CALCULUS
Area between curves, problems
EXAMPLE: The graph below shows the velocity curves for two trains, X and Y, that start side by side and move along parallel tracks.
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The area represents the distance train Y traveled at 40 seconds.
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<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
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<td></td>
<td></td>
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<td></td>
<td></td>
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<tr>
<td>$v_X - v_Y$</td>
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<td>4.1</td>
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Estimate the area using midpoints.

Estimate the area using midpoints.

Train X is ahead of train Y at 40 secs.
Let's choose FOUR subintervals
Mark midpoints.

Estimate the area using midpoints.

area \parallel distance
train X is ahead of
train Y at 40 secs

|$t|$ | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 \\
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<td>$v_X$</td>
<td>0</td>
<td>9.2</td>
<td>14.7</td>
<td>18.2</td>
<td>20.9</td>
<td>22.7</td>
<td>24.1</td>
<td>25.1</td>
<td>25.8</td>
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<tr>
<td>$v_Y$</td>
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<td>5.1</td>
<td>9.3</td>
<td>11.9</td>
<td>13.7</td>
<td>14.7</td>
<td>15.3</td>
<td>15.6</td>
<td>15.9</td>
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<tr>
<td>$v_X - v_Y$</td>
<td>0</td>
<td>4.1</td>
<td>5.4</td>
<td>6.3</td>
<td>7.2</td>
<td>8.0</td>
<td>8.8</td>
<td>9.5</td>
<td>9.9</td>
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Let’s choose FOUR subintervals Mark midpoints.

Estimate the area using midpoints.

area \parallel distance train X is ahead of train Y at 40 secs

£9.1
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Let’s choose FOUR subintervals:
Mark midpoints.

Estimate the area using midpoints.

Distance train $X$ is ahead of train $Y$ at 40 secs.
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Estimate the area using midpoints.

- Train X is ahead of train Y at 40 secs.
- Area: $279\times10$ meters
- Distance: $279\times10$ meters

Skill: Interpret & estimate area between curves.

§9.1
EXAMPLE: Find the shaded area, shown below.

\[-x^2 + 2x = x^2 - x \quad \Leftrightarrow \quad 0 = 2x^2 - 3x = 2x \left( x - \frac{3}{2} \right)\]
EXAMPLE: Find the shaded area, shown below.

\[-x^2 + 2x = x^2 - x \quad \Leftrightarrow \quad 0 = 2x^2 - 3x = 2x \left( x - \frac{3}{2} \right)\]
EXAMPLE: Find the shaded area, shown below.

\[ \int_{0}^{3/2} \left| (-x^2 + 2x) - (x^2 - x) \right| \, dx = \int_{0}^{3/2} (-2x^2 + 3x) \, dx \]

\[ = \left[ -2 \frac{x^3}{3} + 3 \frac{x^2}{2} \right]_{x: \to 0}^{3/2} = \left[ -2 \frac{(3/2)^3}{3} + 3 \frac{(3/2)^2}{2} \right] - [0] \]

\[ = -\frac{18}{8} + \frac{27}{8} = \frac{9}{8} \]
EXAMPLE: Sketch the region enclosed by 
\[ y = \sin x, \quad y = e^{x/9}, \quad x = \pi/2 \quad \text{and} \quad x = 2\pi. \]
Find the area of that region.

\[
\int_{\pi/2}^{2\pi} \left( e^{x/9} - \sin x \right) \, dx = \left[ \frac{e^{x/9}}{1/9} - (-\cos x) \right]_{x=\pi/2}^{x=2\pi} = \left[ \frac{e^{x/9}}{1/9} + \cos x \right]_{x=\pi/2}^{x=2\pi} \\
= \left[ \frac{e^{2\pi/9} - e^{\pi/18}}{1/9} \right] + \left[ (\cos(2\pi)) - (\cos(\pi/2)) \right] \\
= 9 \left[ e^{2\pi/9} - e^{\pi/18} \right] + [1 - 0] = 9 \left[ e^{2\pi/9} - e^{\pi/18} \right] + 1 \]
EXAMPLE: Sketch the region enclosed by 
\[ y = 2 - (\cos x), \quad y = \cos x, \quad x = 0 \quad \text{and} \quad x = 2\pi. \]
Find the area of that region.

\[
\int_{0}^{2\pi} [(2 - \cos x) - (\cos x)] \, dx = \int_{0}^{2\pi} [2 - 2 \cos x] \, dx
\]

\[ = [2x - 2 \sin x]_{x \to 2\pi}^{x \to 0} \]
\[ = 2[2\pi - 0] - 2[(\sin(2\pi)) - (\sin(0))] \]
\[ = 2[2\pi - 0] - 2[0 - 0] = 4\pi \]
EXAMPLE: Sketch the region enclosed by 
\[ y = 2|x| \quad \text{and} \quad y = x^2 - 3. \]
Find the area of that region.

\[
\int_{-3}^{3} \left| (2|x|) - (x^2 - 3) \right| \, dx
\]

\[
= 2 \int_{0}^{3} \left| (2|x|) - (x^2 - 3) \right| \, dx
\]

\[
= 2 \int_{0}^{3} (-x^2 + 2x + 3) \, dx
\]

\[
= 2 \left[ -\frac{x^3}{3} + x^2 + 3x \right]_{x: \to 3}^{x: \to 0}
\]

\[
= 2 \left[ -\frac{3^3}{3} + 3^2 + 3 \cdot 3 \right] - 2 \left[ -\frac{0^3}{3} + 0^2 + 3 \cdot 0 \right]
\]

\[
= 2 \left[ -\frac{27}{3} + 9 + 9 \right] - 2 \cdot 0
\]

\[
= 2 \left[ -9 + 9 + 9 \right] = 18
\]
EXAMPLE: Sketch the region enclosed by \( y = 3x \) and \( y = 4x^3 - x \). Find the area of that region.

Area = \( 2 \int_{0}^{1} 3x - (4x^3 - x) \, dx \)

= \( 2 \int_{0}^{1} 4x - 4x^3 \, dx \)

= \( 2 \left[ 2x^2 - x^4 \right]_{0}^{1} \)

= \( 2 [2 - 1] = 2 \)

**SKILL**

area between curves

§9.1
EXAMPLE: Use calculus to find the area of the triangle whose vertices are: 
(0, 4), (2, -1) and (5, 2).

**SKILL**
area of triangle from vertices

\[
\int_0^2 \left( \frac{21}{10} x \right) \, dx + \int_2^5 \left( -\frac{7}{5} x + 7 \right) \, dx
\]

\[
\int_0^2 \left( \frac{21}{10} x + 4 \right) - \left( -\frac{5}{2} x + 4 \right) \, dx
\]

\[
+ \int_2^5 \left( -\frac{2}{5} x + 4 \right) - (x - 3) \, dx
\]

§9.1
EXAMPLE: An irregular property has been surveyed and is shown below, with measurements made of some cross-sections. Estimate its area.

\[
\begin{align*}
157 \times 100 \\
+ \\
181 \times 100 \\
+ \\
57 \times 100 \\
+ \\
92 \times 100 \\
+ \\
178 \times 100 \\
+ \\
100 \times 100 \\
\end{align*}
\]

\[= 765 \times 100 = 76,500\]

**SKILL**
estimate area, given cross-sections
EXAMPLE: Part of the graph of $y^2 = x^2(x + 7)$ forms a loop. Find the area enclosed by that loop.
EXAMPLE: Part of the graph of \( y^2 = x^2(x + 7) \) forms a loop. Find the area enclosed by that loop.

\[
\int_{-7}^{0} \left| \sqrt{x^2(x + 7)} + \sqrt{x^2(x + 7)} \right| \, dx
\]
EXAMPLE: Part of the graph of \( y^2 = x^2(x + 7) \) forms a loop.

Find the area enclosed by that loop.

\[
\int_{-7}^{0} \left( \sqrt{x^2(x + 7)} \right) + \left( + \sqrt{x^2(x + 7)} \right) \, dx
\]

\[
= \color{red}2 \int_{-7}^{0} \sqrt{x^2(x + 7)} \, dx
\]

\[
= \sqrt{x^2} = -x, \quad \forall x < 0
\]

\[
= 2 \int_{-7}^{0} \sqrt{x^2} \sqrt{x + 7} \, dx
\]

\[
= 2 \int_{-7}^{0} (-x) \sqrt{x + 7} \, dx
\]

\[
= -2 \int_{-7}^{0} x \sqrt{x + 7} \, dx
\]

\[
= -2 \int_{-7+7}^{0+7} (u - 7) \sqrt{u} \, du
\]

\[
x = u - 7 \\
u := x + 7 \\
du = dx
\]

\[
= -2 \int_{0}^{7} (u - 7)u^{1/2} \, du
\]
EXAMPLE: Part of the graph of \( y^2 = x^2(x + 7) \) forms a loop. Find the area enclosed by that loop.

\[
\int_{-7}^{0} \left( \sqrt{x^2(x + 7)} \right) + \left( +\sqrt{x^2(x + 7)} \right) \, dx
\]

\[
= -2 \int_{0}^{7} (u - 7)u^{1/2} \, du
\]

\[
= -2 \int_{0}^{7} u^{3/2} - 7u^{1/2} \, du
\]

\( u \equiv x + 7 \)
\( du = dx \)
EXAMPLE: Part of the graph of \( y^2 = x^2(x + 7) \) forms a loop. Find the area enclosed by that loop.

\[
\int_{-7}^{0} \left( \sqrt{x^2(x + 7)} \right) + \left( -\sqrt{x^2(x + 7)} \right) \, dx
\]

\[
= -2 \int_{0}^{7} (u - 7)u^{1/2} \, du
\]

\[
= -2 \int_{0}^{7} u^{3/2} - 7u^{1/2} \, du
\]

\[
= -2 \left[ \frac{u^{5/2}}{5/2} - 7 \left( \frac{u^{3/2}}{3/2} \right) \right]_{u: \to 0}^{u: \to 7}
\]

\[
= -2 \left[ \frac{7^{5/2}}{5/2} - 7 \left( \frac{7^{3/2}}{3/2} \right) \right]
\]

\[
= -2 \left[ \frac{2}{5} - \frac{2}{3} \right] \cdot 7^{5/2}
\]

\[
= -2 \left[ \frac{6}{15} - \frac{10}{15} \right] \cdot 7^{5/2} = \frac{8}{15} \cdot 7^{5/2}
\]

\[\text{SKILL: area between curves}\]
SKILL
area between curves
Whitman problems
§9.1, p. 182, #1-12

STOP