CALCULUS
Integration by substitution, problems

EXAMPLE: Evaluate $\int x^3 (6+x^4)^{100}\,dx$ via the substitution $u = 6 + x^4$.

$du = 4x^3\,dx$

$\frac{1}{4}\,du = x^3\,dx$

$$\int x^3 (6+x^4)^{100}\,dx = \frac{1}{4}\int u^{100}\,du = \frac{1}{4}\left[\frac{u^{101}}{101}\right] + C = \frac{1}{4}\left[\frac{(6+x^4)^{101}}{101}\right] + C$$

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SKILL
Integration by substitution

EXAMPLE: Evaluate $\displaystyle\int \frac{\sec^2(1/x^3)}{x^4}\,dx$ via the substitution $u = 1/x^3$.

$1/x^3 = x^{-3}$

$$\int \frac{\sec^2(1/x^3)}{x^4}\,dx = \int \sec^2(u)\,\frac{du}{-3} = -\frac{\tan(u)}{3} + C = -\frac{\tan(1/x^3)}{3} + C$$

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$$du = -3x^{-4}\,dx = -3\left[\frac{dx}{x^4}\right]$$

$$\frac{du}{-3} = \frac{dx}{x^4}$$

SKILL
Integration by substitution

EXAMPLE: Find $\int \left[\sqrt{1+x^2}\right][x^7]\,dx$.

$$\int \left[\sqrt{\underbrace{1+x^2}_{u}}\right]\underbrace{[x^7]\,dx} = \frac{1}{2}\int\left[\sqrt{u}\right]\left[(u-1)^3\right]du$$

$u - 1 = x^2$

$(u-1)^3 = x^6$

$du = 2x\,dx$

LINEARITY OF $\int$

$$\left(\frac{1}{2}du = x\,dx\right) \quad \times \quad x^6$$

$$\frac{1}{2}(u-1)^3\,du = \frac{1}{2}x^6\,du = [x^7]\,dx$$

EXAMPLE: Find $\int \left[\sqrt{1+x^2}\right][x^7]\,dx$.

$$\int \left[\sqrt{\underbrace{1+x^2}_{u}}\right][x^7]\,dx = \frac{1}{2}\int \left[\sqrt{u}\right]\left[(u-1)^3\right]du$$

$$= \frac{1}{2}\int \left[\sqrt{u}\right]\left[1u^3 - 3u^2 + 3u - 1\right]du$$

$$= \frac{1}{2}\int \left[u^{1/2}\right]\left[u^{6/2} - 3u^{4/2} + 3u^{2/2} - 1\right]du$$

$$= \frac{1}{2}\int \left[u^{7/2} - 3u^{5/2} + 3u^{3/2} - u^{1/2}\right]du$$

$$= \frac{1}{2}\left[\frac{u^{9/2}}{9/2} - \frac{3u^{7/2}}{7/2} + \frac{3u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right] + C$$

1

1 1

1 2 1

1 3 3 1

EXAMPLE: Find $\int \left[\sqrt{1+x^2}\right]\left[x^7\right] dx$.

$$\int \left[\sqrt{\underbrace{1+x^2}_{u}}\right]\left[x^7\right] dx = \frac{1}{2}\int \left[\sqrt{u}\right]\left[(u-1)^3\right] du$$

$$= \frac{1}{2}\left[\frac{u^{9/2}}{9/2} - \frac{3u^{7/2}}{7/2} + \frac{3u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right] + C$$

$$= \frac{1}{2}\left[\frac{(1+x^2)^{9/2}}{9/2} - \frac{3(1+x^2)^{7/2}}{7/2}\right.$$

$$\left. + \frac{3(1+x^2)^{5/2}}{5/2} - \frac{(1+x^2)^{3/2}}{3/2}\right] + C$$

EXAMPLE: Find $\int \left[\sqrt{1+x^2}\right][x^7]\,dx$.

$$\int \left[\sqrt{\underbrace{1+x^2}_{u}}\right][x^7]\,dx = \frac{1}{2}\int \left[\sqrt{u}\right][(u-1)^3]\,du$$

$$= \frac{1}{2}\left[\frac{u^{9/2}}{9/2} - \frac{3u^{7/2}}{7/2} + \frac{3u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right] + C$$

$$= \frac{1}{2}\left[\frac{(1+x^2)^{9/2}}{9/2} - \frac{3(1+x^2)^{7/2}}{7/2} + \frac{3(1+x^2)^{5/2}}{5/2} - \frac{(1+x^2)^{3/2}}{3/2}\right] + C$$

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SKILL
Integration by substitution

EXAMPLE: Evaluate $\int x^2(\underbrace{x^3-7}_{u})^{100}\,dx$.

$$= \frac{1}{3}\int (x^3-7)^{100}\,(3x^2)\,dx$$

$du = 3x^2\,dx$

$$= \frac{1}{3}\int u^{100}\,du$$

$$= \frac{1}{3}\left[\frac{u^{101}}{101}\right] + C$$

$$= \frac{(x^3-7)^{101}}{303} + C \quad \blacksquare$$

SKILL
Integration by substitution

EXAMPLE: Evaluate $\int (e^x)(\cos(\underbrace{e^x}_{u}))\,dx$.

$$\int (\cos(e^x))\,(e^x)\,dx$$

$$du = e^x\,dx$$

$$\int (\cos(u))\,du$$

$$(\sin(u)) + C$$

$$(\sin(e^x)) + C \quad \blacksquare$$

SKILL
Integration by substitution

EXAMPLE: Evaluate $\int \frac{x^2}{x^3+4}\,dx$.

$x^3 + 4 = u$

$du = 3x^2\,dx$

$$= \frac{1}{3}\int \frac{3x^2\,dx}{x^3+4}$$

$$= \frac{1}{3}\int \frac{du}{u}$$

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$$\frac{\ln(|u|)}{3} + C$$

$$= \frac{\ln(|x^3+4|)}{3} + C \quad \blacksquare$$

SKILL
Integration by substitution

EXAMPLE: Evaluate $\int \left[e^{(e^t)}\right] [e^t]\, dt$.

$u = e^t$

$$\int e^u \, du$$

$$= e^u + C$$

$$= e^{e^t} + C \quad \blacksquare$$

$$du = e^t \, dt$$

SKILL
Integration by substitution

EXAMPLE: Evaluate $\displaystyle\int \frac{dt}{(\sin^2 t)\sqrt{4+\cot t}}$.

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$$\int \frac{\csc^2 t\,dt}{\sqrt{4+\cot t}} \qquad du = -\csc^2 t\,dt \qquad u := 4+\cot t$$

$$\|$$

$$\int \frac{-du}{\sqrt{u}}$$

LINEARITY OF INDEFINITE INTEGRATION

$$\|$$

$$-\int u^{-1/2}\,du = -\frac{u^{1/2}}{1/2} + C$$

$$\|$$

$$■ \quad -2\sqrt{4+\cot t} + C = -\frac{(4+\cot t)^{1/2}}{1/2} + C$$

SKILL

Integration by substitution

EXAMPLE: Evaluate $\int_0^7 x\sqrt{\boxed{2+3x^2}}\,dx$.

$\boxed{2+3x^2} = u$

$$= \frac{1}{6}\int_0^7 \left(\sqrt{2+3x^2}\right)(6x)\,dx$$

$$du = 6x\,dx$$

$$= \frac{1}{6}\int_{2+3\cdot 0^2}^{2+3\cdot 7^2} \sqrt{u}\,du$$

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Integration by substitution

$$= \frac{1}{6}\int_2^{149} u^{1/2}\,du$$

$$= \frac{1}{6}\left[\frac{u^{3/2}}{3/2}\right]_{u:\to 2}^{u:\to 149} = \frac{1}{6}\left[\frac{149^{3/2}-2^{3/2}}{3/2}\right] = \left[\frac{149^{3/2}-2^{3/2}}{9}\right]$$

EXAMPLE: Evaluate $\int_0^1 x^2 e^{-x^3}\, dx$.

$-x^3 = u$

$$= -\frac{1}{3}\int_0^1 \left(e^{-x^3}\right)(-3x^2)\, dx$$

$du = -3x^2\, dx$

$$= -\frac{1}{3}\int_{-0^3}^{-1^3} e^u\, du$$

$$= -\frac{1}{3}\int_0^{-1} e^u\, du$$

$$= -\frac{1}{3}\left[e^u\right]_{u:\to 0}^{u:\to -1} = -\frac{1}{3}\left[e^{-1} - e^0\right] = \frac{1}{3}\left[e^0 - e^{-1}\right]$$

$$= \frac{1}{3}\left[1 - \frac{1}{e}\right]$$

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Integration by substitution

EXAMPLE: Evaluate $\int_0^4 \frac{x}{\sqrt{17-3x}}\,dx.$

$17-3x = u$

$du = -3\,dx$

$u = 17 - 3x$

$3x = 17 - u$

$x = (17-u)/3$

$$= -\frac{1}{3}\int_0^4 \left(\frac{x}{\sqrt{17-3x}}\right)(-3)\,dx$$

$$= -\frac{1}{3}\int_{17-3\cdot 0}^{17-3\cdot 4} \left(\frac{(17-u)/3}{\sqrt{u}}\right) du$$

$$= -\frac{1}{3}\int_{17}^{5} \left(\frac{17-u}{3\sqrt{u}}\right) du$$

$$= -\frac{1}{9}\int_{17}^{5} \left(\frac{17-u}{u^{1/2}}\right) du$$

$$= -\frac{1}{9}\int_{17}^{5} \left(17u^{-1/2} - u^{1/2}\right) du$$

$$= -\frac{1}{9}\left[17\left(\frac{u^{1/2}}{1/2}\right) - \frac{u^{3/2}}{3/2}\right]_{u:\to 17}^{u:\to 5}$$

§8.1

EXAMPLE: Evaluate $\displaystyle\int_0^4 \frac{x}{\sqrt{17-3x}}\,dx.$

$du = -3\,dx$

$17 - 3x = u$

$$= \quad -\frac{1}{9}\left[17\left(\frac{u^{1/2}}{1/2}\right) - \frac{u^{3/2}}{3/2}\right]_{u:\to 17}^{u:\to 5}$$

$u = 17 - 3x$

$3x = 17 - u$

$x = (17 - u)/3$

LINEARITY OF $[\bullet]_{u:\to a}^{u:\to b}$

$$= \quad -\frac{1}{9}\left[17\left(\frac{5^{1/2} - 17^{1/2}}{1/2}\right) - \frac{5^{3/2} - 17^{3/2}}{3/2}\right]$$

$$-\frac{1}{9}\left[17\left(\frac{u^{1/2}}{1/2}\right) - \frac{u^{3/2}}{3/2}\right]_{u:\to 17}^{u:\to 5}$$

EXAMPLE: Evaluate $\displaystyle\int_0^4 \frac{x}{\sqrt{17-3x}}\,dx$.

$u = 17 - 3x$

$du = -3\,dx$

$$= -\frac{1}{9}\left[17\left(\frac{u^{1/2}}{1/2}\right) - \frac{u^{3/2}}{3/2}\right]_{u:\to 17}^{u:\to 5}$$

$u = 17 - 3x$

$3x = 17 - u$

$x = (17 - u)/3$

$$= -\frac{1}{9}\left[17\left(\frac{5^{1/2} - 17^{1/2}}{1/2}\right) - \frac{5^{3/2} - 17^{3/2}}{3/2}\right]$$

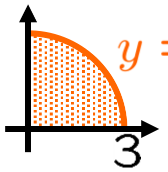
$$= -\frac{34}{9}\left(5^{1/2} - 17^{1/2}\right) + \frac{2}{27}\left(5^{3/2} - 17^{3/2}\right)$$

$$= \frac{34}{9}\left(17^{1/2} - 5^{1/2}\right) - \frac{2}{27}\left(17^{3/2} - 5^{3/2}\right) \blacksquare$$

SKILL

Integration by substitution

EXAMPLE: Write $\int_0^3 (7x+5)\sqrt{9-x^2}\,dx$ as a sum of two integrals. Evaluate one by substitution. Evaluate the other by interpreting it as an area.



$$\int_0^3 (7x+5)\sqrt{9-x^2}\,dx$$

$$= 7\left[\int_0^3 x\sqrt{9-x^2}\,dx\right] + 5\left[\int_0^3 \sqrt{9-x^2}\,dx\right]$$

$$= -\frac{7}{2}\left[\int_0^3 \left(\sqrt{9-x^2}\right)(-2x)\,dx\right] + 5\left[\frac{(\pi)(3^2)}{4}\right]$$

$u := 9 - x^2$
$du = -2x\,dx$

$$= -\frac{7}{2}\left[\int_{9-0^2}^{9-3^2} \sqrt{u}\,du\right] + 5\left[\frac{(\pi)(3^2)}{4}\right]$$

$$= -\frac{7}{2}\left[\int_9^0 u^{1/2}\,du\right] + \frac{45\pi}{4}$$

EXAMPLE: Write $\int_0^3 (7x+5)\sqrt{9-x^2}\,dx$ as a sum of two integrals. Evaluate one by substitution. Evaluate the other by interpreting it as an area.

$$\int_0^3 (7x+5)\sqrt{9-x^2}\,dx$$

$$u := 9 - x^2$$
$$du = -2x\,dx$$

$$= -\frac{7}{2}\left[\int_9^0 u^{1/2}\,du\right] + \frac{45\pi}{4}$$

$$= -\frac{7}{2}\left[\frac{u^{3/2}}{3/2}\right]_{u:\to 9}^{u:\to 0} + \frac{45\pi}{4}$$

$$u := 9 - x^2$$
$$du = -2x\,dx$$

$$= -\frac{7}{2}\left[\int_9^0 u^{1/2}\,du\right] + \frac{45\pi}{4}$$

EXAMPLE: Write $\int_0^3 (7x+5)\sqrt{9-x^2}\,dx$ as a sum of two integrals. Evaluate one by substitution. Evaluate the other by interpreting it as an area.

$$\int_0^3 (7x+5)\sqrt{9-x^2}\,dx$$

$u := 9 - x^2$
$du = -2x\,dx$

$$= -\frac{7}{2}\left[\int_9^0 u^{1/2}\,du\right] + \frac{45\pi}{4}$$

$$= -\frac{7}{2}\left[\frac{u^{3/2}}{3/2}\right]_{u:\to 9}^{u:\to 0} + \frac{45\pi}{4}$$

$$= -\frac{7}{2}\left[\frac{0^{3/2} - 9^{3/2}}{3/2}\right] + \frac{45\pi}{4}$$

SKILL
Integration by substitution

$$= -\frac{7}{2}\left[\frac{0 - 27}{3/2}\right] + \frac{45\pi}{4} = 63 + \frac{45\pi}{4} \quad \blacksquare$$

EXAMPLE: Suppose f is continuous and $\int_0^8 f(x)\,dx = 6$.

Find $\int_0^2 [x^2][f(x^3)]\,dx$.

$$\|$$

$$\frac{1}{3}\int_0^2 [f(x^3)][3x^2]\,dx$$

$u := x^3$

$du = 3x^2\,dx$

$$\|$$

$$\frac{1}{3}\int_{0^3}^{2^3} f(u)\,du$$

$$\|$$

$$\frac{1}{3}\int_0^8 f(u)\,du$$

$$\|$$

■ $2 = \frac{1}{3}[6]$

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Integration by substitution

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Integration by substitution

Whitman problems
§8.1, p. 160, #1-20

STOP