4603 HW6

1. If \( f + g \) has a limit at \( a \), must it be true that both \( f \) and \( g \) have limits at \( a \)? Prove it, or give a counterexample.

\textit{Solution.} No, for example define \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = 0 \) for \( x < 0 \) and \( f(x) = 1 \) for \( x \geq 0 \), and let \( g = -f \). Then \( f + g \equiv 0 \), so \( f + g \) has a limit at 0, but neither \( f \) nor \( g \) has a limit at 0.

2. Define \( f : [0, 1] \to \mathbb{R} \) by

\[
f(x) = \begin{cases} 
0, & x \notin \mathbb{Q} \\
\frac{1}{\sqrt{n}}, & x = \frac{n}{m} \in \mathbb{Q} \text{ where } n \neq 0 \text{ and } n, m \text{ have no common divisors}
\end{cases}
\]

Let \( a \in [0, 1] \). Prove that \( \lim_{x \to a} f(x) = f(a) \) if and only if \( a \notin \mathbb{Q} \).

\textit{Solution.} Let \( \epsilon > 0 \) and consider \( S = \{ x \in [0, 1] : f(x) \geq \epsilon \} \). Observe that \( S \) is finite: if \( x \in S \), then \( x \) is rational and can be written \( x = n/m \) in lowest terms with \( 1 \leq m \leq 1/\epsilon^2 \) and \( 0 \leq n \leq m \). Write \( S = \{x_1, \ldots, x_k\} \) and let \( \delta = \min_{x_j \neq a} |x_j - a| \). Then \( x \in [0, 1] \) and \( 0 < |x - a| < \delta \) implies \( x \neq x_j \) for \( j = 1, \ldots, k \), so that \( |f(x) - 0| = f(x) < \epsilon \). This shows \( \lim_{x \to a} f(x) = 0 \). The result follows from definition of \( f \).

3. Let \( f : [a, b] \to \mathbb{R} \) be a bounded function, define \( g : (a, b) \to \mathbb{R} \) by \( g(x) = \sup\{f(y) : y < x\} \), and let \( c \in (a, b) \). Prove that if \( \lim_{x \to c} f(x) = f(c) \), then \( \lim_{x \to c} g(x) = g(c) \).

\textit{Solution.} We recall the following simple fact:

\((*)\) If \( x, y \in (a, b) \) with \( x < y \), then \( f(x) \leq g(y) \).

Assume \( \lim_{x \to c} f(x) = f(c) \). We will consider two cases: \( g(c) = f(c) \) and \( g(c) \neq f(c) \). Assume \( g(c) = f(c) \). Let \( \epsilon > 0 \) and choose \( \delta > 0 \) such that \( |x - c| < \delta \) implies \( |f(x) - f(c)| < \epsilon \). Pick \( u, v \) such that \( c - \delta < u < c < v < c + \delta \). Note that \( f(x) \leq g(c) \) for \( x < c \) and \( f(x) < f(c) + \epsilon \) for \( c \leq x < v \). Thus \( x \in (u, v) \) and \( (*) \) imply

\[ g(c) - \epsilon = f(c) - \epsilon < f(u) \leq g(x) \leq \max\{g(c), f(c) + \epsilon\} = g(c) + \epsilon. \]

It easily follows that \( \lim_{x \to c} g(x) = g(c) \). Now assume \( g(c) \neq f(c) \) and let \( \epsilon = |f(c) - g(c)| \). Choose \( \delta \) such that \( |x - c| < \delta \) implies \( |f(x) - f(c)| < \epsilon \). Then either \( f(x) > g(c) \) for all \( x \in (c - \delta, c + \delta) \), or \( f(x) < g(c) \) for all \( x \in (c - \delta, c + \delta) \). The former is impossible by \( (*) \), so the latter must hold, which implies \( g \) is constant on \( (c - \delta, c + \delta) \). It easily follows that \( \lim_{x \to c} g(x) = g(c) \).

\(1\)Since \( c \in (a, b) \), \( \delta > 0 \) can be chosen small enough so that \( |x - c| < \delta \) also implies \( x \in [a, b] \).
4. Let \( f \) and \( g \) be defined as in Problem 3, and let \( c \in (a, b) \). If \( f \) has a limit at \( c \), must \( g \) have a limit at \( c \)? Either prove it, or provide a counterexample.

Solution. No. Define \( f : [-1, 1] \to \mathbb{R} \) by \( f(x) = 0 \) for \( x \neq 0 \) and \( f(0) = 1 \). Then \( g : (-1, 1) \to \mathbb{R} \) has the formula \( g(x) = 0 \) for \( x \leq 0 \) and \( g(x) = 1 \) for \( x > 0 \). So \( f \) has a limit at 0, but \( g \) does not have a limit at 0.

5. Prove that if \( f : [a, b] \to \mathbb{R} \) is increasing, then \( f \) has a limit at \( b \).

Solution. Observe that \( f \) is bounded above by \( f(b) \). Let \( \epsilon > 0 \) and \( L = \sup \{ f(x) : x \in [a, b) \} \).
Pick \( y \in [a, b) \) such that \( f(y) > L - \epsilon \), and let \( \delta = |b - y| \). Then \( x \in [a, b] \) and \( 0 < |x - b| < \delta \) implies \( y < x < b \), so that
\[
L - \epsilon < f(y) \leq f(x) \leq L < L + \epsilon.
\]