CALCULUS
The Fundamental Theorems of Calculus, proofs

THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.4

IOU: Rigorous pf

If f is contin. on $[a,b]$, then $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = [f(t)]_{t:\to x} = f(x)$, for $x\in(a,b)$.

Pf: $\forall\Phi$, $\forall x$, $\displaystyle\frac{d}{dx}[\Phi(x)] = \lim_{h\to 0}\frac{1}{h}\,[\Phi(x)]_{x:\to x}^{x:\to x+h}$

$$\frac{d}{dx}\int_a^x f(t)\,dt = \lim_{h\to 0}\frac{1}{h}\left[\int_a^x f(t)\,dt\right]_{x:\to x}^{x:\to x+h}$$

$$= \left[\int_a^{x+h} f(t)\,dt\right] - \left[\int_a^x f(t)\,dt\right]$$

$$= \int_x^{x+h} f(t)\,dt$$

COCYCLE IDENTITY

$$\left[\int_a^x f(t)\,dt\right] + \left[\int_x^{x+h} f(t)\,dt\right] = \int_a^{x+h} f(t)\,dt$$

§7.2

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If f is contin. on $[a,b]$, then $\frac{d}{dx}\int_a^x f(t)\,dt = [f(t)]_{t:\to x} = f(x)$, for $x \in (a,b)$.

Pf: Want: $\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt \;=\; f(x)$

$$\frac{d}{dx}\int_a^x f(t)\,dt \;=\; \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt$$

$$\int_x^{x+h} f(t)\,dt$$

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Pf:

Want: $$\lim_{h\to 0} \underbrace{\frac{1}{h}\int_x^{x+h} f(t)\,dt}_{\substack{\Vert\, h>0}} = f(x)$$

$$⨍_x^{x+h} f(t)\,dt$$

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Pf:

Want: $$\underbrace{\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt}_{\|\, h>0} = f(x)$$

Choose

$c_{x,h} \in (x, x+h)$ s.t. $f(c_{x,h}) = \fint_x^{x+h} f(t)\,dt$

$$x \le c_{x,h} \le \underbrace{x+h}_{\downarrow\, h\to 0^+ \,\to\, x}$$

$$\lim_{h\to 0^+} c_{x,h} = x$$

INTEGRAL MEAN VALUE THEOREM

If f is continuous on $[a, b]$, then there exists $c \in (a, b)$ s.t.

$$f(c) = \fint_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

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If f is contin. on $[a,b]$, then $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = [f(t)]_{t:\to x} = f(x)$, for $x \in (a,b)$.

Pf: Want: $\displaystyle\lim_{h\to 0} \underbrace{\frac{1}{h}\int_x^{x+h} f(t)\,dt}_{\|\, h>0} = f(x)$

Choose

$c_{x,h} \in (x, x+h)$ s.t. $\displaystyle f(c_{x,h}) = ⨍_x^{x+h} f(t)\,dt$

$$\lim_{h\to 0+} c_{x,h} = x$$

$$\lim_{h\to 0+} \frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0+} f(c_{x,h}) = f\left(\lim_{h\to 0+} c_{x,h}\right) = f(x)$$

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cf. §7.2, p. 146 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.4

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If f is contin. on $[a,b]$, then $\frac{d}{dx}\int_a^x f(t)\,dt = [f(t)]_{t:\to x} = f(x)$, for $x \in (a,b)$.

Pf: Want: $\lim\limits_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$ QED

$b_{x,h} \in (x+h, x)$

$$\frac{1}{-h}\int_{x+h}^{x} f(t)\,dt \overset{h<0}{=} ⨍_{x+h}^{x} f(t)\,dt = f(b_{x,h})$$

$||$

$$\lim_{h\to 0-} \frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0^-} f(b_{x,h}) = f\left(\lim_{h\to 0^-} b_{x,h}\right) = f(x)$$

$$\lim_{h\to 0+} \frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0+} f(c_{x,h}) = f\left(\lim_{h\to 0+} c_{x,h}\right) = f(x)$$

INTEGRAL MEAN VALUE THEOREM

If f is continuous on $[a,b]$, then there exists $c \in (a,b)$ s.t.

$$f(c) = ⨍_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

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If f is contin. on $[a,b]$, then $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = [f(t)]_{t:\to x} = f(x)$, for $x \in (a,b)$.

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let f be any function, contin. on $[a,b]$. IOU: Rigorous pf

Let F be an antiderivative of f on $[a,b]$.

Then $\displaystyle\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = (F(b)) - (F(a))$.

Pf: $x \leftrightarrow t$

$$\frac{d}{dt}\int_a^t f(x)\,dx = f(t) = \frac{d}{dt}[F(t)]$$

cf. §7.2, p. 146 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.4

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Then $\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = (F(b)) - (F(a))$.

Pf:

$$\frac{d}{dt}\int_a^t f(x)\,dx = \frac{d}{dt}[F = \frac{d}{dt}[F(t)]$$

MVT corollary:

$$\int_a^t f(x)\,dx = (F(t)) + C$$

cf. §7.2, p. 146 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.4

If f is contin. on $[a,b]$, then $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = [f(t)]_{t:\to x} = f(x)$, for $x \in (a,b)$.

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Then $$\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = (F(b)) - (F(a)).$$

Pf:
$$\frac{d}{dt}\int_a^t f(x)\,dx = \frac{d}{dt}[F(t)]$$

$t :\longrightarrow a$ $$\int_a^t f(x)\,dx = (F(t)) + C = (F(t)) - (F(a))$$

$$0 = \int_a^a f(x)\,dx = (F(a)) + C$$

$$-(F(a)) = C$$

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Let F be an antiderivative of f on $[a,b]$.

Then $\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = (F(b)) - (F(a))$.

Pf:

$$\frac{d}{dt}\int_a^t f(x)\,dx = \frac{d}{dt}[F(t)]$$

$t :\longrightarrow b$

$$\int_a^t f(x)\,dx = (F(t)) - (F = (F(t)) - (F(a))$$

$$\int_a^b f(x)\,dx = (F(b)) - (F(a))$$

cf. §7.2, p. 146 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.4

If f is contin. on $[a,b]$, then $\frac{d}{dx}\int_a^x f(t)\,dt = [f(t)]_{t:\to x} = f(x)$, for $x \in (a,b)$.

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Then $\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = (F(b)) - (F(a))$.

Pf:

$$\frac{d}{dt}\int_a^t f(x)\,dx = \frac{d}{dt}[F(t)]$$

$t :\longrightarrow b$ $$\int_a^t f(x)\,dx = (F(t)) - (F(a))$$

$$\int_a^b f(x)\,dx = (F(b)) - (F(a))$$

QED STOP