CALCULUS
Properties of the definite integral
PROPERTIES OF THE DEFINITE INTEGRAL

\[ \int_a^b c(f(x)) \, dx = c \int_a^b f(x) \, dx \]

Assuming \( a < b \) and \( f \) is continuous on \([a, b]\).

Proof: Let \( F(x) := \int_a^x f(t) \, dt \). Then \( \frac{d}{dx}[F(x)] = f(x) \).

Then
\[ \int_a^b c(f(x)) \, dx = [c(F(x))]_{x=a}^{x=b} \]
\[ = [c(F(b))] - [c(F(a))] \]
\[ = \left[ c \int_a^b f(t) \, dt \right] - \left[ c \int_a^a f(t) \, dt \right] \]
\[ = c \int_a^b f(x) \, dx. \] QED

THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.4

If \( f \) is continuous on \([a, b]\),
\[ \frac{d}{dx} \int_a^x f(t) \, dt = [f(t)]_{t=a}^{t=x} = f(x), \text{ for } x \in (a, b). \]
PROPERTIES OF THE DEFINITE INTEGRAL

\[ \int_a^b c(f(x)) \, dx = c \int_a^b f(x) \, dx \]

\[ \int_a^b (f(x)) + (g(x)) \, dx = \left( \int_a^b f(x) \, dx \right) + \left( \int_a^b g(x) \, dx \right) \]

Similar proof.

\[ \int_a^b \text{ is linear.} \]

\[ \int_a^b [c_1(f_1(x)) + \cdots + c_n(f_n(x))] \, dx = c_1 \left[ \int_a^b f_1(x) \, dx \right] + \cdots + c_n \left[ \int_a^b f_n(x) \, dx \right] \]

\[ \int_a^b [c_1f_1 + \cdots + c_nf_n] = c_1 \left[ \int_a^b f_1 \right] + \cdots + c_n \left[ \int_a^b f_n \right] \]

THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.4

If \( f \) is continuous on \([a, b]\),

then \( \frac{d}{dx} \int_a^x f(t) \, dt = [f(t)]_{t\rightarrow x} = f(x) \), for \( x \in (a, b) \).
PROPERTIES OF THE DEF. INTEGRAL

\[ \int_{a}^{b} c(f(x)) \, dx = c \int_{a}^{b} f(x) \, dx \]

\[ \int_{a}^{b} (f(x)) + (g(x)) \, dx = \left( \int_{a}^{b} f(x) \, dx \right) + \left( \int_{a}^{b} g(x) \, dx \right) \]

\[ \int_{a}^{b} cx \, dx = c(b - a) \text{ QED} \]

Suppose \( c > 0 \) and \( a < b \). If not . . .

THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.4

If \( f \) is continuous on \([a, b]\),

then \( \frac{d}{dx} \int_{a}^{x} f(t) \, dt = [f(t)]_{t: \rightarrow x} = f(x) \), for \( x \in (a, b) \).
PROPERTIES OF THE DEFINITE INTEGRAL

\[
\int_{a}^{b} c \, dx = \int_{a}^{b} c(f(x)) \, dx = c \int_{a}^{b} f(x) \, dx
\]

\[
\int_{a}^{b} (f(x)) + (g(x)) \, dx = \left( \int_{a}^{b} f(x) \, dx \right) + \left( \int_{a}^{b} g(x) \, dx \right)
\]

\[
\int_{a}^{b} c(f(x)) \, dx = c \int_{a}^{b} f(x) \, dx
\]

\[
\int_{a}^{b} (f(x)) - (g(x)) \, dx = \left( \int_{a}^{b} f(x) \, dx \right) - \left( \int_{a}^{b} g(x) \, dx \right)
\]

because subtraction is a linear combination, with coefficients +1 and -1.

THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.4

If \( f \) is continuous on \([a, b]\),

then \( \frac{d}{dx} \int_{a}^{x} f(t) \, dt = [f(t)]_{t \to x} = f(x) \), for \( x \in (a, b) \).
\( \int_a^b c \, dx = c(b - a) \)

\( \int_a^b (f(x)) + (g(x)) \, dx = \left( \int_a^b f(x) \, dx \right) + \left( \int_a^b g(x) \, dx \right) \)

\( \int_a^b c(f(x)) \, dx = c \int_a^b f(x) \, dx \)

\( \int_a^b (f(x)) - (g(x)) \, dx = \left( \int_a^b f(x) \, dx \right) - \left( \int_a^b g(x) \, dx \right) \)

\[ q \geq 0 \text{ on } [a, b] \implies \int_a^b q(x) \, dx \geq 0 \]

**Proof:** \( \forall \) integers \( n \geq 1 \),

\[ h_n := \frac{b - a}{n} \]

\[ R_n S_a^b q = \sum_{j=1}^{n} h_n(q(a + jh_n)) \geq 0. \]

\[ \int_a^b q(x) \, dx = \lim_{n \to \infty} R_n S_a^b q \geq 0. \] QED
PROPERTIES OF THE DEF. INTEGRAL

\[
\int_a^b c \, dx = c(b - a)
\]

\[
\int_a^b (f(x)) + (g(x)) \, dx = \left( \int_a^b f(x) \, dx \right) + \left( \int_a^b g(x) \, dx \right)
\]

\[
\int_a^b c(f(x)) \, dx = c \int_a^b f(x) \, dx
\]

\[
\int_a^b (f(x)) - (g(x)) \, dx = \left( \int_a^b f(x) \, dx \right) - \left( \int_a^b g(x) \, dx \right)
\]

\[q \geq 0 \text{ on } [a, b] \quad \Rightarrow \quad \int_a^b q(x) \, dx \geq 0\]

\[f \geq g \text{ on } [a, b] \quad \Rightarrow \quad \int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx\]

**Proof:** \( q := f - g \geq 0 \) on \( [a, b] \), so

\[
\left( \int_a^b f(x) \, dx \right) - \left( \int_a^b g(x) \, dx \right) = \int_a^b q(x) \, dx \geq 0.
\]

QED
PROPERTIES OF THE DEF. INTEGRAL

\[
\int_{a}^{b} c \, dx = c(b - a)
\]

\[
\int_{a}^{b} (f(x)) + (g(x)) \, dx = \left( \int_{a}^{b} f(x) \, dx \right) + \left( \int_{a}^{b} g(x) \, dx \right)
\]

\[
\int_{a}^{b} c(f(x)) \, dx = c \int_{a}^{b} f(x) \, dx
\]

\[
\int_{a}^{b} (f(x)) - (g(x)) \, dx = \left( \int_{a}^{b} f(x) \, dx \right) - \left( \int_{a}^{b} g(x) \, dx \right)
\]

\[g \leq f \text{ on } [a, b] \quad \Rightarrow \quad \int_{a}^{b} g(x) \, dx \leq \int_{a}^{b} f(x) \, dx\]  

\[\text{monotonicity} \]

\[f \geq g \text{ on } [a, b] \quad \Rightarrow \quad \int_{a}^{b} f(x) \, dx \geq \int_{a}^{b} g(x) \, dx\]  

\[\text{monotonicity} \]

\[m \leq f \leq M \text{ on } [a, b] \quad \Rightarrow \quad m(b - a) \leq \int_{a}^{b} f(x) \, dx \leq M(b - a)\]
PROPERTIES OF THE DEFINITE INTEGRAL

\[ \int_a^b c \, dx = c(b - a) \]

\[ \int_a^b (f(x)) + (g(x)) \, dx = \left( \int_a^b f(x) \, dx \right) + \left( \int_a^b g(x) \, dx \right) \]

\[ \int_a^b c(f(x)) \, dx = c \int_a^b f(x) \, dx \]

\[ \int_a^b (f(x)) - (g(x)) \, dx = \left( \int_a^b f(x) \, dx \right) - \left( \int_a^b g(x) \, dx \right) \]

\[ \int_a^c f(x) \, dx = \left[ \int_a^b f(x) \, dx \right] + \left[ \int_b^c f(x) \, dx \right] \]

\[ f \geq g \text{ on } [a, b] \implies \int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx \]

\[ m \leq f \leq M \text{ on } [a, b] \implies m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a) \]
The "cocycle" identity

\[
\int_{a}^{c} f(x) \, dx = \left[ \int_{a}^{b} f(x) \, dx \right] + \left[ \int_{b}^{c} f(x) \, dx \right]
\]

\(\forall F'(x) = f(x)\)

cocycle identity:

\[
[F(x)]_{x \rightarrow c}^{x \rightarrow a} = ([F(x)]_{x \rightarrow b}^{x \rightarrow a}) + ([F(x)]_{x \rightarrow c}^{x \rightarrow b})
\]
cf. §7.3, p. 150ff PROPERTIES OF THE DEF. INTEGRAL

\[
\int_a^b (f(x)) + (g(x)) \, dx = \left( \int_a^b f(x) \, dx \right) + \left( \int_a^b g(x) \, dx \right)
\]

\[
\int_a^b c(f(x)) \, dx = c \int_a^b f(x) \, dx
\]

\[\int_a^b \text{ is linear.}\]

the "cocycle" identity

\[
\int_a^c f(x) \, dx = \left[ \int_a^b f(x) \, dx \right] + \left[ \int_b^c f(x) \, dx \right]
\]
cf. §7.3, p. 150ff PROPERTIES OF THE DEF. INTEGRAL

\[
\int_a^b (f(x)) + (g(x)) \, dx = \left( \int_a^b f(x) \, dx \right) + \left( \int_a^b g(x) \, dx \right)
\]

\[
\int_a^b c(f(x)) \, dx = c \int_a^b f(x) \, dx
\]

\[
\text{is linear.}
\]

Linear, but not multiplicative: \( \frac{d}{dx}, \sum, \int, \int_a^b, [\bullet]_{x:a \rightarrow b}, \triangle \)

\[
\frac{d}{dx} \left( [f(x)][g(x)] \right) \neq \left[ \frac{d}{dx} (f(x)) \right] \left[ \frac{d}{dx} (g(x)) \right], \quad \text{product rule (differentiation by parts)}
\]

\[
\sum a_j b_j \neq \left[ \sum a_j \right] \left[ \sum b_j \right], \quad \text{summation by parts}
\]

\[
\int [f(x)][g(x)] \, dx \neq \left[ \int f(x) \, dx \right] \left[ \int g(x) \, dx \right], \quad \text{integration by parts}
\]

\[
\int_a^b [f(x)][g(x)] \, dx \neq \left[ \int_a^b f(x) \, dx \right] \left[ \int_a^b g(x) \, dx \right], \quad \text{integration by parts}
\]

\[
[\left( f(x) \right) \left( g(x) \right)]_{x:a \rightarrow b} \neq \left[ \left( f(x) \right)_{x:a \rightarrow b} \right] \left[ \left( g(x) \right)_{x:a \rightarrow b} \right], \quad \text{product rule (eval. by parts)}
\]

\[
\triangle \left[ a_n b_n \right] \neq \left( \triangle a_n \right) \left( \triangle b_n \right), \quad \text{product rule (differencing by parts)}
\]
Linear, and multiplicative: \( \lim, [\bullet]_{x \to a} \)

\[
\lim [f(x)][g(x)] = [\lim f(x)][\lim g(x)]
\]

\[
((f(x)(g(x)))_{x \to a} = ([f(x)]_{x \to a})([g(x)]_{x \to a})
\]

---

Linear, but not multiplicative: \( \frac{d}{dx}, \sum, \int, \int_{a}^{b}, [\bullet]_{x \to a}^{b}, \Delta \)

\[
\frac{d}{dx}([f(x)][g(x)]) \neq \left[ \frac{d}{dx}(f(x)) \right] \left[ \frac{d}{dx}(g(x)) \right], \text{ product rule (differentiation by parts)}
\]

\[
\sum a_j b_j \neq \left[ \sum a_j \right] \left[ \sum b_j \right], \text{ summation by parts}
\]

\[
\int [f(x)][g(x)] \, dx \neq \left[ \int f(x) \, dx \right] \left[ \int g(x) \, dx \right], \text{ integration by parts}
\]

\[
\int_{a}^{b} [f(x)][g(x)] \, dx \neq \left[ \int_{a}^{b} f(x) \, dx \right] \left[ \int_{a}^{b} g(x) \, dx \right] \text{ integration by parts}
\]

\[
((f(x))(g(x)))_{x \to a}^{b} \neq ([f(x)]_{x \to a}^{b})([g(x)]_{x \to a}^{b}), \text{ product rule (eval. by parts)}
\]

\[
\Delta[a_n b_n] \neq (\Delta a_n)(\Delta b_n) \text{ product rule (differencing by parts)}
\]

---

\[^{7.3}\]

\[13\]
EXAMPLE: Use the properties of the integral to evaluate \( \int_0^4 (5 - 8x^2) \, dx \).

\[
\int_0^4 (5 - 8x^2) \, dx = 5 \left( \int_0^4 1 \, dx \right) - 8 \left( \int_0^4 x^2 \, dx \right) \quad \text{(NOT)} \quad \left( \int_0^4 x \, dx \right)^2
\]

\[\int_a^b \text{ is linear, but not multiplicative} \ldots\]
EXAMPLE: Use the properties of the integral to evaluate \( \int_0^4 (5 - 8x^2) \, dx \).

\[
\int_0^4 (5 - 8x^2) \, dx = 5 \left( \int_0^4 1 \, dx \right) - 8 \left( \int_0^4 x^2 \, dx \right)
\]

\[
= 5 \left( [x]_{x=0}^{x=4} \right) - 8 \left( \frac{x^3}{3} \bigg|_{x=0}^{x=4} \right)
\]

\[
= 5 (4 - 0) - 8 \left( \frac{4^3 - 0^3}{3} \right) = -\frac{452}{3}
\]

\([\bullet]_{x=a}^{x=b}\) is also linear, but not multiplicative .
EXAMPLE: Assume \( \int_2^7 f(x) \, dx = 9 \) and \( \int_4^7 f(x) \, dx = 12 \).

Compute \( \int_4^2 f(x) \, dx \).

the "cocycle" identity

\[
\left( \int_2^4 f(x) \, dx \right) + \left( \int_4^7 f(x) \, dx \right) = \int_2^7 f(x) \, dx
\]

\[
\int_2^4 f(x) \, dx = 9 - 12 = -3
\]

\[
\int_4^2 f(x) \, dx = - \int_2^4 f(x) \, dx = -(-3) = 3
\]

\[
\int_a^b f(x) \, dx := - \int_b^a f(x) \, dx, \quad \text{if } a < b
\]

SKILL Properties of integration
EXAMPLE: Compute \( \int_{0}^{7\pi/2} |\cos x| \, dx \).

Where **positive** & **negative** on \([0, 7\pi/2]\)?
EXAMPLE: Compute $\int_0^{7\pi/2} |\cos x| \, dx$.

\[
\left[ \int_0^{\pi/2} |\cos x| \, dx \right] + \left[ \int_{\pi/2}^{3\pi/2} |\cos x| \, dx \right] + \\
\left[ \int_{3\pi/2}^{5\pi/2} |\cos x| \, dx \right] + \left[ \int_{5\pi/2}^{7\pi/2} |\cos x| \, dx \right]
\]

Where positive & negative on $[0, 7\pi/2]$?

$y = \cos x$
EXAMPLE: Compute $\int_0^{7\pi/2} |\cos x| \, dx$.

\[
\left[ \int_0^{\pi/2} |\cos x| \, dx \right] + \left[ \int_{\pi/2}^{3\pi/2} |\cos x| \, dx \right] + \\
\left[ \int_{3\pi/2}^{5\pi/2} |\cos x| \, dx \right] + \left[ \int_{5\pi/2}^{7\pi/2} |\cos x| \, dx \right] + \left[ \int_0^{\pi/2} (\cos x) \, dx \right] + \left[ \int_{\pi/2}^{3\pi/2} (-\cos x) \, dx \right] + \\
\left[ \int_{3\pi/2}^{5\pi/2} (\cos x) \, dx \right] + \left[ \int_{5\pi/2}^{7\pi/2} (-\cos x) \, dx \right]
\]

\[y = \cos x\]
EXAMPLE: Compute $\int_0^{7\pi/2} |\cos x| \, dx$.

\[ \int_{\pi/2}^{\pi/2} \cos x \, dx + \int_{\pi/2}^{3\pi/2} -\cos x \, dx + \int_{3\pi/2}^{5\pi/2} \cos x \, dx + \int_{5\pi/2}^{7\pi/2} -\cos x \, dx + \int_{\pi/2}^{3\pi/2} -\cos x \, dx + \int_{3\pi/2}^{5\pi/2} \cos x \, dx + \int_{5\pi/2}^{7\pi/2} -\cos x \, dx \]

\[ \left[ \left[ \sin x \right]_{\pi/2}^{\pi/2} \right] + \left[ \left[ \sin x \right]_{0}^{\pi/2} \right] + \left[ \left[ \sin x \right]_{\pi/2}^{3\pi/2} \right] + \left[ \left[ \sin x \right]_{3\pi/2}^{5\pi/2} \right] + \left[ \left[ \sin x \right]_{5\pi/2}^{7\pi/2} \right] + \left[ \left[ \sin x \right]_{3\pi/2}^{5\pi/2} \right] + \left[ \left[ \sin x \right]_{5\pi/2}^{7\pi/2} \right] \]
EXAMPLE: Compute \( \int_0^{7\pi/2} |\cos x| \, dx \).
EXAMPLE: A particle moves along a line so that its velocity at time $t$ is $v(t) = t^2 + 2t - 15$ (measured in miles per hour).

(a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

(b) Find the distance traveled during the time period $1 \leq t \leq 4$.

\[
\begin{align*}
(a) \quad & \int_1^4 v(t) \, dt \\
(b) \quad & \int_1^4 |v(t)| \, dt
\end{align*}
\]

ANOTHER particle:

- displacement during $1 \leq t \leq 6$: 0
- distance traveled during $1 \leq t \leq 6$: 40

§8.1
EXAMPLE: A particle moves along a line so that its velocity at time \( t \) is \( v(t) = t^2 + 2t - 15 \) (measured in miles per hour).

(a) Find the displacement of the particle during the time period \( 1 \leq t \leq 4 \).

(b) Find the distance traveled during the time period \( 1 \leq t \leq 4 \).

(a) \[
\int_1^4 v(t) \, dt = \int_1^4 (t^2 + 2t - 15) \, dt
\]

\[
= \left[ \frac{t^3}{3} + t^2 - 15t \right]_{1}^{4}
\]

**LINEARITY OF [\bullet] \Rightarrow [\bullet]**

\[
= \left( \frac{4^3 - 1^3}{3} \right) + (4^2 - 1^2) - 15(4 - 1)
\]

\[
= \frac{63}{3} + 15 - 45
\]

\[
= 21 + 15 - 45
\]

\[
= 36 - 45
\]

\[
= -9
\]
EXAMPLE: A particle moves along a line so that its velocity at time \( t \) is \( v(t) = t^2 + 2t - 15 \) (measured in miles per hour).

(a) Find the displacement of the particle during the time period \( 1 \leq t \leq 4 \).

(b) Find the distance traveled during the time period \( 1 \leq t \leq 4 \).

\[
\int_{1}^{4} v(t) \, dt = -9
\]

Position at \( t = 4 \) is 9 units to the left of position at \( t = 1 \).
EXAMPLE: A particle moves along a line so that its velocity at time $t$ is $v(t) = t^2 + 2t - 15$ (measured in miles per hour).

(a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

(b) Find the distance traveled during the time period $1 \leq t \leq 4$.

$$\int_1^4 v(t) \, dt = -9$$

Position at $t = 4$ is 9 units to the left of position at $t = 1$.

(b) $\int_1^4 |v(t)| \, dt$

It'll turn out that the distance traveled is $> 9$.

$v(t) = (t - 3)(t + 5)$

§8.1
EXAMPLE: A particle moves along a line so that its velocity at time $t$ is $v(t) = t^2 + 2t - 15$ (measured in miles per hour).

(a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

(b) Find the distance traveled during the time period $1 \leq t \leq 4$.

\[
\begin{align*}
\text{(a)} & \quad \int_1^4 v(t) \, dt = -9 \\
& \quad \text{Position at } t = 4 \text{ is 9 units to the left of position at } t = 1.
\end{align*}
\]

\[
\begin{align*}
\text{(b)} & \quad \int_1^4 |v(t)| \, dt = \left[ \int_1^3 |v(t)| \, dt \right] + \left[ \int_3^4 |v(t)| \, dt \right] \\
& \quad = \left[ \int_1^3 -(v(t)) \, dt \right] + \left[ \int_3^4 (v(t)) \, dt \right] \\
& \quad = \left[ \int_1^3 -(t^2 + 2t - 15) \, dt \right] + \left[ \int_3^4 (t^2 + 2t - 15) \, dt \right]
\end{align*}
\]

$v(t) = (t - 3)(t + 5)$

\[
\begin{array}{c|c|c|c|c|c}
& \text{pos} & 0 & \text{neg} & 0 & \text{pos} \\
\hline
-5 & & & & & \\
1 & & & & & \\
3 & & & & & \\
4 & & & & & \\
\end{array}
\]
EXAMPLE: A particle moves along a line so that its velocity at time \( t \) is \( v(t) = t^2 + 2t - 15 \) (measured in miles per hour).

(a) Find the displacement of the particle during the time period \( 1 \leq t \leq 4 \).

(b) Find the distance traveled during the time period \( 1 \leq t \leq 4 \).

(a) \[
\int_{1}^{4} v(t) \, dt = -9 \\
\text{Position at } t = 4 \text{ is 9 units to the left of position at } t = 1.
\]

(b) \[
\int_{1}^{4} |v(t)| \, dt = \left[ \int_{1}^{3} |v(t)| \, dt \right] + \left[ \int_{3}^{4} |v(t)| \, dt \right] \\
= \left[ \int_{1}^{3} -(v(t)) \, dt \right] + \left[ \int_{3}^{4} (v(t)) \, dt \right] \\
= \left[ \int_{1}^{3} -(t^2 + 2t - 15) \, dt \right] + \left[ \int_{3}^{4} (t^2 + 2t - 15) \, dt \right] \\
= \left[ \frac{t^3}{3} + t^2 - 15t \right]_{t:=1}^{t:=3} + \left[ \frac{t^3}{3} + t^2 - 15t \right]_{t:=3}^{t:=4}
\]
EXAMPLE: A particle moves along a line so that its velocity at time \( t \) is \( v(t) = t^2 + 2t - 15 \) (measured in miles per hour).

(a) Find the displacement of the particle during the time period \( 1 \leq t \leq 4 \).

(b) Find the distance traveled during the time period \( 1 \leq t \leq 4 \).

\[
\int_1^4 v(t) \, dt = -9
\]

Position at \( t = 4 \) is 9 units to the left of position at \( t = 1 \).

\[
\int_1^4 |v(t)| \, dt = \left[- \left(\frac{t^3}{3} + t^2 - 15t\right)\right]_{t: \to 1}^{t: \to 3} + \left[\frac{t^3}{3} + t^2 - 15t\right]_{t: \to 3}^{t: \to 4}
\]

\[
= - \left[ \left(\frac{3^3}{3} + 3^2 - 15 \cdot 3\right) - \left(\frac{1^3}{3} + 1^2 - 15 \cdot 1\right)\right]
\]

\[
+ \left[ \left(\frac{4^3}{3} + 4^2 - 15 \cdot 4\right) - \left(\frac{3^3}{3} + 3^2 - 15 \cdot 3\right)\right]
\]

\[
= - \left[ \frac{3^3}{3} + t^2 - 15t\right]_{t: \to 1}^{t: \to 3} + \left[\frac{3^3}{3} + t^2 - 15t\right]_{t: \to 3}^{t: \to 3}
\]

§8.1
EXAMPLE: A particle moves along a line so that its velocity at time \( t \) is \( v(t) = t^2 + 2t - 15 \) (measured in miles per hour).

(a) Find the displacement of the particle during the time period \( 1 \leq t \leq 4 \).

(b) Find the distance traveled during the time period \( 1 \leq t \leq 4 \).

(a) \[ \int_{1}^{4} v(t) \, dt = -9 \] Position at \( t = 4 \) is 9 units to the left of position at \( t = 1 \).

(b) \[ \int_{1}^{4} |v(t)| \, dt = - \left[ \frac{t^3}{3} + t^2 - 15t \right]_{t \rightarrow 1}^{t \rightarrow 3} + \left[ \frac{t^3}{3} + t^2 - 15t \right]_{t \rightarrow 3}^{t \rightarrow 4} \]

\[ = - \left[ \left( \frac{3^3}{3} + 3^2 - 15 \cdot 3 \right) - \left( \frac{1^3}{3} + 1^2 - 15 \cdot 1 \right) \right] + \left[ \left( \frac{4^3}{3} + 4^2 - 15 \cdot 4 \right) - \left( \frac{3^3}{3} + 3^2 - 15 \cdot 3 \right) \right] \]

\[ = - \left[ -\frac{40}{3} \right] + \left[ \frac{13}{3} \right] = \frac{53}{3} \approx 17.667 \] 

SKILL: compute displacement and distance traveled.

§8.1
EXAMPLE:

(a) Compute \( \int_{-3}^{3} -\sqrt{9 - x^2} \, dx \).
(b) Compute \( \int_{-3}^{3} -\sqrt{9 - x^2} \, dx \).

Area = \( \frac{9\pi}{2} \)

\[ y = \sqrt{9 - x^2} \]

\[ \int_{-3}^{3} \sqrt{9 - x^2} \, dx = \frac{9\pi}{2} \]

\[ y = -\sqrt{9 - x^2} \]

\[ \int_{-3}^{3} -\sqrt{9 - x^2} \, dx = -\frac{9\pi}{2} \]

§7.3
EXAMPLE:

(a) Compute \( \int_{-3}^{3} -\sqrt{9 - x^2} \, dx \).

(b) Compute \( \int_{3}^{-3} -\sqrt{9 - x^2} \, dx \).

(a) \( \int_{-3}^{3} -\sqrt{9 - x^2} \, dx = -\frac{9\pi}{2} \)

\[ \int_{-3}^{3} -\sqrt{9 - x^2} \, dx = -\frac{9\pi}{2} \]
EXAMPLE:

(a) Compute \( \int_{-3}^{3} -\sqrt{9-x^2} \, dx \).

(b) Compute \( \int_{3}^{-3} -\sqrt{9-x^2} \, dx \).

\[
(a) \int_{-3}^{3} -\sqrt{9-x^2} \, dx = -9\pi/2
\]

**SKILL**

Properties of integration

\[
(b) \int_{3}^{-3} -\sqrt{9-x^2} \, dx = - \int_{-3}^{3} -\sqrt{9-x^2} \, dx = 9\pi/2
\]

\[
\int_{b}^{a} f(x) \, dx := - \int_{a}^{b} f(x) \, dx, \quad \text{if} \ a < b
\]
SKILL
Properties of integration
Whitman problems
§7.3, p. 154, #1-6

STOP