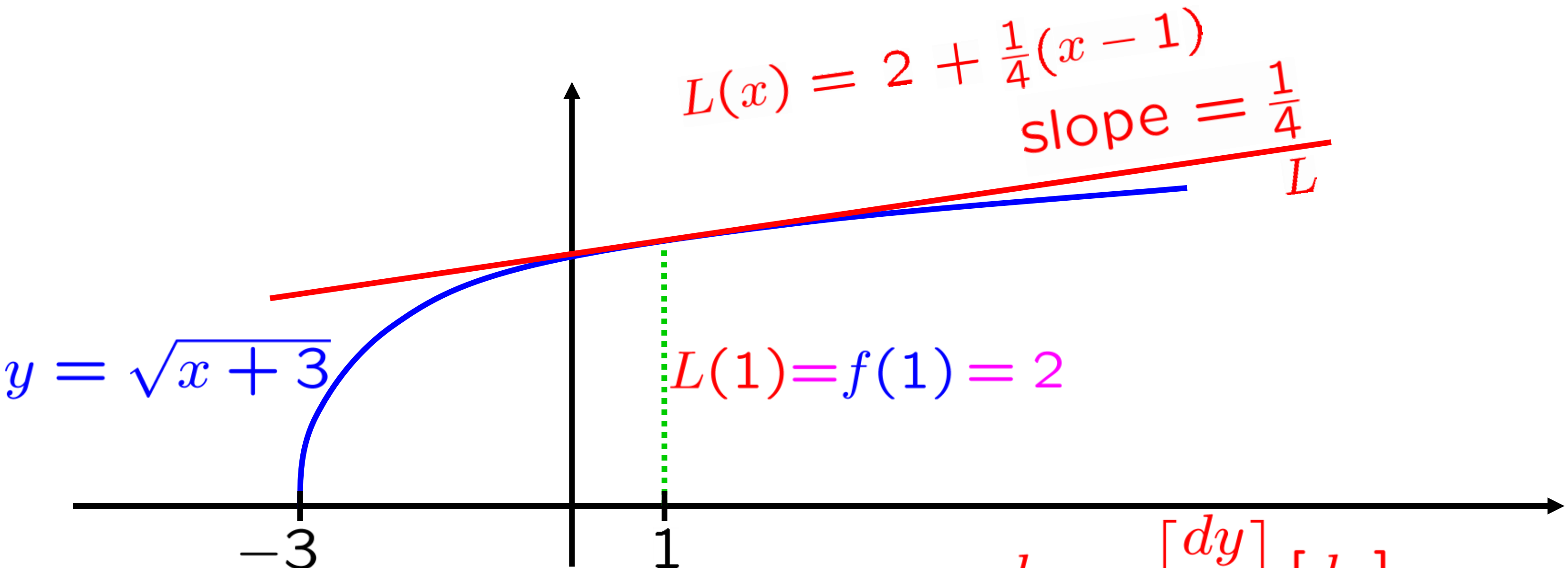


$$dy = \left[\frac{dy}{dx}\right][dx]$$

Another approach via $\triangle \approx d$ . . .

$$[y]_{x:\to 1}^{x:\to 1.05} = [\triangle y]_{x:\to 1, \ \triangle x:\to 0.05} \approx [dy]_{x:\to 1, \ dx:\to 0.05} = \frac{1}{4}[0.05]$$

$$\sqrt{4.05} - \sqrt{4} \approx \frac{1}{4}[0.05]$$

$$L(2.05) = 2 + \frac{1}{4}[0.05] = 2 + 0.0125 = 2.0125$$

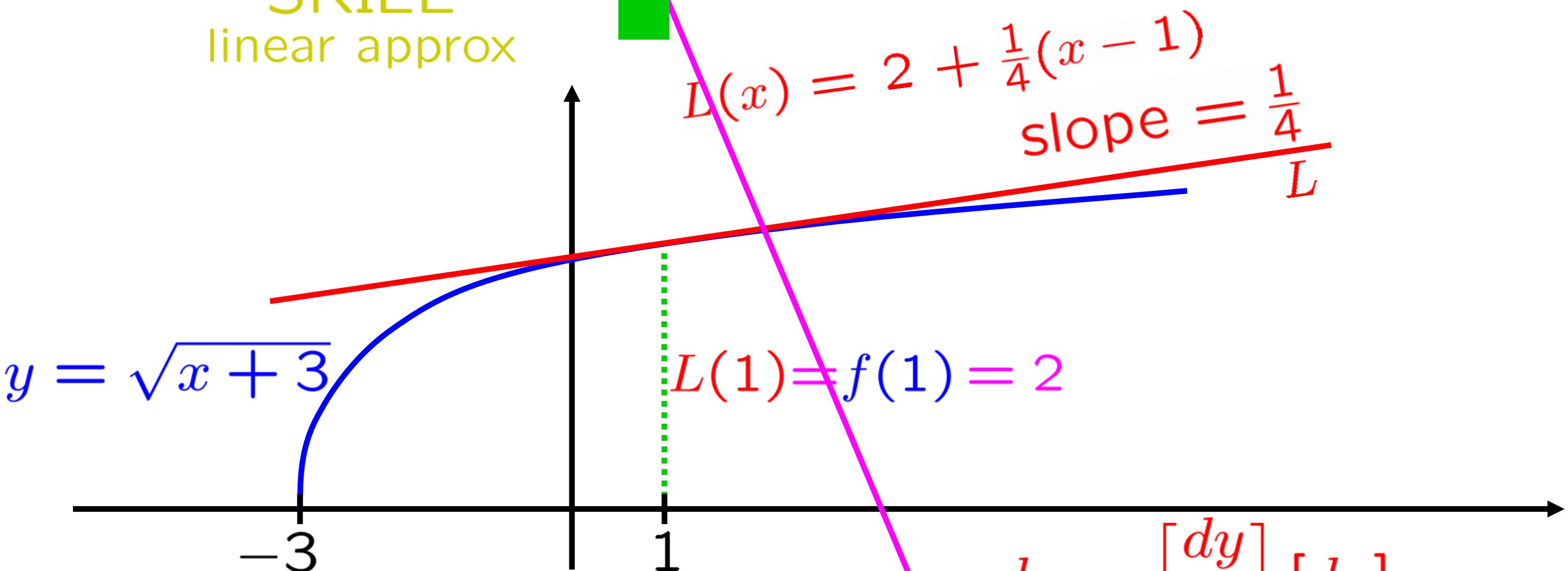
§6.4

EXAMPLE: Find the linearization of $f(x) = \sqrt{x+3}$ at $x = 1$ and use it to approximate $\sqrt{4.05}$.

Is this amount an overestimate or an underestimate?

Goal: Estimate $f(1.05)$.

SKILL
linear approx



$$dy = \left[\frac{dy}{dx}\right][dx]$$

Another approach via $\triangle \approx d$ . . .

$$[y]_{x:\to 1}^{x:\to 1.05} = [\triangle y]_{x:\to 1,\ \triangle x:\to 0.05} < \approx [dy]_{x:\to 1,\ dx:\to 0.05} = \frac{1}{4}[0.05]$$

$$\sqrt{4.05} \approx \sqrt{4} + \frac{1}{4}[0.05] = 2 + 0.0125 = 2.0125$$

$$L(2.05) = 2 + \frac{1}{4}[0.05] = 2 + 0.0125 = 2.0125$$

EXAMPLE: The radius of a sphere was measured and found to be 23 cm with a possible error in measurement of at most 0.07 cm. Approximate the maximum error in using 23 cm to compute the volume in the sphere.

max error $\approx 465\text{cm}^3$

$$V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dr} = 4\pi r^2$$

467, −464

$$[\triangle V]_{r:\to 23,\ \triangle r:\to \pm 0.07} = \frac{4}{3}\pi(23 \pm 0.07)^3 - \underbrace{\frac{4}{3}\pi(23)^3}_{\doteq\, 50965}$$

$$\approx$$

$$[dV]_{r:\to 23,\ dr:\to \pm 0.07} = \left[\left(\frac{dV}{dr}\right) dr\right]_{r:\to 23,\ dr:\to \pm 0.07}$$

$$= \left[\left(4\pi r^2\right) dr\right]_{r:\to 23,\ dr:\to \pm 0.07}$$

$$= 4\pi(23^2)(\pm 0.07)$$

$$\doteq \pm 465 \ \blacksquare$$

SKILL
linear approx

§6.4

EXAMPLE: Explain in terms of linear approximations why the following approximation is reasonable.

$$(1.001)^7 \approx 1.007$$

$$(1.001)^7 = 1^7 + [x^7]_{x:\to 1}^{x:\to 1.001}$$

$$= 1^7 + [\triangle(x^7)]_{x:\to 1, \triangle x:\to 0.001}$$

$$\approx 1^7 + [d(x^7)]_{x:\to 1, dx:\to 0.001}$$

$$= 1^7 + [(7x^6)(dx)]_{x:\to 1, dx:\to 0.001}$$

$$= 1^7 + [(7 \cdot 1^6)(0.001)]$$

$$= 1 + [0.007] = 1.007 \blacksquare$$

$$[f(x)]_{x:\to a}^{x:\to a+h} = [\triangle(f(x))]_{x:\to a, \triangle x:\to h}$$

SKILL
linear approx

EXAMPLE: Use a linear approximation (or differentials) to estimate $e^{-0.032}$.

$$e^{-0.032} = e^0 + [e^x]_{x:\to 0}^{x:\to -0.032}$$

$$= e^0 + [\triangle(e^x)]_{x:\to 0, \triangle x:\to -0.032}$$

$$\approx e^0 + [d(e^x)]_{x:\to 0, dx:\to -0.032}$$

$$= e^0 + [(e^x)(dx)]_{x:\to 0, dx:\to -0.032}$$

$$= e^0 + [e^0(-0.032)]$$

$$= 1 + (-0.032)$$

$$= 0.968 \blacksquare$$

SKILL
linear approx

$$[f(x)]_{x:\to a}^{x:\to a+h} = [\triangle(f(x))]_{x:\to a, \triangle x:\to h}$$

§6.4

SKILL
linear approx

Whitman problems
§6.4, p. 133, #1-5