54.a. Let \( A, B \subseteq \mathbb{R} \), let \( f : A \to \mathbb{R} \) and let \( g : B \to \mathbb{R} \). Let \( x \in A \cap B \). Assume that \( x \) is an accumulation point of \( A \cap B \). Assume that \( f \) and \( g \) are both differentiable at \( x \). Show that \( f + g \) is differentiable at \( x \) and that \((f + g)'(x) = [f'(x)] + [g'(x)]\).

**Proof:** We wish to show that \( SS^{x}_{f+g}(y) \to [f'(x)] + [g'(x)] \), as \( y \to x \). As \( y \to x \), we have both \( SS^{x}_{f}(y) \to [f'(x)] \) and \( SS^{x}_{g}(y) \to [g'(x)] \), so, by additivity of limits, it suffices to show, for all \( y \in (A \cap B) \setminus \{x\} \), that \( SS^{x}_{f+g}(y) = [SS^{x}_{f}(y)] + [SS^{x}_{g}(y)] \).

For all \( y \in (A \cap B) \setminus \{x\} \), we have

\[
SS^{x}_{f+g}(y) = \frac{[(f + g)(y)] - [(f + g)(x)]}{y - x} = \frac{[f(y)] + [g(y)] - [f(x)] - [g(x)]}{y - x} = \left[ \frac{[f(y)] - [f(x)]}{y - x} \right] + \left[ \frac{[g(y)] - [g(x)]}{y - x} \right] = [SS^{x}_{f}(y)] + [SS^{x}_{g}(y)].
\]

b. Let \( A \subseteq \mathbb{R} \), let \( f : A \to \mathbb{R} \) and let \( c \in \mathbb{R} \). Let \( x \in A \). Assume that \( x \) is an accumulation point of \( A \). Assume that \( f \) is differentiable at \( x \). Show that \( cf \) is differentiable at \( x \) and that \((cf)'(x) = c \cdot [f'(x)]\).

**Proof:** We wish to show that \( SS^{x}_{cf}(y) \to c \cdot [f'(x)] \), as \( y \to x \). As \( y \to x \), we have \( SS^{x}_{f}(y) \to [f'(x)] \), so, because limits respect scalar multiplication, it suffices to show, for all \( y \in A \setminus \{x\} \), that \( SS^{x}_{cf}(y) = [SS^{x}_{f}(y)] \).

For all \( y \in A \setminus \{x\} \), we have

\[
SS^{x}_{cf}(y) = \frac{[(cf)(y)] - [(cf)(x)]}{y - x} = \frac{[c \cdot [f(y)]] - [c \cdot [f(x)]]}{y - x} = c \cdot \left[ \frac{[f(y)] - [f(x)]}{y - x} \right] = c \cdot [SS^{x}_{f}(y)].
\]

55. Define \( r : \mathbb{R} \setminus \{0\} \to \mathbb{R} \) by \( r(x) = x^{-1} \). Let \( x \in \mathbb{R} \setminus \{0\} \). Show that \( r \) is differentiable at \( x \) and that \( r'(x) = -x^{-2} \).

**Proof:** We have \( r(x) = x^{-1} = 1/x \). For all \( y \in \mathbb{R} \setminus \{0, x\} \), we have \( r(y) = y^{-1} = 1/y \), so

\[
SS^{x}_{r}(y) = \frac{[r(y)] - [r(x)]}{y - x} = \frac{[1/y] - [1/x]}{y - x} = \frac{[(x - y)/xy]}{y - x} = \frac{-y^{-2}}{1} = -1/xy.
\]

Then \( SS^{x}_{r}(y) \to \frac{-1}{x^2} \), as \( y \to x \). Then \( r \) is differentiable at \( x \) and \( r'(x) = -\frac{1}{x^2} = -x^{-2} \).

53. Let \( A \subseteq \mathbb{R} \) be connected. Let \( f : A \to \mathbb{R} \) be continuous. Let \( x, y, z \in A \). Assume that \( x < y < z \) and that \( f(x) > f(y) \leq f(z) \). Show that \( f \) is not one-to-one.
**Proof:** Assume that \( f \) is one-to-one. We aim for a contradiction.

Since \( y < z \), it follows that \( y \neq z \). So, since \( f \) is one-to-one, we conclude that \( f(y) \neq f(z) \). So, since \( f(y) \leq f(z) \), we see that \( f(y) < f(z) \). Let \( u := f(y) \). Then \( u < f(z) \). By assumption, \( f(x) > f(y) \), i.e., \( f(y) < f(x) \). That is, \( u < f(x) \).

Let \( w := \min\{f(x), f(z)\} \). Then \( w \leq f(x) \) and \( w \leq f(z) \). Recall that \( u < f(x) \) and that \( u < f(z) \). So, since \( w \in \{f(x), f(z)\} \), it follows that \( u < w \). Let \( v := (u + w) / 2 \). Then \( u < v < w \). Then \( f(y) = u < v < w \leq f(x) \) and that \( f(y) = u < v < w \leq f(z) \).

Since \( f(y) < v < f(x) \), by the Intermediate Value Theorem, choose \( s \in (x, y) \) such that \( f(s) = v \). Since \( f(y) < v < f(z) \), by the Intermediate Value Theorem, choose \( t \in (y, z) \) such that \( f(t) = v \). Then \( x < s < y < t < z \). In particular, \( s \neq t \). So, since \( f(s) = v = f(t) \), we see that \( f \) is not one-to-one, contradiction.

---

51. Let \( \{y_n\}_{n \in \mathbb{N}} \) be a sequence in \( \mathbb{R} \). Let \( z \in \mathbb{R} \). Assume that \( y_n \to z \) as \( n \to \infty \). Assume, for all \( n \in \mathbb{N} \), that \( y_n \leq 0 \). Show that \( z \leq 0 \).

**Proof:** Assume that \( z > 0 \). We aim for a contradiction.

Let \( \varepsilon := z \). Choose \( N \in \mathbb{N} \) such that, for all integers \( n \geq N \), we have: \( |y_n - z| < \varepsilon \). Then, as \( N \geq N \), we see that \( |y_N - z| < \varepsilon \). Then \( z - \varepsilon < y_N < z + \varepsilon \). Recall, for all \( n \in \mathbb{N} \), that \( y_n \leq 0 \). Then \( y_N \leq 0 \). Then \( 0 = z - z < z - \varepsilon < y_N \leq 0 \), so \( 0 < 0 \), contradiction.

---

52. Let \( S \subseteq \mathbb{R} \). Let \( a, b \in \mathbb{R} \). Assume that \( S \neq \emptyset \) and that \( a \leq S \leq b \). Show that \( a \leq \inf S \leq \sup S \leq b \).

**Proof:** Since \( a \leq S \), it follows that \( a \leq \inf S \). Since \( b \geq S \), it follows that \( b \geq \sup S \). It remains to show that \( \inf S \leq \sup S \).

Since \( S \neq \emptyset \), choose \( x \in S \). Since \( \inf S \leq S \), we get \( \inf S \leq x \). Since \( \sup S \geq S \), we get \( \sup S \geq x \). Then \( \inf S \leq x \leq \sup S \).

---

47. Let \( E \subseteq \mathbb{R} \) be bounded and let \( \delta > 0 \). Show that there exist \( n \in \mathbb{N} \) and \( S_1, \ldots, S_n \subseteq \mathbb{R} \) such that

\[
\begin{align*}
(i) & \text{ for all } j \in \{1, \ldots, n\}, \text{ we have: } \operatorname{diam}(S_j) \leq \delta; \\
(ii) & E \subseteq S_1 \cup \cdots \cup S_n.
\end{align*}
\]

(Recall that, for all \( S \subseteq \mathbb{R} \), that we define \( \operatorname{diam}(S) := \sup\{y - x | x, y \in S\} \).

**Proof:** Choose \( a, b \in \mathbb{R} \) such that \( a \leq E \leq b \). Choose \( n \in \mathbb{N} \) such that \( n > (a - b)/\delta \). Let \( \delta_0 := (a - b)/n \). Then \( \delta_0 < \delta \). For all integers \( j \in [1, n] \), let \( E_j := [a + (j - 1)\delta_0, a + j\delta_0] \). Then, for all integers \( j \in [1, n] \), \( \operatorname{diam}(E_j) = \delta_0 < \delta \). We wish to prove \( E \subseteq S_1 \cup \cdots \cup S_n \).

We have \( S_1 \cup \cdots \cup S_n = [a, a + n\delta_0] = [a, a + (b - a)] = [a, b] \). By the choice of \( a \) and \( b \), we have \( E \subseteq [a, b] \). Then \( E \subseteq [a, b] = S_1 \cup \cdots \cup S_n \), as desired.

---

48. Let \( n \in \mathbb{N} \) and let \( E_1, \ldots, E_n \subseteq \mathbb{R} \) be bounded sets. Show that \( E := E_1 \cup \cdots \cup E_n \) is bounded.

**Proof:** Let \( E := E_1 \cup \cdots \cup E_n \). We wish to show that \( E \) is bounded.
For all integers \( j \in [1, n] \), choose \( a_j, b_j \in \mathbb{R} \) such that \( a_j \leq E_j \leq b_j \). We define \( a := \min\{a_1, \ldots, a_n\} \) and \( b := \max\{b_1, \ldots, b_n\} \). We wish to show that \( a \leq E \leq b \). That is, we wish to show that \( E \subseteq [a, b] \). It suffices to show, for all integers \( j \in [1, n] \), that \( E_j \subseteq [a, b] \). Given an integer \( j \in [1, n] \). We wish to show that \( E_j \subseteq [a, b] \).

We have \( a_j \leq E_j \leq b_j \), i.e., \( E_j \subseteq [a_j, b_j] \). We have \( a \leq a_j \) and \( b_j \leq b \), so \( [a_j, b_j] \subseteq [a, b] \). Then \( E_j \subseteq [a_j, b_j] \subseteq [a, b] \), as desired.

49. Let \( E \) be a closed subset of \( \mathbb{R} \). Let \( \{y_n\}_{n \in \mathbb{N}} \) be a sequence in \( E \) and let \( y_0 \in \mathbb{R} \). Assume that \( y_n \to y_0 \) as \( n \to \infty \). Show that \( y_0 \in E \).

**Proof:** Since \( E \) is closed, it suffices to show that \( y_0 \) is a closure point of \( E \). Given a neighborhood \( U \) of \( y_0 \). We wish to show that \( U \cap E \neq \emptyset \).

Choose \( \delta > 0 \) such that \( (y_0 - \delta, y_0 + \delta) \subseteq U \). Since \( y_n \to y_0 \) as \( n \to \infty \), choose \( N \in \mathbb{N} \) such that, for any integer \( n \geq N \), we have \( |y_n - y_0| < \delta \). Then \( |y_N - y_0| < \delta \), so \( y_N \in (y_0 - \delta, y_0 + \delta) \). Then \( y_N \in (y_0 - \delta, y_0 + \delta) \subseteq U \). Since \( \{y_n\}_{n \in \mathbb{N}} \) is a sequence in \( E \), it follows that \( y_N \in E \). Then \( y_N \in U \cap E \). Then \( U \cap E \neq \emptyset \).

50. Let \( E \) and \( C \) be sets and let \( f : E \to C \) be a function. Let \( V \subseteq f(E) \). Show that \( f(f^{-1}(V)) = V \).

**Proof:**

**Proof of \( \subseteq \):** Given \( c \in f(f^{-1}(V)) \). We wish to show that \( c \in V \).

Choose \( e \in f^{-1}(V) \) such that \( c = f(e) \). Since \( e \in f^{-1}(V) \), we get \( f(e) \in V \). Then \( c = f(e) \in V \). End of proof of \( \subseteq \).

**Proof of \( \supseteq \):** Given \( c \in V \). We wish to show that \( c \in f(f^{-1}(V)) \).

We have \( c \in V \subseteq f(E) \). Choose \( e \in E \) such that \( f(e) = c \). We have \( f(e) = c \in V \), so \( e \in f^{-1}(V) \). Then \( c = f(e) \in f(f^{-1}(V)) \). End of proof of \( \supseteq \).

43. Let \( I \) be a set. Let \( \{U_\alpha\}_{\alpha \in I} \) be a family of subsets of \( \mathbb{R} \). Assume, for all \( \alpha \in I \), that \( U_\alpha \) is open. Show that \( \bigcup_{\alpha \in I} U_\alpha \) is open.

**Proof:** Given \( x \in \bigcup_{\alpha \in I} U_\alpha \). We wish to show that there is a neighborhood \( N \) of \( x \) such that \( N \subseteq \bigcup_{\alpha \in I} U_\alpha \).

Choose \( \beta \in I \) such that \( x \in U_\beta \). Then, because \( U_\beta \) is open, choose a neighborhood \( N \) of \( x \) such that \( N \subseteq U_\beta \). Then \( N \subseteq U_\beta \subseteq \bigcup_{\alpha \in I} U_\alpha \).

44. Let \( U, V \subseteq \mathbb{R} \). Assume that \( U \) and \( V \) are both open. Show that \( U \cap V \) is open.

**Proof:** Given \( x \in U \cap V \). We wish to show that there is a neighborhood \( N \) of \( x \) such that \( N \subseteq U \cap V \).

Let \( R \) be a neighborhood of \( x \) such that \( R \subseteq U \). Let \( S \) be a neighborhood of \( x \) such that \( S \subseteq V \). Choose \( \rho > 0 \) such that \( (x - \rho, x + \rho) \subseteq R \). Choose \( \sigma > 0 \) such that
\[(x - \sigma, x + \sigma) \subseteq S. \] Let \( \nu := \min\{\rho, \sigma\} \). Let \( N := (x - \nu, x + \nu) \). We wish to show that \( N \subseteq U \cap V \).

We have \( N \subseteq (x - \rho, x + \rho) \subseteq R \subseteq U \). We have \( N \subseteq (x - \sigma, x + \sigma) \subseteq S \subseteq V \). Since we have both \( N \subseteq U \) and \( N \subseteq V \), it follows that \( N \subseteq U \cap V \), as desired.

---

45. Find a sequence \( U_1, U_2, \ldots \) of subsets of \( \mathbb{R} \) such that
- for all \( j \in \mathbb{N} \), \( U_j \) is open; and
- \( \bigcap_{j=1}^{\infty} U_j \) is NOT open.

**Answer:** For all \( j \in \mathbb{N} \), let \( U_j := (-1/j, 1/j) \). Then \( \bigcap_{j=1}^{\infty} U_j = \{0\} \), which is NOT open.

---

46. Let \( X \subseteq \mathbb{R} \) and let \( A \subseteq X \). Show that:

| A is compact | \iff | A is \( X \)-compact |

**Proof:** Proof of \( \Rightarrow \): Assume that \( A \) is compact. Given an \( X \)-open cover \( S \) of \( X \). We wish to show that there exists a finite \( F \subseteq S \) such that \( \bigcup F \supseteq A \).

For all \( S \in S \), let \( U_S \) be an open \( X \)-extension of \( S \). Then, for all \( S \in S \), we have \( U_S \cap X = S \), and so \( U_S \cap X \subseteq S \). Let \( U := \{U_S \mid S \in S\} \). Then \( U \) is an open cover of \( A \).

Also, for all \( U \in U \), we have \( U \cap X \subseteq S \). Choose a finite \( F_0 \subseteq U \) such that \( \bigcup F_0 \supseteq A \). Let \( F := \{U \cap X \mid U \in U\} \). Then \( F \subseteq S \) and \( F \) is finite. It remains to show that \( \bigcup F \supseteq A \).

We have \( \bigcup F = (\bigcup U) \cap X \supseteq A \cap X \). Since \( A \subseteq X \), it follows that \( A \cap X = A \). Then \( \bigcup F \supseteq A \cap X = A \), as desired. *End of proof of \( \Rightarrow \).*

Proof of \( \Leftarrow \): Assume that \( A \) is \( X \)-compact. Given an open cover \( U \) of \( A \). We wish to show that there exists a finite \( F \subseteq U \) such that \( \bigcup F \supseteq A \).

We have \( \bigcup U \supseteq A \). For all \( U \in U \), \( U \) is an open \( X \)-extension of \( U \cap X \), so \( U \cap X \) is \( X \)-open. Let \( S := \{U \cap X \mid U \in U\} \). Then \( \bigcup S = (\bigcup U) \cap X \). Since \( A \subseteq X \), it follows that \( A \cap X = A \). Then \( \bigcup S = (\bigcup U) \cap X \supseteq A \cap X = A \). Then \( S \) is an \( X \)-open cover of \( A \).

Choose a finite \( F_0 \subseteq S \) such that \( \bigcup F_0 \supseteq A \). For all \( S \in S \), there exists \( U \in U \) such that \( U \cap X = S \). For all \( S \in F_0 \), choose \( U_S \in U \) such that \( U_S \cap X = S \). Let \( F := \{U_S \mid S \in F_0\} \). Then \( F \) is finite and \( F \subseteq U \). It remains to show that \( \bigcup F \supseteq A \).

For all \( S \in F_0 \), we have \( \bigcup F \supseteq U_S \supseteq U_S \cap X = S \). Then \( \bigcup F \supseteq \bigcup F_0 \supseteq A \), as desired. *End of proof of \( \Leftarrow \).*

---

38. Let \( D \subseteq \mathbb{R} \). Assume that \( D \) is not bounded below. Let \( f : D \to \mathbb{R} \). Assume that \( f(x) \to \infty, \) as \( x \to -\infty \). Show that \( 1/|f(x)| \to 0, \) as \( x \to -\infty \).

**Proof:** Given \( \varepsilon > 0 \). We wish to show that there exists \( N \in \mathbb{R} \) such that, for all \( x \in D \), we have: \( [x < N] \Rightarrow [1/|f(x)| - 0] < \varepsilon \].

Choose \( N \in \mathbb{R} \) such that, for all \( x \in D \), we have: \( [x < N] \Rightarrow [f(x) > 1/\varepsilon] \).

Given \( x \in D \). Assume that \( x < N \). We wish to show that \( |1/|f(x)|| - 0| < \varepsilon \). That is, we wish to show that \( 1/|f(x)| < \varepsilon \).
We have \( f(x) > 1/\varepsilon > 0 \), so \( |f(x)| = f(x) \). Then \( |f(x)| = f(x) > 1/\varepsilon \). Then \( 1/|f(x)| < 1/[1/\varepsilon] = \varepsilon \).

39. Find a function \( f: \mathbb{R} \rightarrow \mathbb{R}\setminus\{0\} \) such that:
   (i) \( f(x) \rightarrow 0 \), as \( x \rightarrow -\infty \); and
   (ii) NOT \( [1/|f(x)|] \rightarrow \infty \), as \( x \rightarrow -\infty \).

Answer: Define \( f: \mathbb{R} \rightarrow \mathbb{R}\setminus\{0\} \) by \( f(x) = -e^x \).

40. Let \( D \subseteq \mathbb{R} \). Let \( f: D \rightarrow \mathbb{R} \). Let \( x_0 \in D \). Assume that \( x_0 \) is an accumulation point of \( D \). Show that:
   \[
   \text{[ } f \text{ is continuous at } x_0 \text{ ] } \quad \text{iff} \quad \text{[ } \lim_{x \rightarrow x_0} f(x) = f(x_0) \text{ ]}.
   \]

Proof: Proof of “only if”: Say \( f \) is continuous at \( x_0 \). We wish to prove: \( \lim_{x \rightarrow x_0} f(x) = f(x_0) \).

   Given \( \varepsilon > 0 \). We wish to show that there exists \( \delta > 0 \) such that, for all \( x \in D \), we have:
   \[
   \left[ 0 < |x - x_0| < \delta \right] \Rightarrow \left[ |(f(x)) - (f(x_0))| < \varepsilon \right].
   \]

   Choose \( \delta > 0 \) such that, for all \( x \in D \), we have:
   \[
   [x-x_0] < \delta \Rightarrow [(f(x)) - (f(x_0))] < \varepsilon.
   \]

   Given \( x \in D \). Assume that \( 0 < |x - x_0| < \delta \). We wish to show that \( |(f(x)) - (f(x_0))| < \varepsilon \).

   Since \( |x - x_0| < \delta \), it follows, from the choice of \( \delta \), that \( |(f(x)) - (f(x_0))| < \varepsilon \), as desired. End of proof of “only if”.

   Proof of “if”: Say \( \lim_{x \rightarrow x_0} f(x) = f(x_0) \). We wish to prove: \( f \) is continuous at \( x_0 \).

   Given \( \varepsilon > 0 \). We wish to show that there exists \( \delta > 0 \) such that, for all \( x \in D \), we have:
   \[
   \left[ |x - x_0| < \delta \right] \Rightarrow \left[ |(f(x)) - (f(x_0))| < \varepsilon \right].
   \]

   Choose \( \delta > 0 \) such that, for all \( x \in D \), \( 0 < |x - x_0| < \delta \) \Rightarrow \[(f(x)) - (f(x_0))] < \varepsilon\].

   Given \( x \in D \). Assume that \( |x - x_0| < \delta \). We wish to show that \( |(f(x)) - (f(x_0))| < \varepsilon \).

   If \( x = x_0 \), then \( |(f(x)) - (f(x_0))| = |0| = 0 < \varepsilon \), and we are done. We therefore assume that \( x \neq x_0 \). Then \( 0 < |x - x_0| \).

   Since \( 0 < |x - x_0| < \delta \), it follows, from the choice of \( \delta \), that \( |(f(x)) - (f(x_0))| < \varepsilon \), as desired. End of proof of “if”.

41. Define \( f: \mathbb{R}\setminus\{0\} \rightarrow \mathbb{R} \) by \( f(x) = 1/x \). Show that \( f \) is continuous at \( 3 \).

Proof: By Homework #35, \( f(x) \rightarrow 1/3 \), as \( x \rightarrow 3 \). That is, \( \lim_{x \rightarrow 3} f(x) = 1/3 \). Then \( \lim_{x \rightarrow 3} f(x) = f(3) \). Also, \( 3 \in \mathbb{R}\setminus\{0\} \) and \( 3 \) is an accumulation point of \( \mathbb{R}\setminus\{0\} \). Then, by Homework #40, we see that \( f \) is continuous at \( 3 \).

42. Let \( D, E \subseteq \mathbb{R} \). Let \( f: D \rightarrow E \) and \( g: E \rightarrow \mathbb{R} \). Let \( x_0 \) be an accumulation point of \( D \). Let \( L \in E \). Assume that \( f(x) \rightarrow L \), as \( x \rightarrow x_0 \). Assume that \( g \) is continuous at \( L \). Show that \( g(f(x)) \rightarrow g(L) \), as \( x \rightarrow x_0 \).

Proof: Given \( \varepsilon > 0 \). We wish to prove that there exists \( \delta > 0 \) such that, for all \( x \in D \), we have:
   \[
   \left[ 0 < |x - x_0| < \delta \right] \Rightarrow \left[ |g(f(x))| - |g(L)| < \varepsilon \right].
   \]

   Choose \( \gamma > 0 \) such that, for all \( y \in E \), we have:
   \[
   [y-L] < \gamma \Rightarrow ||g(y)| - |g(L)|| < \varepsilon.
   \]

   Choose \( \delta > 0 \) such that, for all \( x \in D \), we have:
   \[
   0 < |x - x_0| < \delta \Rightarrow |(f(x)) - L| < \gamma.
   \]
Given \( x \in D \). Assume that \( 0 < |x-x_0| < \delta \). We wish to show that \( |g(f(x))| - |g(L)| < \varepsilon \).

By the choice of \( \delta \), we have: \( |f(x)| - L| < \gamma \). Let \( y := f(x) \). Then \( |y - L| < \gamma \). Also, \( y \in \text{im}[f] \subseteq E \). Then, by the choice of \( \gamma \), we get: \( |g(y)| - |g(L)| < \varepsilon \). That is, we get \( |g(f(x))| - |g(L)| < \varepsilon \), as desired.

---

35. Define \( f : \mathbb{R}\{0\} \to \mathbb{R} \) by \( f(x) = 1/x \). Show that \( f(x) \to 1/3 \), as \( x \to 3 \).

**Proof:** Given \( \varepsilon > 0 \). We wish to prove that there exists \( \delta > 0 \) such that, for all \( x \in D \),

\[
0 < |x - 3| < \delta \quad \Rightarrow \quad |(f(x)) - (1/3)| < \varepsilon.
\]

Let \( \delta := \min\{1, 6\varepsilon\} \). Given \( x \in D \). Assume that \( 0 < |x - 3| < \delta \). We wish to prove that \( |(f(x)) - (1/3)| < \varepsilon \).

We have \( (f(x)) - (1/3) = (1/x) - (1/3) = (3-x)/(3x) \), so

\[
|(f(x)) - (1/3)| = |3-x| \cdot |1/3| \cdot |1/|x||.
\]

We have \( |3-x| = |x-3| < \delta \leq 6\varepsilon \). Because \( 3-x \leq |x-3| < \delta \leq 1 \), we have \( 2 \leq x \), and so \( |x| = x \geq 2 \), and so \( 1/(|x|) \leq 1/2 \). Then \( |(f(x)) - (1/3)| < 6\varepsilon \cdot 1/3 \cdot 1/2 = \varepsilon \).

---

36. Define \( g : \mathbb{R}\{0\} \to \mathbb{R} \) by \( g(x) = x[\sin(1/x)] \). Show that \( g(x) \to 0 \), as \( x \to 0 \).

**Proof:** Given \( \varepsilon > 0 \). We wish to prove that there exists \( \delta > 0 \) such that, for all \( x \in D \),

\[
0 < |x - 0| < \delta \quad \Rightarrow \quad |(g(x)) - 0| < \varepsilon.
\]

Let \( \delta := \varepsilon \). Given \( x \in D \). Assume that \( 0 < |x - 0| < \delta \). We wish to show that \( |(g(x)) - 0| < \varepsilon \).

We have \( |x| < \delta \) and \( x \neq 0 \). Because \( -1 \leq \sin(1/x) \leq 1 \), we get \( |\sin(1/x)| \leq 1 \). Then \( |(g(x)) - 0| = |g(x)| = (|x|) \cdot (|\sin(1/x)|) \leq (|x|) \cdot 1 = |x| < \delta = \varepsilon \), as desired.

---

37. Let \( \{a_n\}_{n \in \mathbb{N}} \) be a sequence in \( \mathbb{R} \) and let \( x \in \mathbb{R} \). Assume that \( a_n \to x \), as \( n \to \infty \). Let \( \{b_n\}_{n \in \mathbb{N}} \) be a subsequence of \( \{a_n\}_{n \in \mathbb{N}} \). Show that \( b_n \to x \), as \( n \to \infty \).

**Proof:** Given \( \varepsilon > 0 \). We wish to show that there exists \( N \in \mathbb{N} \) such that, for all integers \( n \geq N \), we have \( |b_n - x| < \varepsilon \).

Choose \( N \in \mathbb{N} \) such that, for all integers \( n \geq N \), we have \( |a_n - x| < \varepsilon \). Given an integer \( n \geq N \). We wish to prove that \( |b_n - x| < \varepsilon \).

Since \( \{b_m\}_{m \in \mathbb{N}} \) is a subsequence of \( \{a_m\}_{m \in \mathbb{N}} \), choose an integer \( q \geq n \) such that \( a_q = b_n \). Since \( q \geq n \geq N \), it follows, from the choice of \( N \), that \( |a_q - x| < \varepsilon \). Then \( |b_n - x| = |a_q - x| < \varepsilon \), as desired.

---

32. Compute \( \lim_{n \to \infty} \left( (\sqrt[n]{n+1}) - (\sqrt[n]{n}) \right) \left[ \sqrt[n]{n^2} \right] \).

**Hint:** \( (b^{1/3} - a^{1/3})(b^{2/3} + b^{1/3}a^{1/3} + a^{2/3}) = b - a \).

**Answer:** For all \( n \in \mathbb{N} \), using \( b = n + 1 \) and \( a = n \) in the hint, we get

\[
((n+1)^{1/3} - n^{1/3})((n+1)^{2/3} + (n+1)^{1/3}n^{1/3} + n^{2/3}) = (n+1) - n,
\]
so, since \((\sqrt[3]{n+1}) - (\sqrt[3]{n}) = (n+1)^{1/3} - n^{1/3}\) and since \((n+1) - n = 1\), we get

\[
(\sqrt[3]{n+1}) - (\sqrt[3]{n}) = \frac{1}{(n+1)^{2/3} + (n+1)^{1/3}n^{1/3} + n^{2/3}}.
\]

Then, for all \(n \in \mathbb{N}\), we have

\[
[(\sqrt[3]{n+1}) - (\sqrt[3]{n})] [\sqrt[3]{n^2}] = \frac{n^{2/3}}{(n+1)^{2/3} + (n+1)^{1/3}n^{1/3} + n^{2/3}} \cdot \frac{n^{-2/3}}{n^{-2/3}}
\]

\[
= \frac{1}{(n+1)^{2/3}n^{-2/3} + (n+1)^{1/3}n^{-1/3} + 1}
\]

\[
= \frac{1}{[1 + (1/n)^{2/3} + [1 + (1/n)]^{1/3} + 1}.
\]

As \(n \to \infty\), we have \(1/n \to 0\), so

\[
[(\sqrt[3]{n+1}) - (\sqrt[3]{n})] [\sqrt[3]{n^2}] \to \frac{1}{[1+0]^{2/3} + [1+0]^{1/3} + 1} = \frac{1}{3}.
\]

---

33. Let \(\{a_n\}_{n \in \mathbb{N}}\) be a bounded sequence in \(\mathbb{R}\). Show that there exists a subsequence \(\{b_n\}_{n \in \mathbb{N}}\) of \(\{a_n\}_{n \in \mathbb{N}}\) such that \(\{b_n\}_{n \in \mathbb{N}}\) is convergent.

**Proof:** Because a bounded semi-monotone sequence is convergent, and because any subsequence of a bounded sequence is again bounded, the result follows from the following lemma. QED

**Lemma.** Let \(\{a_n\}_{n \in \mathbb{N}}\) be a bounded sequence in \(\mathbb{R}\). Then there exists a subsequence \(\{b_n\}_{n \in \mathbb{N}}\) of \(\{a_n\}_{n \in \mathbb{N}}\) such that \(\{b_n\}_{n \in \mathbb{N}}\) is semi-monotone.

**Proof:** We wish to show that there is a strictly increasing sequence \(\{k_n\}_{n \in \mathbb{N}}\) in \(\mathbb{N}\) such that \(\{a_{k_n}\}_{n \in \mathbb{N}}\) is semi-monotone.

Let \(S := \{n \in \mathbb{N} | a_n \geq a_{n+1} \geq a_{n+2} \geq \cdots\}\).

**Special case:** \(S\) is infinite. **Proof in special case:** Define a function \(F : S \to S\) by \(F(s) = \min\{t \in S | t \geq s + 1\}\). Let \(k_1 := \min S\). For all \(n \in \mathbb{N}\), let \(k_{n+1} := F^n(k_1)\). For all \(n \in \mathbb{N}\), we have \(k_{n+1} \geq k_n + 1\). Thus \(\{k_n\}_{n \in \mathbb{N}}\) is strictly increasing. It suffices to show that \(\{a_{k_n}\}_{n \in \mathbb{N}}\) is semi-decreasing. Given \(n \in \mathbb{N}\). We wish to show that \(a_{k_n} \geq a_{k_{n+1}}\).

Because \(k_n \in S\), we have \(a_{k_n} \geq a_{k_{n+1}} \geq a_{k_{n+2}} \geq \cdots\). So, since \(k_{n+1} \geq k_n\), this yields \(a_{k_n} \geq a_{k_{n+1}}\), as desired. **End of proof in special case.**

We may therefore assume that \(S\) is finite. Let \(N := 1 + (\max S)\). Then, for all integers \(n \geq N\), we have \(n \notin S\), so there exists an integer \(q \geq n + 1\) such that \(a_n < a_q\).

Let \(T := \{N, N+1, N+2, \ldots\}\). Define \(G : T \to T\) by \(G(k) = \min\{q \geq k+1 | a_k < a_q\}\). For all \(k \in \mathbb{N}\), we have \(G(k) \geq k + 1\) and \(a_k < a_{G(k)}\).

Let \(k_1 := N\). For all \(n \in \mathbb{N}\), let \(k_{n+1} := G^n(k_1)\). Then, for all \(n \in \mathbb{N}\), we have \(k_{n+1} \geq k_n + 1\). Thus \(\{k_n\}_{n \in \mathbb{N}}\) is strictly increasing. It suffices to show that \(\{a_{k_n}\}_{n \in \mathbb{N}}\) is strictly increasing. Fix \(n \in \mathbb{N}\). We wish to show that \(a_{k_n} < a_{k_{n+1}}\).
We have \( a_{k_n} < a_{G(k_n)} \), so, as \( G(k_n) = k_{n+1} \), this yields \( a_{k_n} < a_{k_{n+1}} \), as desired. QED

34. Find a sequence \( \{a_n\}_{n \in \mathbb{N}} \) in \( \mathbb{R} \) such that \( \{a_n\}_{n \in \mathbb{N}} \) is NOT convergent, and such that for all convergent subsequences \( \{b_n\}_{n \in \mathbb{N}} \) of \( \{a_n\}_{n \in \mathbb{N}} \), \( b_n \to 0 \), as \( n \to \infty \).

**Answer:** Define \( \{a_n\}_{n \in \mathbb{N}} \) by \( a_n = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ n, & \text{if } n \text{ is even.} \end{cases} \)

28. Let \( a, b \in \mathbb{R} \) and let \( \varepsilon > 0 \). Assume that \( a < b \). Show that there exists \( m \in \mathbb{N} \) such that \( \frac{b - a}{2^m} < \varepsilon \).

**Proof:** **Claim:** For all \( m \in \mathbb{N} \), \( m < 2^m \). **Proof of claim:** We have \( 1 < 2 = 2^1 \). Given \( m \in \mathbb{N} \), we assume \( m < 2^m \) and, by induction, we wish to prove that \( m + 1 < 2^{m+1} \).

We have \( 1 \leq m \), so \( m + 1 \leq 2m \). Since \( m < 2^m \), we get \( 2m < 2 \cdot 2^m \). Then \( m + 1 \leq 2m < 2 \cdot 2^m = 2^{m+1} \). End of proof of claim.

Because \( \mathbb{N} \) is not bounded above, choose \( m \in \mathbb{N} \) such that \( m > \frac{b - a}{\varepsilon} \). We wish to prove that \( \frac{b - a}{2^m} < \varepsilon \).

By the claim, \( m < 2^m \), so \( \frac{b - a}{2^m} < \frac{b - a}{m} \). Because \( m > \frac{b - a}{\varepsilon} \), we get \( \frac{b - a}{m} < \varepsilon \).

Therefore, we have \( b - a < \varepsilon \).

29. Let \( \{a_n\}_{n \in \mathbb{N}} \) be a Cauchy sequence of real numbers and let \( \{b_n\}_{n \in \mathbb{N}} \) be a subsequence of \( \{a_n\}_{n \in \mathbb{N}} \). Let \( x \in \mathbb{R} \). Assume that \( b_n \to x \), as \( n \to \infty \). Show that \( a_n \to x \), as \( n \to \infty \).

**Proof:** Given \( \varepsilon > 0 \), we wish to prove that there exists \( N \in \mathbb{N} \) such that, for all integers \( n \geq N \), we have: \( |a_n - x| < \varepsilon \).

Choose \( N_1 \in \mathbb{N} \) such that, for all \( n \geq N_1 \), we have: \( |b_n - x| < \varepsilon/2 \). Choose \( N_2 \in \mathbb{N} \) such that, for all \( m, n \geq N_2 \), we have: \( |a_m - a_n| < \varepsilon/2 \). Let \( N := \max\{N_1, N_2\} \). Given an integer \( n \geq N \), we wish to prove: \( |a_n - x| < \varepsilon \).

Because \( n \geq N \geq N_1 \), we have \( |b_n - x| < \varepsilon/2 \). Because \( \{b_k\}_{k \in \mathbb{N}} \) be a subsequence of \( \{a_k\}_{k \in \mathbb{N}} \), choose an integer \( q \geq n \) such that \( b_n = a_q \). We have \( n \geq N \geq N_2 \) and \( q \geq n \geq N \geq N_2 \), so \( |a_q - a_n| < \varepsilon/2 \). So, since \( b_n = a_q \), we get \( |b_n - a_n| < \varepsilon/2 \).

Then \( |a_n - x| \leq |a_n - b_n| + |b_n - x| < (\varepsilon/2) + (\varepsilon/2) = \varepsilon \).

30. Let \( f, g : \mathbb{R} \to \mathbb{R} \) be two functions. Let \( u, v, a \in \mathbb{R} \). Assume that \( f(x) \to u \), as \( x \to a \). Assume that \( g(x) \to v \), as \( x \to a \). Prove that \( [f(x)] + [g(x)] \to u + v \), as \( x \to a \).

**Proof:** Let \( \varepsilon > 0 \) be given. We wish to prove that there exists \( \delta > 0 \) such that, for all \( x \in (a - \delta, a + \delta) \setminus \{a\} \), we have: \( |[(f(x)) + (g(x))] - [u + v]| < \varepsilon \).

Choose \( \delta_1 > 0 \) such that, for all \( x \in (a - \delta_1, a + \delta_1) \setminus \{a\} \), we have: \( |(f(x)) - u| < \varepsilon/2 \).

Choose \( \delta_2 > 0 \) such that, for all \( x \in (a - \delta_2, a + \delta_2) \setminus \{a\} \), we have: \( |(g(x)) - v| < \varepsilon/2 \).

Let \( \delta := \min\{\delta_1, \delta_2\} \). Given \( x \in (a - \delta, a + \delta) \setminus \{a\} \).
We wish to prove that \[|((f(x)) + (g(x))) - (u + v)| < \varepsilon.\]

Because \(\delta \leq \delta_1\), it follows that \((a - \delta, a + \delta) - \{a\} \subseteq (a - \delta_1, a + \delta_1) - \{a\}\), and so \(x \in (a - \delta_1, a + \delta_1) - \{a\}\). Then \(|(f(x)) - u| < \varepsilon/2\).

Because \(\delta \leq \delta_2\), it follows that \((a - \delta, a + \delta) - \{a\} \subseteq (a - \delta_2, a + \delta_2) - \{a\}\), and so \(x \in (a - \delta_2, a + \delta_2) - \{a\}\). Then \(|(g(x)) - v| < \varepsilon/2\).

We have \(|((f(x)) + (g(x))) - (u + v)| = |(f(x)) - u| + |(g(x)) - v|\). It follows that \(|((f(x)) + (g(x))) - (u + v)| \leq |(f(x)) - u| + |(g(x)) - v| < \varepsilon/2 + \varepsilon/2 = \varepsilon|.

31. Find two functions \(f, g : \mathbb{R} \to \mathbb{R}\) such that all of the following are true:
   (i) \(f(x) \to 2\), as \(x \to 1\).
   (ii) \(g(x) \to 3\), as \(x \to 2\).
   (iii) NOT \([g(f(x)) \to 3\), as \(x \to 1\)]

Answer: Define \(f : \mathbb{R} \to \mathbb{R}\) by \(f(x) = 2\). Define \(g : \mathbb{R} \to \mathbb{R}\) by \(g(x) = \begin{cases} 3, & \text{if } x \neq 2; \\ 4, & \text{if } x = 2. \end{cases}\)

21. On a real number line, graph:
   all real numbers \(x\) such that \(|x - 3| < 1\).

Answer: Too much trouble to insert a graph. See me if you have questions.

22. On a real number line, graph:
   all real numbers \(x\) such that \(2 < x < 4\).

Answer: Too much trouble to insert a graph. See me if you have questions.

23. On a real number line, graph:
   all real numbers \(x\) such that \(0 < |x - 3| < 1\).

Answer: Too much trouble to insert a graph. See me if you have questions.

24. On a real number line, graph:
   all real numbers \(x\) such that \([2 < x < 4) \text{ and } (x \neq 3)\].

Answer: Too much trouble to insert a graph. See me if you have questions.

25. Let \(\{a_n\}_{n \in \mathbb{N}}\) and \(\{b_n\}_{n \in \mathbb{N}}\) be two sequences of real numbers. Let \(c \in \mathbb{R}\). Assume that \(a_n \to c\), as \(n \to \infty\), and that \(b_n \to c\), as \(n \to \infty\). Let \(\{c_n\}_{n \in \mathbb{N}}\) be the sequence given by

\[
c_n = \begin{cases} 
a_{(n+1)/2}, & \text{if } n \text{ is odd;} \\ 
b_{n/2}, & \text{if } n \text{ is even.} 
\end{cases}
\]

Show that \(c_n \to c\), as \(n \to \infty\).

Proof: Given \(\varepsilon > 0\). Want: \(\exists N \in \mathbb{N}\) such that \(\forall\) integers \(n \geq N\), we have: \(|c_n - c| < \varepsilon\).

Choose \(N_1 \in \mathbb{N}\) such that \(\forall\) integers \(n \geq N_1\), we have: \(|a_n - c| < \varepsilon\).
Choose \( N_2 \in \mathbb{N} \) such that \( \forall \) integers \( n \geq N_2 \), we have: \( |b_n - c| < \varepsilon \).

Let \( N := \max\{2N_1 - 1, 2N_2\} \). Given an integer \( n \geq N \). Want \( |c_n - c| < \varepsilon \).

**Special case:** \( n \) is odd. Proof in special case: Let \( m := (n + 1)/2 \). Then \( c_n = a_m \). We wish to prove \( |a_m - c| < \varepsilon \).

Since \( 2m - 1 = n \geq N \geq 2N_1 - 1 \), we see that \( m \geq N_1 \). Then, by definition of \( N_1 \), we have \( |a_m - c| < \varepsilon \), as desired. **End of proof in special case.**

We now assume that \( n \) is even. Let \( m := n/2 \). Then \( c_n = b_m \). We wish to prove: \( |b_m - c| < \varepsilon \).

Since \( 2m = n \geq N \geq 2N_2 \), we see that \( m \geq N_2 \). Then, by definition of \( N_2 \), we have \( |b_m - c| < \varepsilon \), as desired.

26. Let \( \{a_n\}_{n \in \mathbb{N}} \) and \( \{b_n\}_{n \in \mathbb{N}} \) be two sequences of real numbers. Let \( x \in \mathbb{R} \). Assume that \( a_n \to x \), as \( n \to \infty \), and that \( b_n \to \infty \), as \( n \to \infty \). Show that \( a_n + b_n \to \infty \), as \( n \to \infty \).

**Proof:** Given \( M \in \mathbb{R} \). Want: \( \exists N \in \mathbb{N} \) such that \( \forall \) integers \( n \geq N \), we have: \( a_n + b_n > M \).

Choose \( N_1 \in \mathbb{N} \) such that, \( \forall \) integers \( n \geq N_1 \), we have: \( |a_n - x| < 1 \). Then, \( \forall \) integers \( n \geq N_1 \), we have: \( a_n > x - 1 \).

Choose \( N_2 \in \mathbb{N} \) such that, \( \forall \) integers \( n \geq N_1 \), we have: \( b_n > M - (x - 1) \).

Let \( N := \max\{N_1, N_2\} \).

Given an integer \( n \geq N \). Want: \( a_n + b_n > M \).

Since \( n \geq N \geq N_1 \), we get \( a_n > x - 1 \). Since \( n \geq N \geq N_2 \), we get \( b_n > M - (x - 1) \). Then \( a_n + b_n > (x - 1) + M - (x - 1) = M \), as desired.

27. Let \( \{a_n\}_{n \in \mathbb{N}} \) be a sequence of real numbers. Let \( x \in \mathbb{R} \). Let \( \{c_n\}_{n \in \mathbb{N}} \) be the sequence given by

\[
c_n = \begin{cases} 
a_{(n+1)/2}, & \text{if } n \text{ is odd;} 
x, & \text{if } n \text{ is even.}
\end{cases}
\]

Assume that \( \{c_n\}_{n \in \mathbb{N}} \) is Cauchy. Show that \( a_n \to x \), as \( n \to \infty \).

**Proof:** Given \( \varepsilon > 0 \). Want: \( \exists N \in \mathbb{N} \) such that, \( \forall \) integers \( n \geq N \), we have: \( |a_n - x| < \varepsilon \).

Choose \( N \in \mathbb{N} \) such that, \( \forall \) integers \( m, n \geq N \), we have: \( |c_m - c_n| < \varepsilon \).

Given an integer \( n \geq N \). Want: \( |a_n - x| < \varepsilon \).

Let \( m := 2n - 1 \). Then \( c_m = a_n \) and \( c_{m+1} = x \). We have \( N - 1 \geq 0 \), so \( N + (N - 1) \geq N \).

Then \( m = 2n - 1 \geq 2N - 1 = N + (N - 1) \geq N \). Then \( m \geq N \) and \( m + 1 \geq N \). Then, by definition of \( N \), we have \( |c_m - c_{m+1}| < \varepsilon \). Then \( |a_n - x| = |c_m - c_{m+1}| < \varepsilon \), as desired.

17. Let \( n \in \mathbb{Z} \). Show both of the following:

(i) If \( n \) is even, then \( n^2 \) is even.

(ii) If \( n \) is odd, then \( n^2 \) is odd.

**Proof of (i):** Assume \( n \) is even. Want: \( n^2 \) is even.

Choose \( k \in \mathbb{Z} \) such that \( n = 2k \).

Then \( n^2 = (2k)^2 = 2(2k^2) \), so \( n^2 \) is even.

**Proof of (ii):** Assume \( n \) is odd. Want: \( n^2 \) is odd.
Choose $k \in \mathbb{Z}$ such that $n = 2k + 1$. Then $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, so $n^2$ is odd.

18. Let $S \subseteq \mathbb{R}$ and let $a_0, b_0 \in \mathbb{R}$. Assume that $a_0 \geq S$ and that $b_0 \geq S$. Assume, for all $a \geq S$, that $a \geq a_0$. Assume, for all $b \geq S$, that $b \geq b_0$. Prove that $a_0 = b_0$.

Proof: For all $a \geq S$, we have $a \geq a_0$. So, since $b_0 \geq S$, we have $b_0 \geq a_0$. It therefore suffices to show that $a_0 \geq b_0$.

For all $b \geq S$, we have $b \geq b_0$. So, since $a_0 \geq S$, we have $a_0 \geq b_0$.

19. Find a nonempty set $I$ of intervals such that
   
   (i) for all $I, J \in I$, either $I \subseteq J$ or $J \subseteq I$; and
   
   (ii) $\bigcap_{I \in I} I = \emptyset$.

Answer: Let $I := \{(0, 1/n) \mid n \in \mathbb{N}\}$.

20. An interval $I$ is compact if there exist $a, b \in \mathbb{R}$ such that $a < b$ and such that $I = [a, b]$. Let $I$ be a nonempty set of compact intervals such that
    
    for all $I, J \in I$, either $I \subseteq J$ or $J \subseteq I$.

Show that $\bigcap_{I \in I} I \neq \emptyset$.

Answer: For all $I \in I$, let $a_I := \min I$ and let $c_I := \max I$. Then, for all $I \in I$, we have: $I = [a_I, c_I]$. Let $A := \{a_I \mid I \in I\}$. Then $A \neq \emptyset$.

Claim 1: For all $I \in I$, we have $A \leq c_I$. Proof of Claim 1: Given $I \in I$. We wish to prove that $A \leq c_I$. That is, we wish to prove, for all $a \in A$, that $a \leq c_I$.

Given $a \in A$. We wish to prove that $a \leq c_I$. Choose $J \in I$ such that $a = a_J$. We wish to prove that $a_J \leq c_I$.

Either $J \subseteq I$ or $I \subseteq J$. If $J \subseteq I$, then $a_J \in [a_J, c_J] = J \subseteq I = [a_I, c_I]$, and so $a_J \leq c_I$, as desired. We may therefore assume that $I \subseteq J$.

Then $c_I \in [a_I, c_I] = I \subseteq J = [a_J, c_J]$. Then $c_I \geq a_J$, as desired. End of proof of Claim 1.

By Claim 1, $A$ is bounded above. Let $b := \sup A$.

Claim 2: For all $I \in I$, we have $a_I \leq b \leq c_I$. Proof of Claim 2: Given $I \in I$. We wish to prove that $a_I \leq b \leq c_I$.

Since $b = \sup A$, we have $A \leq b$. So, since $a_I \in A$, we get $a_I \leq b$. By Claim 1, $A \leq c_I$. So, since $b = \sup A$, we get $b \leq c_I$. Then $a_I \leq b \leq c_I$. End of proof of Claim 2.

By Claim 2, for all $I \in I$, we have $b \in [a_I, c_I] = I$. Then $b \in \bigcap_{I \in I} I$. Then $\bigcap_{I \in I} I \neq \emptyset$.

12. Display a bijection from $\{1, 2, 3, \ldots\}$ onto $\{-3, -5, -7, \ldots\}$.

Answer: Define $f : \{1, 2, 3, \ldots\} \to \{-3, -5, -7, \ldots\}$ by $f(n) = -2n - 1$. Then $f$ is a bijection from $\{1, 2, 3, \ldots\}$ onto $\{-3, -5, -7, \ldots\}$.
13. Display a bijection from $\mathbb{N}$ onto $\mathbb{Z}$.

Answer: Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even;} \\ (1-n)/2, & \text{if } n \text{ is odd.} \end{cases}$$

Then $f$ is a bijection from $\mathbb{N}$ onto $\mathbb{Z}$.

14. Display a bijection from $[0, 1]$ onto $[0, 1)$.

Answer: Let $S := \{1, 1/2, 1/3, \ldots\}$. Define $f : S \to S$ by $f(x) = 1/((1/x) + 1)$. Define $g : [0, 1] \to [0, 1)$ by

$$g(x) = \begin{cases} f(x), & \text{if } x \in S; \\ x, & \text{if } x \notin S. \end{cases}$$

Then $g$ is a bijection from $[0, 1]$ onto $[0, 1)$.

15. By $\mathbb{Q}[x]$, we denote the set of polynomials in $x$ with rational coefficients. Show that $\mathbb{Q}[x]$ is countable.

Proof: For all $n \in \mathbb{N}$, $\mathbb{Q}^n$ is a finite product of countable sets, and is therefore countable. Let $Y := [\mathbb{Q}] \cup [\mathbb{Q}^2] \cup [\mathbb{Q}^3] \cup \cdots$. Then $Y$ is a countable union of countable sets, and is therefore countable. Fix a surjection $f : \mathbb{N} \to Y$.

Define $g : Y \to \mathbb{Q}[x]$ by $g(q_1, \ldots, q_n) = q_1x^{n-1} + q_2x^{n-2} + \cdots + q_{n-2}x^2 + q_{n-1}x + q_n$. Then $g$ is a surjection from $Y$ onto $\mathbb{Q}[x]$. Then $g \circ f : \mathbb{N} \to \mathbb{Q}[x]$ is a surjection from $\mathbb{N}$ onto $\mathbb{Q}[x]$. Then $\mathbb{Q}[x]$ is countable.

16. The zero polynomial is denoted $0 \in \mathbb{Q}[x]$. A number $\alpha \in \mathbb{R}$ is said to be real algebraic if there exists $p(x) \in (\mathbb{Q}[x])\{0\}$ such that $p(\alpha) = 0$. Let $A$ denote the set of all real algebraic numbers. Show that $A$ is countable.

Proof: Let $C := (\mathbb{Q}[x])\{0\}$. By Problem 15, $\mathbb{Q}[x]$ is countable, and so $C$ is a subset of a countable set, and is therefore countable.

For all $p(x) \in C$, let $R_{p(x)} := \{\alpha \in \mathbb{R} | p(\alpha) = 0\}$. For all $p(x) \in C$, the set $R_{p(x)}$ is finite, and therefore countable. Then, because $A = \bigcup_{p(x) \in C} R_{p(x)}$, we see that $A$ is a countable union of countable sets, and is therefore countable.

6. Let $X$ and $Y$ be sets and let $f : X \to Y$ be a function. Let $A, B \subseteq X$. Show that:

$$f(A \cup B) = (f(A)) \cup (f(B)).$$

Lemma 1: Let $X$ and $Y$ be sets and let $f : X \to Y$ be a function. Let $P$ and $Q$ be subsets of $X$. Assume that $P \subseteq Q$. Then $f(P) \subseteq f(Q)$. 

Proof of Lemma 1: Given $y \in f(P)$. Want: $y \in f(Q)$.

Choose $x \in P$ such that $y = f(x)$. We have $x \in P \subseteq Q$. Then $y = f(x) \in f(Q)$, as desired. End of proof of Lemma 1.

Lemma 2: Let $U$, $V$ and $W$ be sets. Assume that $U \subseteq W$ and that $V \subseteq W$. Then $U \cup V \subseteq W$.

Proof of Lemma 2: Given $y \in U \cup V$. Want: $y \in W$.

We know: $(y \in U) \Rightarrow (y \in W)$.
Because $V \subseteq W$, we know: $(y \in V) \Rightarrow (y \in W)$.
So, because

\[(P \text{ or } Q) \quad \text{and} \quad (P \Rightarrow R) \quad \text{and} \quad (Q \Rightarrow R) \quad \Rightarrow \quad R,
\]

we conclude that $y \in W$, as desired. End of proof of Lemma 2.

Main proof: Proof of $\subseteq$: Let $y \in f(A \cup B)$. Want: $y \in (f(A)) \cup (f(B))$.

Choose $x \in A \cup B$ such that $y = f(x)$.
We know: $(x \in A) \text{ or } (x \in B)$.
We know: $[x \in A] \Rightarrow [y = f(x) \in f(A) \subseteq (f(A)) \cup (f(B))]$.
We know: $[x \in B] \Rightarrow [y = f(x) \in f(B) \subseteq (f(A)) \cup (f(B))]$.
So, because

\[\{(P \text{ or } Q) \quad \text{and} \quad (P \Rightarrow R) \quad \text{and} \quad (Q \Rightarrow R)\} \quad \Rightarrow \quad R,
\]

we conclude that $y \in (f(A)) \cup (f(B))$, as desired. End of proof of $\subseteq$.

Proof of $\supseteq$: Because $A \subseteq A \cup B$, by Lemma 1, we see that $f(A) \subseteq f(A \cup B)$. Because $B \subseteq A \cup B$, by Lemma 1, we see that $f(B) \subseteq f(A \cup B)$. Because $f(A) \subseteq f(A \cup B)$ and $f(B) \subseteq f(A \cup B)$, by Lemma 2, we see that $(f(A)) \cup (f(B)) \subseteq f(A \cup B)$, as desired. End of proof of $\supseteq$.

7. Find sets $X$ and $Y$, a function $f : X \to Y$ and subsets $A, B \subseteq X$ such that:

\[f(A \cap B) \neq (f(A)) \cap (f(B)).\]

Answer: Let $X := \mathbb{R}$, let $Y := \mathbb{R}$ and let $f : X \to Y$ be defined by $f(x) = x^2$. Let $A := \{-1\}$ and let $B := \{1\}$.

Then $A \cap B = \emptyset$, so $f(A \cap B) = \emptyset$. On the other hand, $f(A) = \{1\} = f(B)$, so $(f(A)) \cap (f(B)) = \{1\}$.

Then $f(A \cap B) = \emptyset \neq \{1\} = (f(A)) \cap (f(B))$.

8. Let $f : X \to Y$ and $g : Y \to Z$. Assume $g \circ f : X \to Z$ is 1-1. Show $f : X \to Y$ is 1-1.

Proof: Let $x', x'' \in X$ and assume that $(g \circ f)(x') = (g \circ f)(x'')$. Want: $x' = x''$.
Let \( y' := f(x') \) and let \( y'' := f(x'') \). Then
\[
 g(y') = g(f(x')) = (g \circ f)(x') = (g \circ f)(x'') = g(f(x'')) = g(y'').
\]
So, since \( g \) is 1-1, we conclude that \( y' = y'' \).
Then \( f(x') = y' = y'' = f(x'') \). So, since \( f \) is 1-1, we conclude that \( x' = x'' \).

9. Prove by induction: For all integers \( n \geq 1 \),
\[
 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n+1)(2n-1)}{3}.
\]

**Proof:** We have \( 1^2 = \frac{1 \cdot (2 \cdot 1 + 1)(2 \cdot 1 - 1)}{3} \). Fix an integer \( k \geq 1 \), and make the induction assumption that
\[
 (*) \quad 1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 = \frac{k(2k+1)(2k-1)}{3}.
\]
By induction, it suffices to show that
\[
 1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 + (2k+1)^2 = \frac{(k+1)(2(k+1) + 1)(2(k+1) - 1)}{3}.
\]
Let \( S := 1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 \). Then we wish to prove:
\[
 S + (2k+1)^2 = \frac{(k+1)(2k+2 + 1)(2k+2 - 1)}{3}.
\]
By \(*\), we have: \( S = \frac{k(2k+1)(2k-1)}{3} \). So we wish to prove:
\[
 \frac{k(2k+1)(2k-1)}{3} + (2k+1)^2 = \frac{(k+1)(2k+3)(2k+1)}{3}.
\]
Multiplying by 3 and dividing by \( 2k+1 \), we wish to prove:
\[
 k(2k-1) + (6k+3) = (k+1)(2k + 3).
\]
We have
\[
 k(2k-1) + (6k+3) = 2k^2 - k + 6k + 3
 = 2k^2 + 5k + 3
 = (k+1)(2k + 3).
\]

10. Prove by modified induction: For all integers \( n \geq 20 \), \( 200n + 100 < 2^n \).

**Proof:** We have \( 200 \cdot 20 + 100 = 4100 < (1000)^2 < (1024)^2 = (2^{10})^2 = 2^{20} \). Fix an integer \( k \geq 20 \), and make the induction assumption that
By modified induction, it suffices to show that $200(k + 1) + 100 < 2^{k+1}$. Dividing by 2, we wish to prove that $100(k + 1) + 50 < 2^k$. That is, we wish to prove that $100k + 150 < 2^k$.

Because $k \geq 20$, it follows that $50 < 100k$. So, adding $100k + 100$ to both sides, we get $100k + 150 < 200k + 100$. So, from (*), we get $100k + 150 < 2^k$, as desired.

11. Prove by modified induction: For all integers $n \geq 20$, $100n^2 < 2^n$.

**Proof:** We have $100 \cdot (20^2) = 40000 < (1000)^2 < (1024)^2 = (2^{10})^2 = 2^{20}$. Fix an integer $k \geq 20$, and make the induction assumption that

\[(*) \quad 100k^2 < 2^k.\]

By modified induction, it suffices to show that $100(k + 1)^2 < 2^{k+1}$. That is, we wish to prove that $100k^2 + 200k + 100 < 2^{k+1}$.

By Problem 10, we have $200k + 100 < 2^k$. Combining this with (*), we conclude that $100k^2 + 200k + 100 < 2^k + 2^k = 2 \cdot (2^k) = 2^{k+1}$, as desired.

1. Let $A$ and $B$ be sets. Show that: $A = B \Rightarrow B \subseteq A$.

**Proof:** Assume $A = B$. We wish to show that $B \subseteq A$.

Given $a_0 \in A$. We wish to show that $a_0 \in B$.

Since $A = B$, we know, $\forall a$, that: $a \in A \iff a \in B$. So, as $a_0 \in A$, we get $a_0 \in B$, as desired.

2. Let $A$, $B$ and $C$ be sets. Show that: $(A \cap B) \cap C = A \cap (B \cap C)$.

**Proof:** We must prove both $\subseteq$ and $\supseteq$.

We first prove $\subseteq$. Given $x \in (A \cap B) \cap C$. We wish to show that $x \in A \cap (B \cap C)$.

Since $x \in (A \cap B) \cap C$, we know that $x \in A \cap B$ and that $x \in C$. Since $x \in A \cap B$, we know that $x \in A$ and that $x \in B$. Since $x \in B$ and $x \in C$, we know that $x \in B \cap C$. Since $x \in A$ and $x \in B \cap C$, we know that $x \in A \cap (B \cap C)$, concluding the proof of $\subseteq$.

It remains to prove $\supseteq$. Given $x \in A \cap (B \cap C)$. We wish to show that $x \in (A \cap B) \cap C$.

Since $x \in A \cap (B \cap C)$, we know that $x \in A$ and that $x \in B \cap C$. Since $x \in B \cap C$, we know that $x \in B$ and that $x \in C$. Since $x \in A$ and $x \in B$, we know that $x \in A \cap B$. Since $x \in A \cap B$ and $x \in C$, we know that $x \in (A \cap B) \cap C$, concluding the proof of $\supseteq$.

3. Let $A$, $B$ and $C$ be sets. Show that: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

**Proof:** We must prove both $\subseteq$ and $\supseteq$.

We first prove $\subseteq$. Given $x \in A \cup (B \cap C)$. We wish to show that $x \in (A \cup B) \cap (A \cup C)$.

Because $x \in A \cup (B \cap C)$, it follows that: $(x \in A)$ or $(x \in B \cap C)$. Therefore, we have $(x \in A)$ or $((x \in B)$ and $(x \in C))$. So, since

$$ P \lor (Q \land R) \Rightarrow (P \lor Q) \land (P \lor R),$$

we get: $((x \in A)$ or $(x \in B))$ and $((x \in A)$ or $(x \in C))$. So $(x \in A \cup B)$ and $(x \in A \cup C)$. Then $x \in (A \cup B) \cap (A \cup C)$, concluding the proof of $\subseteq$. 
It remains to prove $\supseteq$. Let $x \in (A \cup B) \cap (A \cup C)$ be given. We wish to show that $x \in A \cup (B \cap C)$.

Since $x \in (A \cup B) \cap (A \cup C)$, we have: $(x \in A \cup B)$ and $(x \in A \cup C)$. Therefore, we have $((x \in A) \text{ or } (x \in B))$ and $((x \in A) \text{ or } (x \in C))$. So, since

$$P \text{ or } (Q \text{ and } R) \iff (P \text{ or } Q) \text{ and } (P \text{ or } R),$$

we get: $(x \in A) \text{ or } ((x \in B) \text{ and } (x \in C))$. So $x \in A$ or $(x \in B \cap C)$. Then $x \in A \cup (B \cap C)$, concluding the proof of $\supseteq$. 
4. Compute $\bigcap_{j=1}^{\infty} \left[ -\frac{1}{j}, \frac{1}{j} \right]$.

Answer: $\{0\}$

5. Compute $\bigcap_{j=1}^{\infty} \left( -\frac{1}{j}, \frac{1}{j} \right)$.

Answer: $\{0\}$