CALCULUS
Area between curves
REMARK:
Suppose $f(x) \geq g(x)$, for all $x \in [a, b]$.
Then the area of the region bounded by the curves $y = f(x)$ and $y = g(x)$ and by the lines $x = a$ and $x = b$
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Suppose \( f(x) \geq g(x) \), for all \( x \in [a, b] \).
Then the area of the region bounded by the curves \( y = f(x) \) and \( y = g(x) \) and by the lines \( x = a \) and \( x = b \) is equal to ??
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\[
\int_{a}^{b} f(x) \, dx
\]
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REMARK:
Suppose $f(x) \geq g(x)$, for all $x \in [a, b]$.
Then the area of the region bounded
by the curves $y = f(x)$ and $y = g(x)$
and by the lines $x = a$ and $x = b$
is equal to $\int_a^b f(x) \, dx - \int_a^b g(x) \, dx$
Suppose $f(x) \geq g(x)$, for all $x \in [a, b]$. Then the area of the region bounded by the curves $y = f(x)$ and $y = g(x)$ and by the lines $x = a$ and $x = b$ is equal to $\int_a^b [((f(x)) - (g(x))] \, dx$. What happens if we drop this hypothesis?
REMARK:
The area of the region bounded
by the curves $y = f(x)$ and $y = g(x)$
and by the lines $x = a$ and $x = b$
is equal to $\int_a^b \underbrace{[\max\{f(x), g(x)\}] - [\min\{f(x), g(x)\}]}_{|[f(x)] - [g(x)]|} \, dx.$
REMARK:

The area of the region bounded by the curves \( y = f(x) \) and \( y = g(x) \) and by the lines \( x = a \) and \( x = b \)
is equal to \( \int_{a}^{b} [f(x)] - [g(x)] \, dx \).
EXAMPLE: Find the area enclosed by the parabolas $y = x^2$ and $y = 4x - x^2$.

Solution:

$x^2 - (4x - x^2) = 2x^2 - 4x = 2x(x - 2)$

$2x(x - 2)$

pos 0 neg 0 pos

$x$

0 2

$\int_0^2 2x - 4x^2 \, dx = \int_0^2 -(2x^2 - 4x) \, dx$

$= - \int_0^2 2x^2 - 4x \, dx$

$= - \left[ \frac{x^3}{3} - 4 \frac{x^2}{2} \right]_{x: \to 2}$

$= - \left( \left[ \frac{16}{3} - 4 \frac{16}{2} \right] - [0] \right) = \frac{8}{3}$

SKILL: area between curves
EXAMPLE: Find the area enclosed by the curves \( y = x^5 + x^4 \) and \( y = 2x^4 + 2x^3 \).

Solution:

\[
(x^5 + x^4) - (2x^4 + 2x^3) = x^5 - x^4 - 2x^3 = x^3(x^2 - x - 2) = x^3(x - 2)(x + 1)
\]

\[
\int_{-1}^{2} \left( x^5 - x^4 - 2x^3 \right) \, dx
\]

COCYCLE IDENTITY

\[
= \left[ \int_{-1}^{0} \left( x^5 - x^4 - 2x^3 \right) \, dx \right] + \left[ \int_{0}^{2} \left( x^5 - x^4 - 2x^3 \right) \, dx \right]
\]

\[
= \left[ \int_{-1}^{0} \left( x^5 - x^4 - 2x^3 \right) \, dx \right] + \left[ \int_{0}^{2} \left( x^5 - x^4 - 2x^3 \right) \, dx \right]
\]

\[
= \left[ \int_{-1}^{0} \left( x^5 - x^4 - 2x^3 \right) \, dx \right] - \left[ \int_{0}^{2} \left( x^5 - x^4 - 2x^3 \right) \, dx \right]
\]
EXAMPLE: Find the area enclosed by the curves \( y = x^5 + x^4 \) and \( y = 2x^4 + 2x^3 \).

Solution:

\[
\int_{-1}^{2} \left| x^5 - x^4 - 2x^3 \right| \, dx
= \left[ \int_{-1}^{0} x^5 - x^4 - 2x^3 \, dx \right] - \left[ \int_{0}^{2} x^5 - x^4 - 2x^3 \, dx \right]
- \left[ \left. \frac{x^6}{6} - \frac{x^5}{5} - \frac{2x^4}{4} \right|_{x: \to 0}^{x: \to 2} \right]
- \left[ \left. \frac{x^6}{6} - \frac{x^5}{5} - \frac{2x^4}{4} \right|_{x: \to -1}^{x: \to 0} \right]
\]

\[
= \left[ \int_{0}^{2} x^5 - x^4 - 2x^3 \, dx \right] - \left[ \int_{-1}^{0} x^5 - x^4 - 2x^3 \, dx \right]
\]
EXAMPLE: Find the area enclosed by the curves \( y = x^5 + x^4 \) and \( y = 2x^4 + 2x^3 \).

Solution:

\[
\int_{-1}^{2} \left| x^5 - x^4 - 2x^3 \right| \, dx
\]

\[
= \left[ \int_{-1}^{0} x^5 - x^4 - 2x^3 \, dx \right] - \left[ \int_{0}^{2} x^5 - x^4 - 2x^3 \, dx \right]
\]

\[
= \left[ \frac{x^6}{6} - \frac{x^5}{5} - \frac{2x^4}{4} \right]_{x:-1}^{x:0} - \left[ \frac{x^6}{6} - \frac{x^5}{5} - \frac{2x^4}{4} \right]_{x:0}^{x:2}
\]

\[
= \left[ 0 - \left( \frac{(-1)^6}{6} - \frac{(-1)^5}{5} - \frac{2(-1)^4}{4} \right) \right] - \left[ \left( \frac{2^6}{6} - \frac{2^5}{5} - \frac{2^4}{4} \right) - 0 \right]
\]

\[
= \left[ \frac{1}{6} - \frac{1}{5} + \frac{1}{2} \right] - \left[ \frac{64}{6} - \frac{32}{5} - \frac{32}{4} \right]
\]

\[
= \left[ \frac{8}{60} \right] - \left[ -\frac{224}{60} \right] = \frac{58}{15}
\]

\[\text{SKILL} \quad \text{area between curves} \]
EXAMPLE: Find the area enclosed by the line \( y = \frac{1}{2}x - 1 \) and the parabola \( y^2 = x + 6 \).

not hard to solve for \( x \): \( x = y^2 - 6 \)

hard to solve for \( y \)
EXAMPLE: Find the area enclosed by the line \(y = \frac{1}{2}x - 1\) and the parabola \(y^2 = x + 6\).

Solution:

\[x = 2y + 2\]
\[x = y^2 - 6\]

\[(2y + 2) - (y^2 - 6) = -y^2 + 2y + 8 = -(y + 2)(y - 4)\]

\[-(y + 2)(y - 4)\]

\(y\)  \(\text{neg}\)  \(0\)  \(\text{pos}\)  \(0\)  \(\text{neg}\)

\(-2\) \(4\)

We’ll work this problem with expressions of \(y\) instead of expressions of \(x\).

Slightly uncommon, but doable . . .
EXAMPLE: Find the area enclosed by the line \( y = \frac{1}{2}x - 1 \) and the parabola \( y^2 = x + 6 \).

Solution:

\[
x = 2y + 2 \quad \quad \quad \quad x = y^2 - 6
\]

\[
(2y + 2) - (y^2 - 6) = -y^2 + 2y + 8 = -(y + 2)(y - 4)
\]

\[-(y + 2)(y - 4) \quad \text{neg} \quad 0 \quad \text{pos} \quad 0 \quad \text{neg}
\]

\[
\int_{-2}^{4} -y^2 + 2y + 8 \, dy = + \int_{-2}^{4} -y^2 + 2y + 8 \, dy
\]

\[
= \left[ -\frac{1}{3}y^3 + y^2 + 8y \right]_{y: \rightarrow -2}^{y: \rightarrow 4}
\]

\[
= \left[ -\frac{1}{3}(4^3) + 4^2 + 8 \cdot 4 \right] - \left[ -\frac{1}{3}(-2)^3 + (-2)^2 + 8(-2) \right]
\]

\[
= \left[ -\frac{64}{3} + 16 + 32 \right] - \left[ \frac{8}{3} + 4 - 16 \right] = 36
\]

SKILL: area between curves

You might choose to interchange \( x \) and \( y \)...
EXAMPLE: Find the area enclosed by the line $x = \frac{1}{2}y - 1$ and the parabola $x^2 = y + 6$.

Solution:

$y = 2x + 2$

$y = x^2 - 6$

$(2x + 2) - (x^2 - 6) = -x^2 + 2x + 8 = -(x + 2)(x - 4)$

$-(x + 2)(x - 4)$

<table>
<thead>
<tr>
<th>neg</th>
<th>0</th>
<th>pos</th>
<th>0</th>
<th>neg</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>-2</td>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\int_{-2}^{4} \left| -x^2 + 2x + 8 \right| \, dx &= + \int_{-2}^{4} -x^2 + 2x + 8 \, dx \\
&= \left[ -\frac{1}{3}x^3 + x^2 + 8x \right]_{-2}^{4} \\
&= \left[ -\frac{1}{3}4^3 + 4^2 + 8 \cdot 4 \right] - \left[ +\frac{1}{3}(+2)^3 + (+2)^2 + 8(-2) \right] \\
&= \left[ -\frac{64}{3} + 16 + 32 \right] - \left[ \frac{8}{3} + 4 - 16 \right] = 36 \quad \blacksquare
\end{align*}
\]

SKILL

§9.1 area between curves
EXAMPLE  Find the area enclosed in a circle of radius 7.

\[ y^2 = 7^2 - x^2 \quad y = \pm \sqrt{7^2 - x^2} \]
EXAMPLE Find the area enclosed in a circle of radius 7.

\[ y^2 = 7^2 - x^2 \quad y = \pm \sqrt{7^2 - x^2} \]

\[ x^2 + y^2 = 7^2 \]

\[ y = \sqrt{7^2 - x^2} \quad y = -\sqrt{7^2 - x^2} \]

\[ \int_{-7}^{7} \left( \sqrt{7^2 - x^2} \right) + \left( +\sqrt{7^2 - x^2} \right) \, dx \]

UNNEEDED \[ \int_{-7}^{7} 2\sqrt{7^2 - x^2} \, dx \]

UNNEEDED

LINEARITY OF DEFINITE INTEGRATION

\[ 2 \int_{-7}^{7} \sqrt{7^2 - x^2} \, dx \]
EXAMPLE Find the area enclosed in a circle of radius 7.

\[ 2 \int_{-7}^{7} \sqrt{7^2 - x^2} \, dx \]

\[ y = \sqrt{7^2 - x^2} \]
EXAMPLE  Find the area enclosed in a circle of radius 7.

\[ 2 \int_{-7}^{7} \sqrt{7^2 - x^2} \, dx = 4 \int_{0}^{7} \sqrt{7^2 - x^2} \, dx \]

\[ y = \sqrt{7^2 - x^2} \]

INTEGRATING SYMMETRIC FUNCTIONS

Suppose \( f \) is continuous on \([−a, a]\).

(a) If \( f \) is even, i.e., \( f(−x) = f(x) \),
then \( \int_{−a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx \).

(b) If \( f \) is odd, i.e., \( f(−x) = −(f(x)) \),
then \( \int_{−a}^{a} f(x) \, dx = 0 \).
EXAMPLE Find the area enclosed in a circle of radius 7.

\[ 2 \int_{-7}^{7} \sqrt{7^2 - x^2} \, dx = 4 \int_{0}^{7} \sqrt{7^2 - x^2} \, dx \]

\[ x^2 + y^2 = 7^2 \]

\[ x = 7 \sin \theta \]

\[ 0 \leq \theta \leq \pi/2 \]

\[ 0 \leq x \leq 7 \]
EXAMPLE Find the area enclosed in a circle of radius 7.

\[
2 \int_{-7}^{7} \sqrt{7^2 - x^2} \, dx = 4 \int_{0}^{7} \sqrt{7^2 - x^2} \, dx
\]

\[
= 4 \int_{0}^{\pi/2} \sqrt{7^2 - (7 \sin \theta)^2} \, [7 \cos \theta] \, d\theta
\]

\[0 \leq \theta \leq \pi/2\]
EXAMPLE  Find the area enclosed in a circle of radius 7.

\[
2 \int_{-7}^{7} \sqrt{7^2 - x^2} \, dx = 4 \int_{0}^{7} \sqrt{7^2 - x^2} \, dx
\]

\[
= 4 \int_{0}^{\pi/2} \sqrt{7^2 - (7 \sin \theta)^2} \, [7 \cos \theta] \, d\theta
\]

\[
= 4(7^2) \int_{0}^{\pi/2} \cos^2 \theta \, d\theta
\]

\[
0 \leq \theta \leq \pi/2 \quad \Rightarrow \quad \cos \theta \geq 0
\]

\[
\Rightarrow \quad 7 \cos \theta \geq 0
\]

\[
\Rightarrow \quad \sqrt{(7 \cos \theta)^2} = 7 \cos \theta
\]
EXAMPLE Find the area enclosed in a circle of radius 7.

\[ 2 \int_{-7}^{7} \sqrt{7^2 - x^2} \, dx = 4 \int_{0}^{7} \sqrt{7^2 - x^2} \, dx \]

\[ = 4 \int_{0}^{\pi/2} \sqrt{7^2 - (7 \sin \theta)^2} \, [7 \cos \theta] \, d\theta \]

\[ = 4 \int_{0}^{\pi/2} [7 \cos \theta] \, [7 \cos \theta] \, d\theta \]

\[ = 4(7^2) \int_{0}^{\pi/2} \cos^2 \theta \, d\theta \]

\[ = 4(7^2) \int_{0}^{\pi/2} \frac{1 + \cos(2\theta)}{2} \, d\theta \]

\[ \cos^2 \theta = \frac{1 + \cos(2\theta)}{2} \]

\[ 1 - \cos^2 \theta = \frac{1 + \cos(2\theta)}{2} \]

\[ \cos(2\theta) = \frac{1 - \cos^2 \theta}{\cos^2 \theta} \]

\[ \cos^2 \theta = 2 \cos^2 \theta - 1 \]

\[ \cos^2 \theta = \frac{1 + \cos(2\theta)}{2} \]
EXAMPLE  Find the area enclosed in a circle of radius 7.

\[ 2 \int_{-7}^{7} \sqrt{7^2 - x^2} \, dx = 4 \int_{0}^{7} \sqrt{7^2 - x^2} \, dx \]

\[ = 4 \int_{0}^{\pi/2} \sqrt{7^2 - (7 \sin \theta)^2} \, [7 \cos \theta] \, d\theta \]

\[ = 4 \int_{0}^{\pi/2} [7 \cos \theta] \, [7 \cos \theta] \, d\theta \]

\[ = 4(7^2) \int_{0}^{\pi/2} \cos^2 \theta \, d\theta \]

\[ = 4(7^2) \int_{0}^{\pi/2} \frac{1 + \cos(2\theta)}{2} \, d\theta \]

\[ = 2(7^2) \left[ \theta + \frac{\sin(2\theta)}{2} \right]_{\theta:0}^{\pi/2} \]

\[ = 2(7^2) \left[ \left( \frac{\pi}{2} + \frac{\sin \pi}{2} \right) - \left[ 0 + \frac{\sin 0}{2} \right] \right] \]
EXAMPLE  Find the area enclosed in a circle of radius 7.

\[ 2 \int_{-7}^{7} \sqrt{7^2 - x^2} \, dx = 2(7^2) \left[ \left[ \frac{\pi}{2} + \frac{\sin \pi}{2} \right] - \left[ 0 + \frac{\sin 0}{2} \right] \right] \]

\[ = 2(7^2) \left[ \left[ \frac{\pi}{2} + \frac{0}{2} \right] - \left[ 0 + \frac{0}{2} \right] \right] \]

\[ = 2(7^2) \left[ \frac{\pi}{2} \right] \]
EXAMPLE  Find the area enclosed in a circle of radius 7.

\[
2 \int_{-7}^{7} \sqrt{7^2 - x^2} \, dx = 2(7^2) \left[ \left( \frac{\pi}{2} + \frac{\sin \pi}{2} \right) - \left( 0 + \frac{\sin 0}{2} \right) \right]
\]

\[
= 2(7^2) \left[ \left( \frac{\pi}{2} + \frac{0}{2} \right) - \left( 0 + \frac{0}{2} \right) \right]
\]

\[
= 2(7^2) \left( \frac{\pi}{2} \right) = (\pi)(7^2)
\]

SKILL
Area by integration