CALCULUS
Volume by slices and
the disk and washer methods,
problems
EXAMPLE: A wedge shape is cut out of a right circular cylinder of radius 7 by two cuts. One cut is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 45° along a diameter of the cylinder. Find the volume of the wedge.

Solution:
EXAMPLE: A wedge shape is cut out of a right circular cylinder of radius 7 by two cuts. One cut is perpendicular to the axis of the cylinder. The other intersects the first at an angle of $45^\circ$ along a diameter of the cylinder. Find the volume of the wedge.

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Solution:

Goal: \[ \int_0^7 A(u) \, du \]

width \(du\)

Area \(A(u) = ?\)

\[ \sqrt{49 - u^2} \]

\[ 0 \leq u \leq 7 \]
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Goal: $\int_0^7 A(u) \, du$

Area $A(u) = ?$

$\tan(\pi/4) = \frac{\sqrt{49 - u^2}}{u}$

$? = u[\tan(\pi/4)]$

$u \leq 7$


**EXAMPLE:** A wedge shape is cut out of a right circular cylinder of radius 7 by two cuts. One cut is perpendicular to the axis of the cylinder. The other intersects the first at an angle of $45^\circ$ along a diameter of the cylinder. **Find** the volume of the wedge.

**Solution:**

\[
\int_0^7 A(u) \, du = 2 \int_0^7 u \sqrt{49 - u^2} \, du
\]

\[
v := 49 - u^2
\]

\[
dv = -2u \, du
\]

Area \( A(u) = \left[ 2\sqrt{49 - u^2} \right] [u] \)

\[
\tan(\pi/4) = \frac{?}{u}
\]

\[
? = u[\tan(\pi/4)] = u
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dv = -2u \, du
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\[
= 2 \int_0^{49} \sqrt{v} \, \frac{dv}{2}
\]

\[
= \int_0^{49} v^{1/2} \, dv
\]

\[
= \frac{(49)^{3/2}}{3/2}
\]

\[
= \frac{686}{3}
\]

\[
\approx 343
\]

Skill vol solid
EXAMPLE: Find the volume of the solid obtained by revolving the region bdd by \(y = 2 - x^2\) and \(y = 0\) about the \(x\)-axis.

\[
\int_{-\sqrt{2}}^{\sqrt{2}} \pi (2 - x^2)^2 \, dx = 2 \int_{0}^{\sqrt{2}} \pi (2 - x^2)^2 \, dx
\]

**SKILL**
disk method

\[
= 2\pi \int_{0}^{\sqrt{2}} 4 - 4x^2 + x^4 \, dx
\]

\[
= 2\pi \left[ 4\sqrt{2} - \frac{4(\sqrt{2})^3}{3} + \frac{(\sqrt{2})^5}{5} \right]
\]

\[\text{§9.3}\]

\[11\]
EXAMPLE: Find the volume of the solid obtained by revolving the region bdd by $y = 2 - x^2$ and $y = 1$ about the $x$-axis.

Volume = $\int_{-1}^{1} \pi (2 - x^2)^2 - \pi (1)^2 \, dx$

SYMMETRY

$= 2\pi \int_{0}^{1} (2 - x^2)^2 - 1 \, dx$

$= 2\pi \int_{0}^{1} (4 - 4x^2 + x^4) - 1 \, dx$
EXAMPLE: Find the volume of the solid obtained by revolving the region bdd by $y = 2 - x^2$ and $y = 1$ about the $x$-axis.

![Diagram showing the region and the washer method](image)

Outer radius = $2 - x^2$

Inner radius = $1$

Volume = $2\pi \int_{0}^{1} (4 - 4x^2 + x^4) - 1 \, dx$

= $2\pi \int_{0}^{1} 3 - 4x^2 + x^4 \, dx$

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EXAMPLE: Find the volume of the solid obtained by revolving the region bdd by $y = 2 - x^2$ and $y = 1$ about the $x$-axis.

Volume = $2\pi \int_0^1 (4 - 4x^2 + x^4) - 1 \, dx$

= $2\pi \int_0^1 3 - 4x^2 + x^4 \, dx$

= $2\pi \left[ 3x - \frac{4x^3}{3} + \frac{x^5}{5} \right]_{x: \to 0}^{1}$

= $2\pi \left[ 3 - \frac{4}{3} + \frac{1}{5} \right]$
EXAMPLE: Find the volume of the solid obtained by revolving the region bounded by

\[ y = 2x^2 \quad \text{and} \quad y = 20 - 3x^2 \]

about the \( x \)-axis.

\[
\int_{-2}^{2} \left[ \pi (20 - 3x^2)^2 - \pi (2x^2)^2 \right] \, dx
\]

\[ [2x^2 = 20 - 3x^2] \iff [x = 2 \text{ or } x = -2] \]
EXAMPLE: Set up an integral that computes the volume of the solid obtained by revolving the region bounded by 

\[ y = x^4 \quad \text{and} \quad y = 8x \]

about the line \( x = 3 \). **Don’t evaluate the integral.**

**WASHER**

Outer radius = \( 3 - \frac{y}{8} \)

Inner radius = \( 3 - \frac{4\sqrt{y}}{8} \)

\[
\int_0^{16} \pi \left( 3 - \frac{y}{8} \right)^2 - \pi \left( 3 - \frac{4\sqrt{y}}{8} \right)^2 \, dy
\]

**SKILL**

washer method
EXAMPLE: Find the volume of the solid obtained by revolving the region in the first quadrant bounded by $y = \frac{3}{\sqrt[3]{x}}$ and $y = 2x$ about the line $x = -1$.

$$\int_{0}^{1/\sqrt{2}} \left[ \pi \left(1 + \left(\frac{y}{2}\right)^2\right) - \pi \left(1 + y^3\right)^2 \right] dy = \cdots$$

SKILL
washer method
EXAMPLE: Describe a solid whose volume is

\[ \pi \int_{0}^{\pi} \left[ 1 + \sin^2 x \right]^2 \, dx. \]

\[ \pi \int_{0}^{\pi} \left[ 1 + \sin^2 x \right]^2 \, dx = \int_{0}^{\pi} \pi \left[ 1 + \sin^2 x \right]^2 \, dx \]

is the volume of the solid obtained by revolving, about the \( x \)-axis, the region bounded by

\( y = 1 + \sin^2 x \), \( y = 0 \), \( x = 0 \) and \( x = \pi \).
EXAMPLE: Describe a solid whose volume is $\pi \int_{4}^{8} e^{2x} \, dx$.

\[ \pi \int_{4}^{8} e^{2x} \, dx = \int_{4}^{8} \pi (e^{x})^2 \, dx \]

is the volume of the solid obtained by revolving, about the $x$-axis, the region bounded by $y = e^{x}$, $y = 0$, $x = 4$ and $x = 8$.

Difficult to draw this with a uniform scale on the $y$-axis.

**SKILL**
region from integral
EXAMPLE: Describe a solid whose volume is

\[ \pi \int_0^\pi \left[ 3 + \sin^2 x \right]^2 - 4 \, dx. \]

\[ \pi \int_0^\pi \left[ 3 + \sin^2 x \right]^2 - 4 \, dx = \int_0^\pi \pi \left[ 3 + \sin^2 x \right]^2 - \pi 2^2 \, dx \]

is the volume of the solid obtained by revolving, about the \( x \)-axis, the region bounded by

\[ y = 3 + \sin^2 x, \ y = 2, \ x = 0 \text{ and } x = \pi. \]
EXAMPLE: Describe a solid whose volume is \( \pi \int_{3}^{7} y^4 - 9 \, dy \).

\[
\pi \int_{3}^{7} y^4 - 9 \, dy = \int_{3}^{7} \pi \left(y^2 \right)^2 - \pi 3^2 \, dy
\]

is the volume of the solid obtained by revolving, about the y-axis, the region bounded by

\[
x = y^2, \quad x = 3, \quad y = 3 \text{ and } y = 7.\]

\section*{SKILL region from integral}

\section*{STOP}