CALCULUS
The disk and washer methods
EXAMPLE: Find the volume enclosed in a sphere of radius 7.

\[ \text{Vol(cone)} + \text{Vol(hemisphere)} = \text{Vol(cylinder)} \]

3 \rightarrow 7
EXAMPLE: Find the volume enclosed in a sphere of radius 7.

\[ \text{Vol(cone)} + \text{Vol(hemisphere)} = \text{Vol(cylinder)} \]

\[ \frac{\pi 7^3}{3} + 3 \rightarrow 7 = (\pi 7^2)7 = \pi 7^3 \]

In 3-D, \( \text{Vol(generalized cone)} = \frac{\text{Vol(generalized cylinder)}}{3} \)
EXAMPLE: Find the volume enclosed in a sphere of radius 7.

\[
\text{Vol(cone)} + \text{Vol(hemisphere)} = \text{Vol(cylinder)}
\]

\[
\frac{\pi 7^3}{3} + \left( \pi 7^3 - \frac{\pi 7^3}{3} \right) = \left( 1 - \frac{1}{3} \right) \pi 7^3 = \frac{2}{3} \pi 7^3
\]

\[
\text{Vol(sphere)} = \frac{4}{3} \pi 7^3
\]

A MORE DIRECT APPROACH ...
EXAMPLE: Find the volume enclosed in a sphere of radius 7.

Solution:

2-D is easier to draw, but you have to imagine the 3-D visualization.
EXAMPLE: Find the volume enclosed in a sphere of radius 7.

Solution:

\[
\text{Area} \quad \pi (7^2 - u^2) \quad \text{radius} \quad \sqrt{7^2 - u^2}
\]

3-D

\[
\int_{-7}^{7} \pi (7^2 - u^2) \, du
\]

2-D is easier to draw, but you have to imagine the 3-D visualization.

\[\text{Definition:} \quad \text{A solid obtained by revolving, about a line, a subset of a plane, is called a solid of revolution.}\]
EXAMPLE: Find the volume enclosed in a sphere of radius 7.

Solution:

\[
\text{Area} \quad \pi (7^2 - u^2)
\]

\[
\text{radius} \quad \sqrt{7^2 - u^2}
\]

Goal:

\[
\int_{-7}^{7} \pi (7^2 - u^2) \, du = 2 \int_{0}^{7} \pi (7^2 - u^2) \, du
\]

\[
= 2 \left[ \pi \left( 7^2 u - \frac{u^3}{3} \right) \right]_{u:\to 7}^{u:\to 0}
\]

DEFINITION:

A solid obtained by revolving, about a line, a subset of a plane, is called a solid of revolution.
EXAMPLE: Find the volume enclosed in a sphere of radius 7.

Solution:

\[ \pi (7^2 - u^2) \]
\[ \sqrt{7^2 - u^2} \]

Area

Goal:
\[ \int_{-7}^{7} \pi (7^2 - u^2) \, du = 2 \left[ \pi \left( 7^2 u - \frac{u^3}{3} \right) \right]_{u: \to -7}^{u: \to 7} \]
\[ = 2 \left[ \pi \left( 7^2 \cdot 7 - \frac{7^3}{3} \right) \right]_{u: \to -7}^{u: \to 0} \]

DEFINITION:
A solid obtained by revolving, about a line, a subset of a plane, is called a **solid of revolution**.
EXAMPLE: Find the volume enclosed in a sphere of radius 7.

Solution:

\[ \pi (7^2 - u^2) \sqrt{7^2 - u^2} \]

Goal: \( \int_{-7}^{7} \pi (7^2 - u^2) \, du = 2 \left[ \pi \left( 7^2 \cdot 7 - \frac{7^3}{3} \right) \right] \)

\[ = 2 \left[ \pi \left( 7^3 - \frac{7^3}{3} \right) \right] \]

DEFINITION:
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EXAMPLE: Find the volume enclosed in a sphere of radius 7.

Solution:

\[ \pi \left( 7^2 - u^2 \right) \]

Goal:

\[
\int_{-7}^{7} \pi (7^2 - u^2) \, du = 2 \left[ \pi \left( 7^2 \cdot 7 - \frac{7^3}{3} \right) \right] = 2 \left[ \pi (7^3) \left( 1 - \frac{1}{3} \right) \right]
\]

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\text{Area} \quad \pi (7^2 - u^2) \\
\text{radius} \quad \sqrt{7^2 - u^2}
\]

Goal: \[ \int_{-7}^{7} \pi (7^2 - u^2) \, du = 2 \left[ \pi (7^3) \left( 1 - \frac{1}{3} \right) \right] \]

\[ = 2 \left[ \pi (7^3) \left( \frac{2}{3} \right) \right] = 2 \left[ \pi (7^3) \left( 1 - \frac{1}{3} \right) \right] \]

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\text{Area} \quad \pi(7^2 - u^2) \quad \text{radius} \quad \sqrt{7^2 - u^2} \quad \text{Area} \quad \pi(7^2 - u^2) \quad \text{radius} \quad \sqrt{7^2 - u^2}
\]

Disk

\[
\int_{-7}^{7} \pi(7^2 - u^2) \, du = 2 \left[ \pi(7^3) \left(1 - \frac{1}{3}\right)\right] \quad \text{Vol solid}
\]

\[
= 2 \left[ \pi(7^3) \left(\frac{2}{3}\right)\right] = \frac{4}{3}\pi 7^3
\]

Could we use vertical disks?

DEFINITION:
A solid obtained by revolving, about a line, a subset of a plane, is called a solid of revolution.
EXAMPLE: Find the volume enclosed in a sphere of radius 7.

Solution:

Area \( \pi (7^2 - u^2) \)

\[
\text{disk} \quad \frac{\text{radius}}{\sqrt{7^2 - u^2}} \quad \text{disk}
\]

Goal:

\[
\int_{-7}^{7} \pi (7^2 - u^2) \, du = 2 \left[ \pi (7^3) \left( 1 - \frac{1}{3} \right) \right] = 2 \left[ \pi (7^3) \left( \frac{2}{3} \right) \right] = \frac{4}{3} \pi 7^3
\]

Could we use vertical disks? Yes.  

Next: Generalization...

DEFINITION:

A solid obtained by revolving, about a line, a subset of a plane, is called a **solid of revolution**.
THE DISK METHOD:

Assume $f \geq 0$ on $[a, b]$.

Let $R$ be the region between the $x$-axis and the graph of $y = f(x)$ from $x = a$ to $x = b$.

If $S$ is the solid obtained by revolving $R$ about the horizontal axis, then the volume of $S$ is $\int_a^b \pi [f(x)]^2 \, dx$.

area of disk $= \pi [f(x)]^2$
THE DISK METHOD:

Assume $f \geq 0$ on $[a, b]$.

Let $R$ be the region between the $x$-axis and the graph of $y = f(x)$ from $x = a$ to $x = b$.

If $S$ is the solid obtained by revolving $R$ about the horizontal axis, then the volume of $S$ is $\int_a^b \pi[f(x)]^2 \, dx$.

Next: General setup with horizontal disks . . .

$x \leftrightarrow y$
THE DISK METHOD:

Assume $f \geq 0$ on $[a, b]$.

Let $R$ be the region between the $y$-axis and the graph of $x = f(y)$ from $y = a$ to $y = b$.

If $S$ is the solid obtained by revolving $R$ about the vertical axis, then the volume of $S$ is $\int_a^b \pi [f(y)]^2 \, dy$. 
THE DISK METHOD:

Assume $f \geq 0$ on $[a, b]$.

Let $R$ be the region between the $y$-axis and the graph of $x = f(y)$ from $y = a$ to $y = b$.

If $S$ is the solid obtained by revolving $R$ about the vertical axis, then the volume of $S$ is \[ V = \int_a^b \pi [f(y)]^2 \, dy. \]

The axis of rotation may be neither horizontal nor vertical.

It might point left or down.

The variable may be neither "$x$" nor "$y$".

We only require that all cross-sections be disks.
EXAMPLE: Find the volume of the solid obtained by revolving, about the $x$-axis, the region under the curve $y = \sqrt[4]{x}$ from 0 to 5. Illustrate the definition of volume by sketching a typical approximating cylinder.

Left endpoints:

$$x_j = 0 + (j - 1)(5/n), \quad j = 1, \ldots, n$$
EXAMPLE: Find the volume of the solid obtained by revolving, about the x-axis, the region under the curve \( y = \sqrt[4]{x} \) from 0 to 5. Illustrate the definition of volume by sketching a typical approximating cylinder.

**Volume of Solid of Revolution**

\[
\text{total volume of solid of revolution} \approx \sum_{j=1}^{n} \pi \left( \sqrt[4]{x_j} \right)^2 \Delta x
\]

**Area**

\[
\text{area} = \pi \left( \sqrt[4]{x_j} \right)^2
\]

**Radius**

\[
\text{radius} = \sqrt[4]{x_j}
\]

**Volume**

\[
\text{volume} = \pi \left( \sqrt[4]{x_j} \right)^2 \Delta x
\]

Left endpoints:

\[
x_j = 0 + (j - 1)(5/n), \quad j = 1, \ldots, n
\]

\[\text{Now take limit as } n \to \infty.\]

\[\text{§ 9.3}\]
EXAMPLE: Find the volume of the solid obtained by revolving, about the $x$-axis, the region under the curve $y = \sqrt[4]{x}$ from 0 to 5.

Total volume of solid of revolution $\equiv \int_{0}^{5} \pi \left( \sqrt[4]{x} \right)^2 \, dx$

Area $= \pi \left( \sqrt[4]{x} \right)^2$

Radius $= \sqrt[4]{x}$

Volume $= \pi \left( \sqrt[4]{x} \right)^2 \, dx$

Intuitive: $dx$
EXAMPLE: Find the volume of the solid obtained by revolving, about the $x$-axis, the region under the curve $y = \frac{4}{\sqrt{x}}$ from $0$ to $5$.

Total volume of solid of revolution = $\int_0^5 \pi \left( \frac{4}{\sqrt{x}} \right)^2 \, dx$

$\pi \int_0^5 \left( x^{1/4} \right)^2 \, dx$

$\pi \int_0^5 x^{1/2} \, dx$

$\pi \left[ \frac{x^{3/2}}{3/2} \right]_{x : \rightarrow 0}^{x : \rightarrow 5}$

$\pi \left[ \frac{5^{3/2}}{3/2} \right]$

**Skill**

Disk method
THE DISK METHOD:

Assume $f \geq 0$ on $[a, b]$. Let $R$ be the region between the $x$-axis and the graph of $y = f(x)$ from $x = a$ to $x = b$.

If $S$ is the solid obtained by revolving $R$ about the horizontal axis, then the volume of $S$ is $\int_a^b \pi [f(x)]^2 \, dx$. 

Let's try rotating a region between two functions...
THE WASHER METHOD:

Assume \( g \geq f \geq 0 \) on \([a, b]\).

Let \( R \) be the region between the graphs of \( y = f(x) \) and \( y = g(x) \) from \( x = a \) to \( x = b \).

If \( S \) is the solid obtained by revolving \( R \) about the horizontal axis, then the volume of \( S \) is

\[
V = \int_a^b \pi ([g(x)]^2 - [f(x)]^2) \, dx.
\]

area of washer = \( \pi ([g(x)]^2 - [f(x)]^2) \)

Horizontal washers . . .

\( x \leftrightarrow y \)

area of inner disk = \( \pi [f(x)]^2 \)

area of outer disk = \( \pi [g(x)]^2 \)
THE WASHER METHOD:
Assume \( g \geq f \geq 0 \) on \([a, b]\).
Let \( R \) be the region between the graphs of \( x = f(y) \) and \( x = g(y) \) from \( y = a \) to \( y = b \).

If \( S \) is the solid obtained by revolving \( R \) about the vertical axis, then the volume of \( S \) is
\[
\int_a^b \pi ([g(y)]^2 - [f(y)]^2) \, dy.
\]

The axis of rotation may be **neither** horizontal nor vertical.
It might point left or down.
The variable may be **neither** “\( x \)” nor “\( y \)”.
We only require that all cross-sections be washers.
EXAMPLE: Find the volume of the solid obtained by revolving, about the $y$-axis, the region bounded by $y = x^2 + 3$, $y = 7$ and $x = 1$.

- Inner radius of the washer: $\sqrt{y - 3}$
- Outer radius of the washer: $1$

\[
\text{vol}(\text{washer}) = \int_4^7 \left[ \pi (\sqrt{y - 3})^2 - \pi (1)^2 \right] dy
\]

\[
\text{vol}(\text{solid}) = \int_4^7 \left[ \pi (\sqrt{y - 3})^2 - \pi (1)^2 \right] dy
\]
EXAMPLE: Find the volume of the solid obtained by revolving, about the $y$-axis, the region bounded by $y = x^2 + 3$, $y = 7$ and $x = 1$.

$$
\text{vol(solid)} = \int_4^7 \left[ \pi \left( \sqrt{y - 3} \right)^2 - \pi (1)^2 \right] dy
$$

$$
= \int_4^7 \left[ \pi \left( \sqrt{y - 3} \right)^2 - \pi \right] dy
$$

$$
= \left[ \frac{\pi y^2}{2} - 4\pi y \right]_4^7
$$

$$
= \left[ \int_4^7 [\pi y - 4\pi] \, dy \right]
$$

$$
= \left[ \frac{\pi y^2}{2} - 4\pi y \right]_4^7
$$

$$
= \left[ \int_4^7 [\pi y - 3\pi - \pi] \, dy \right]
$$

$$
= \left[ \int_4^7 [\pi (y - 3) - \pi] \, dy \right]
$$
EXAMPLE: Find the volume of the solid obtained by revolving, about the $y$-axis, the region bounded by $y = x^2 + 3$, $y = 7$ and $x = 1$.

$\text{vol}(\text{solid}) = \pi \left[ \frac{y^2}{2} - 4\pi y \right]_{y \to 7} - 4\pi \left( 7 - 4 \right)$

$\text{vol}(\text{solid}) = \frac{\pi(7^2 - 4^2)}{2} - 4\pi(7 - 4)$
EXAMPLE: Find the volume of the solid obtained by revolving, about the $y$-axis, the region bounded by $y = x^2 + 3$, $y = 7$ and $x = 1$.

$$\text{vol(solid)} = \left[ \frac{\pi y^2}{2} - 4\pi y \right]_{y:4}^{y:7}$$

$$\pi(7^2 - 4^2) \quad \frac{\pi(49 - 16)}{2} - 12\pi$$

$$\frac{33\pi}{2} - \frac{24\pi}{2} = \frac{9\pi}{2}$$