PRINT YOUR NAME:

SOLUTIONS

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Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.

Turn off all mobile electronic devices.
I. Definitions

A. (5 pts) Let $D \subseteq \mathbb{R}$. Let $x_0 \in \mathbb{R}$. We say $x_0$ is an accumulation point of $D$ if...

\[ \forall \text{nbd } U \text{ of } x_0, \quad U \cap D \text{ is infinite} \]

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B. (5 pts) Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ and let $x_0$ be an accumulation point of $D$. Let $L \in \mathbb{R}$. We say that $f(x) \to L$ as $x \to x_0$ if...

\[ \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad \forall x \in D \quad \left( 0 < |x - x_0| < \delta \right) \implies \left( |f(x) - L| < \varepsilon \right) \]

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C. (5 pts) Let $D \subseteq \mathbb{R}$. Let $f : D \to \mathbb{R}$. Let $x_0 \in D$. We say $f$ is continuous at $x_0$ if...

\[ \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad \forall x \in D \quad \left( |x - x_0| < \delta \right) \implies \left( |f(x) - f(x_0)| < \varepsilon \right) \]
D. (5 pts) Let \( D \subseteq \mathbb{R} \), let \( f : D \rightarrow \mathbb{R} \) and let \( E \subseteq D \). We say that \( f \) is uniformly continuous on \( E \) if...

\[
\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t., } \forall x, y \in E,
\left| x - y \right| < \delta \Rightarrow \left| f(x) - f(y) \right| < \varepsilon
\]

E. (5 pts) Let \( C \subseteq \mathbb{R} \) and let \( \mathcal{U} \subseteq 2^{\mathbb{R}} \). We say that \( \mathcal{U} \) is an open cover of \( C \) if...

\[
\left( \forall U \in \mathcal{U}, \ U \text{ is open} \right) \text{ and } \left( U \mathcal{U} = C \right)
\]

F. (5 pts) Let \( C \subseteq \mathbb{R} \). We say that \( C \) is compact if...

\[
\forall \text{ open cover } \mathcal{U} \text{ of } C, \exists \text{ finite } \mathcal{F} \subseteq \mathcal{U} \text{ s.t. } \bigcup \mathcal{F} = C
\]
II. True or false (no partial credit):

a. (5 pts) Any closed subset of \( \mathbb{R} \) is compact.
   
   \text{False}

b. (5 pts) Every subset of \( \mathbb{R} \) is either open or closed (or both).
   
   \text{False}

c. (5 pts) A nonempty open subset of \( \mathbb{R} \) cannot be compact.
   
   \text{True}

d. (5 pts) Any compact subset of \( \mathbb{R} \) is both closed and bounded.
   
   \text{True}

e. (5 pts) Let \( I \) be a set and let \( \{C_\alpha\}_{\alpha \in I} \) be a family of closed subsets of \( \mathbb{R} \). Then \( \bigcap_{\alpha \in I} C_\alpha \) is closed.
   
   \text{True}

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PLEASE DO NOT WRITE BELOW THE LINE

I. A, B, C

I. D, E, F

II. a, b, c, d, e

III. 1

III. 2

III. 3

III. 4
III. Hand-graded problems. Show work.

1. (10 pts) Find an open cover of the interval \([0, 1]\) that has no finite subcover.

\[\left\{ (-1, 1-\frac{1}{n}) \mid n \in \mathbb{N} \right\}\]
2. (10 pts.) Find a continuous injective function \( f : \mathbb{R} \to \mathbb{R} \) and a compact set \( C \subseteq \mathbb{R} \) such that \( f^{-1}(C) \) is noncompact. State clearly what your function \( f \) is (and be sure that its domain and codomain are both \( \mathbb{R} \)). State clearly what your compact set \( C \) is. Also, state clearly what \( f^{-1}(C) \) is.

Define \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = \arctan x \).

Let \( C := \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \).

Then \( f^{-1}(C) = \mathbb{R} \).
3. (10 pts) Let \( a, b \in \mathbb{R} \). Assume that \( a \neq 0 \). Define \( f : \mathbb{R} \rightarrow \mathbb{R} \) by \( f(x) = ax + b \). Show that \( f : \mathbb{R} \rightarrow \mathbb{R} \) is uniformly continuous.

Given \( \epsilon > 0 \)

\[
\text{Want: } \exists \delta > 0 \text{ s.t., } \forall x, y \in \mathbb{R}, \quad |x-y| < \delta \implies |(f(x) - (f(y))| < \epsilon
\]

Let \( \delta := \frac{\epsilon}{|a|} \). Given \( x, y \in \mathbb{R} \)

Assume \( |x-y| < \delta \). \( \text{Want: } |(f(x) - (f(y))| < \epsilon \)

\[
|f(x) - f(y)| = |ax + b - ay - b|
\]

\[
= |a(x - y)|
\]

\[
= |a| \cdot |x - y|
\]

\[
< |a| \cdot \delta
\]

\[
= |a| \cdot \frac{\epsilon}{|a|}
\]

\[
= \epsilon. \quad \square
\]
4. (15 pts) Let $A, B \subseteq \mathbb{R}$. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be uniformly continuous functions. Assume that $\text{im}[f] \subseteq B$. Show that $g \circ f : A \to \mathbb{R}$ is uniformly continuous.

Given $\varepsilon > 0$.

Want: $\exists \delta > 0 \text{ s.t. } \forall x, y \in A$

\[(|x-y| < \delta) \implies (|((g \circ f)(x)) - ((g \circ f)(y))| < \varepsilon).\]

Choose $\gamma > 0 \text{ s.t. } \forall \alpha, \beta \in B$

\[(|\alpha - \beta| < \gamma) \implies (|g(\alpha) - g(\beta)| < \varepsilon).\]

Choose $\delta > 0 \text{ s.t. } \forall x, y \in A$

\[(|x-y| < \delta) \implies (|f(x) - f(y)| < \gamma).\]

Given $x, y \in A$. Assume $|x-y| < \delta$.

Want: $|((g \circ f)(x)) - ((g \circ f)(y))| < \varepsilon$.

Let $\alpha := f(x)$. Let $\beta := f(y)$.

Want: $|g(\alpha) - g(\beta)| < \varepsilon$

By choice of $\gamma$, $|f(x) - f(y)| < \gamma$, i.e., $|\alpha - \beta| < \gamma$.

So, by choice of $\gamma$, $|g(\alpha) - g(\beta)| < \varepsilon$.