CALCULUS
The Integral Mean Value Theorem
Calculate the mean (or average) of 3, 5, 9, 7.

\[ \frac{3 + 5 + 9 + 7}{4} = 6 \]

\[ 6 + 6 + 6 + 6 = 24 \]

max

mean

min

Total area = 24

mean = 6

Total area = 24

SAME AREA!!

\[ \text{§9.4} \]
Let \( f(x) = x^2 \). Calculate the mean (or average) of \( f \) on \([1, 3]\).

\[
\int_{1}^{3} f(x) \, dx = \frac{3^3}{3} - \frac{1^3}{3} = \frac{26}{3}
\]

\[
\left[ \frac{1}{3-1} \right] \left[ \frac{26}{3} \right] = \frac{13}{3}
\]

\[
\int_{1}^{3} \frac{13}{3} \, dx = \frac{26}{3}
\]
$y = f(x)$
DEFINITION: The **average (or mean)** of $f$ on $[a, b]$ is

$$
\int_{a}^{b} f(x) \, dx := \frac{1}{b - a} \int_{a}^{b} f(x) \, dx.
$$

$v = \left(\text{mean value}\right)$ \Rightarrow \int_{a}^{b} f(x) \, dx = \int_{a}^{b} v \, dx = v(b - a)$

$$
\frac{1}{b - a} \int_{a}^{b} f(x) \, dx = v
$$
DEFINITION: The average (or mean) of $f$ on $[a, b]$ is

$$\int_{a}^{b} f(x) \, dx := \frac{1}{b - a} \int_{a}^{b} f(x) \, dx.$$
**DEFINITION:** The average (or mean) of \( f \) on \([a, b]\) is

\[
\int_a^b f(x) \, dx := \frac{1}{b-a} \int_a^b f(x) \, dx.
\]

**INTEGRAL MEAN VALUE THEOREM**

If \( f \) is continuous on \([a, b]\), then there exists \( c \in (a, b) \) s.t.

\[
f(c) = \int_a^b f(x) \, dx = \frac{1}{b-a} \int_a^b f(x) \, dx.
\]

"the mean value is attained"
DEFINITION: The average (or mean) of $f$ on $[a, b]$ is

$$
\int_{a}^{b} f(x) \, dx := \frac{1}{b - a} \int_{a}^{b} f(x) \, dx.
$$

$\int_{a}^{b}$ is linear, but not multiplicative.

$$
\int_{a}^{c} \neq \int_{a}^{b} + \int_{b}^{c}
$$

$\int_{a}^{b} f(t) \, dt := \frac{1}{b - a} \int_{a}^{b} f(t) \, dt$

$\int_{a}^{b} f(s) \, ds := \frac{1}{b - a} \int_{a}^{b} f(s) \, ds$

$\int_{a}^{b} f := \frac{1}{b - a} \int_{a}^{b} f$

INTEGRAL MEAN VALUE THEOREM

If $f$ is continuous on $[a, b]$, then there exists $c \in (a, b)$ s.t.

$$
f(c) = \int_{a}^{b} f(x) \, dx = \frac{1}{b - a} \int_{a}^{b} f(x) \, dx.
$$
DEFINITION: The average (or mean) of $f$ on $[a, b]$ is

$$\int_a^b f(x) \, dx := \frac{1}{b-a} \int_a^b f(x) \, dx.$$ 

$\int_a^b$ is linear, but not multiplicative.

$$\int_a^c \neq \int_a^b + \int_b^c$$

etc., etc., etc.

$f \rightarrow v$ and $x \rightarrow t$

$$\int_a^b f$$

INTTEGRAL MEAN VALUE THEOREM

If $v$ is continuous on $[a, b]$, then there exists $c \in (a, b)$ s.t.

$$v(c) = \int_a^b v(t) \, dt = \frac{1}{b-a} \int_a^b v(t) \, dt.$$
INTERPRETATION VIA MOTION ON A LINE

Think of \( v \) as the velocity of a particle traveling on a line. 

\[
\int_{a}^{b} v(t) \, dt \text{ is the displacement from time } a \text{ to time } b.
\]

\[
\int_{a}^{b} v(t) \, dt \text{ is the average velocity from time } a \text{ to time } b.
\]

The integral MVT says that average velocity is \( ATTAINED \) at some time \( c \).

On my trip to Chicago, \( a = 0, b = 8 \) and \( \int_{a}^{b} v(t) \, dt = 400 \),

so \( \int_{a}^{b} v(t) \, dt = \frac{400}{8 - 0} = 50 \) was my avg velocity.

The \( \int \) MVT says: I \( ATTAINED \) that velocity at some time(s).

INTEGRAL MEAN VALUE THEOREM

If \( v \) is continuous on \([a, b]\), then there exists \( c \in (a, b) \) s.t.

\[
v(c) = \frac{1}{b - a} \int_{a}^{b} v(t) \, dt.
\]
The jMVT says: I ATTAINED that velocity at some time(s).

Expect: Every avg. velocity is an instantaneous velocity. Expect: Every sec. slope is a tangent slope.

Average velocity is 50 mph from 0 hrs to 8 hrs.

§6.5
The MVT says: I ATTAINED that velocity at some time(s).

Switch to: Graph of velocity ...

Expect: Every avg. velocity is an instantaneous velocity.
Expect: Every sec. slope is a tangent slope.

Average velocity is 50 mph from 0 hrs to 8 hrs.

instantaneous velocity is 50 mph at some time (45 min) (4 hrs 19 min 30 sec) (6 hrs 5 sec)
The MVT says: I ATTAINED that velocity at some time(s).

Switch to: Graph of velocity ...
EXAMPLE: Find the average value of the function \(2 + x^3\) on the interval \([-1, 5]\).

Solution: 
\[
\int_{-1}^{5} (2 + x^3) \, dx = \frac{1}{5 - (-1)} \left[ \int_{-1}^{5} (2 + x^3) \, dx \right]
\]
\[
= \frac{1}{6} \left[ 2x + \frac{x^4}{4} \right]_{x \to 5}^{x \to -1}
\]
\[
= \frac{1}{6} \left[ \left( 10 + \frac{5^4}{4} \right) - \left( -2 + \frac{(-1)^4}{4} \right) \right]
\]
\[
= \frac{1}{6} \left[ 12 + \frac{625}{4} \right] = \frac{1}{6} \left[ 12 + \frac{624}{4} \right] = \frac{1}{6} [12 + 156]
\]
\[
= \frac{1}{6} [168] = 28
\]
EXAMPLE: Find a number $c \in (-1, 5)$ s.t. 

$$\int_{-1}^{5} (2 + x^3) \, dx = 2 + c^3.$$ 

Solution: $\int_{-1}^{5} (2 + x^3) \, dx$ 

Solution: $\int_{-1}^{5} (2 + x^3) \, dx = 28$
EXAMPLE: Find a number \( c \in (-1, 5) \) s.t.

\[
\int_{-1}^{5} (2 + x^3) \, dx = 2 + c^3.
\]

Solution: \( \int_{-1}^{5} (2 + x^3) \, dx = 28 \)

\[28 = 2 + c^3\]

\[26 = c^3\]

\[c = \sqrt[3]{26}\]

SKILL where avg attained
EXAMPLE: Find a number \( c \in (-6, 6) \) s.t.

\[
\int_{-6}^{6} (4 + x^2) \, dx = 4 + c^2.
\]

Solution:

\[
\int_{-6}^{6} (4 + x^2) \, dx = \left[ \int_{-6}^{6} 4 \, dx \right] + \left[ \int_{-6}^{6} x^2 \, dx \right] \\
= 4 + \left[ \frac{1}{12} \int_{-6}^{6} x^2 \, dx \right] \\
= 4 + \left[ \frac{1}{12} \left[ \frac{x^3}{3} \right]_{x=-6}^{x=6} \right] \\
= 4 + \left[ \frac{1}{12} \left( \frac{6^3 - (-6)^3}{3} \right) \right] \\
= 16
\]

\( \int_{a}^{b} \) is linear, BUT NOT multiplicative.
EXAMPLE: Find a number $c \in (-6, 6)$ s.t.

$$\int_{-6}^{6} (4 + x^2) \, dx = 4 + c^2.$$ 

Solution: \[\int_{-6}^{6} (4 + x^2) \, dx = 16\]

\[16 = 4 + c^2\]
EXAMPLE: Find a number $c \in (-6, 6)$ s.t.

$$\int_{-6}^{6} (4 + x^2) \, dx = 4 + c^2.$$  

Solution:  

$$\int_{-6}^{6} (4 + x^2) \, dx = 16$$  

$$16 = 4 + c^2$$  

$$12 = c^2$$  

$$c = \pm \sqrt{12}$$  

$$= \pm 2\sqrt{3}$$

**SKILL**

where avg attained
EXAMPLE: Find a number \( c \in (0, 6) \) s.t.

\[
\int_0^6 (4 + x^2) \, dx = 4 + c^2.
\]

Solution: \[
\int_0^6 (4 + x^2) \, dx = 16
\]

\[
16 = 4 + c^2
\]

\[
12 = c^2
\]

\[
c = \pm \sqrt{12} = \pm 2\sqrt{3}
\]

\( c \in (0, 6) \)

\[
c = 2\sqrt{3}
\]

SKILL
where avg attained

\[\text{§9.4}\]
EXAMPLE: Find the average value of the function $f(x) = \sin(6x)$ on $[-2, 2]$.

Odd function

Solution: $\int_{-2}^{2} \sin(6x) \, dx = \left[ \frac{1}{2 - (-2)} \right] \left[ \int_{-2}^{2} \sin(6x) \, dx \right]$

$= \left[ \frac{1}{4} \right] \left[ -\frac{\cos(6x)}{6} \right]_{x:-2}^{x:2}$

LINEARITY OF $\int_{x:a}^{x:b} \cdot \, dx$

$= \left[ \frac{1}{4} \right] \left[ -\frac{\cos(12)}{6} - \frac{\cos(-12)}{6} \right]$  

$= 0$  

SKILL

find avg value
EXAMPLE: Find the average value of the function \( f(x) = \cos(6x) \) on \([-2, 2]\).

Solution: \( \int_{-2}^{2} \cos(6x) \, dx = \left[ \frac{1}{2 - (-2)} \right] \left[ \int_{-2}^{2} \cos(6x) \, dx \right] \)

\[ = \left[ \frac{1}{4} \right] \left[ \left. \frac{\sin(6x)}{6} \right|_{2}^{-2} \right] \]

\[ = \left[ \frac{1}{4} \right] \left[ \frac{\sin(12)}{6} + \frac{\sin(-12)}{6} \right] \]

\[ = \left[ \frac{1}{4} \right] \left[ \frac{2\sin(12)}{6} \right] \]

\[ = \frac{\sin(12)}{12} \approx -0.0447 \]

\( \text{SKILL} \) find avg value

\( \text{LINEARITY OF } \int_{a}^{b} f(x) \, dx \)
EXAMPLE: Find the average value of the function \( f(x) = \cos(6x) \) on \([0, \pi]\).

Exercise: Graph \( y = \cos(6x) \) on \([0, \pi]\).

Hint: First, graph \( y = \cos(x) \) on \([0, 6\pi]\).

Solution: \[
\int_{0}^{\pi} \cos(6x) \, dx = \left[ \frac{1}{\pi - 0} \right] \left[ \int_{0}^{\pi} \cos(6x) \, dx \right]
\]

\[
= \left[ \frac{1}{\pi} \right] \left[ \frac{\sin(6x)}{6} \right]_{x: \rightarrow \pi}^{x: \rightarrow 0}
\]

\[
= \left[ \frac{1}{\pi} \right] \left[ \frac{\sin(6\pi) - \sin(0)}{6} \right]
\]

\[
= \left[ \frac{1}{\pi} \right] \left[ \frac{0 - 0}{6} \right] = 0 \quad \text{FINISH FOR find avg value}
\]
EXAMPLE: Find the average value of the function \( f(x) = \cos(6x) \) on \([0, \frac{\pi}{12}]\).

Solution: \[
\int_0^{\pi/12} \cos(6x) \, dx = \left[ \frac{1}{(\pi/12) - 0} \right] \left[ \int_0^{\pi/12} \cos(6x) \, dx \right]
\]

\[
= \left[ \frac{12}{\pi} \right] \left[ \sin(6x) \right]_{x \to 0}^{x \to \pi/12} \left[ \frac{6}{6} \right]_{x \to 0}^{x \to \pi/12}
\]

\[
= \left[ \frac{12}{\pi} \right] \left[ \frac{\sin(\pi/2)}{6} - \frac{\sin(0)}{6} \right]
\]

\[
= \left[ \frac{12}{\pi} \right] \left[ \frac{1 - 0}{6} \right] = \frac{2}{\pi}
\]

\(\text{SKILL}\) find avg value
EXAMPLE: Find the average value of the function
\[ f(\theta) = \sec^2(\theta/4) \] on \([0, \pi]\).

Solution:
\[
\int_0^\pi \sec^2\left(\frac{\theta}{4}\right) \, d\theta = \left[ \frac{1}{\pi} \right] \left[ \int_0^\pi \sec^2\left(\frac{\theta}{4}\right) \, d\theta \right]
\]

= \left[ \frac{1}{\pi} \right] \left[ \tan\left(\frac{\theta}{4}\right) \right]_{\theta: \rightarrow \pi}^{\theta: \rightarrow 0}

= \left[ \frac{1}{\pi} \right] \left[ \frac{\tan(\pi/4)}{1/4} \right]

= \left[ \frac{1}{\pi} \right] \left[ \frac{[\tan(\pi/4)] - [\tan(0)]}{1/4} \right]

= \left[ \frac{1}{\pi} \right] \left[ \frac{[1] - [0]}{1/4} \right] = \frac{4}{\pi}

{\text{SKILL: find avg value}}
EXAMPLE: Find the average value of the function
\[ h(w) = (5 - 2w)^{-1} \text{ on } [-2, 2]. \]

Solution:
\[
\int_{-2}^{2} \left[ \frac{1}{5 - 2w} \right] \, dw = \left[ \frac{1}{2 - (-2)} \right] \left[ \int_{-2}^{2} \left[ \frac{1}{5 - 2w} \right] \, dw \right]
\]
\[
= \left[ \frac{1}{4} \right] \left[ \ln(|5 - 2w|) \right]_{w: \to 2}^{w: \to -2}
\]
\[
= \left[ \frac{1}{4} \right] \left[ \ln(|5 - 2w|) \right]_{w: \to 2}^{w: \to -2}
\]
\[
= \left[ \frac{1}{4} \right] \left[ \frac{\ln(1)}{2} + \frac{\ln(9)}{2} \right]
\]
\[
= \left[ \frac{1}{4} \right] \left[ \ln(9) \right] = \frac{\ln(9)}{8} \approx 0.2747
\]

\[\text{SKILL} \quad \text{find avg value}\]
EXAMPLE: Let \( f(x) = \sqrt[3]{x} \).

(a) Find \( \int_0^8 f(x) \, dx \).

(b) Find \( c \in (0, 8) \) such that \( \int_0^8 f(x) \, dx = f(c) \).

(c) Sketch the graph of \( f \) and a rectangle whose area is the same as the area under the graph of \( f \).

Sol’n: (a) \[
\frac{1}{8} \int_0^8 x^{1/3} \, dx = \frac{1}{8} \left[ \frac{x^{4/3}}{4/3} \right]_{x: \to 0}^{x: \to 8} = \frac{1}{8} \left[ \frac{8^{4/3}}{4/3} \right] = \frac{1}{8} \left[ \frac{2^4}{4/3} \right] = \frac{3}{2}
\]

(b) \( 3/2 = c^{1/3} \implies c = \frac{27}{8} \)

\[\text{SKILL} \quad \text{find avg value} \quad \text{where avg attained} \quad \text{show avg rectangle}\]
EXAMPLE: Let \( f(x) = \frac{2x}{(1 + x^2)^2} \).

(a) Find \( \int_0^2 f(x) \, dx \).

(b) Find \( c \in (0, 2) \) such that \( \int_0^2 f(x) \, dx = f(c) \).

(c) Sketch the graph of \( f \) and a rectangle whose area is the same as the area under the graph of \( f \).

Sol’n: (a) \[
\frac{1}{2} \int_0^2 \frac{2x}{(1 + x^2)^2} \, dx = \frac{1}{2} \left[ \frac{(1 + x^2)^{-1}}{-1} \right]_x=0 \to_2 = \frac{2}{5} = 0.4
\]

(b) \[
\frac{2}{5} = \frac{2c}{(1 + c^2)^2} \Rightarrow c \in \{0.220, 1.207\}
\]

online root finder
SKILL
average of a function
Whitman problems
§9.4, p. 195, #1-6

STOP